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Hartogs-Bochner extension theorem for L^2_{loc}-functions on unbounded domains

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ABSTRACT: We prove an L^2_{loc}-Hartogs-Bochner type extension theorem for unbounded domain D in a complex manifold X of complex dimension $n \geqslant 2$. More precisely, we show that if Φ is a paracompactifying family of closed subsets of X not containing X, then the $\bar{\partial}$-cohomology group of $(0,1)$-currents of class $\mathscr{C}^\infty$ on X with supports in Φ is isomorphic to the $\bar{\partial}$-cohomology group of $(0,1)$-forms with $L^2_{\text{loc}}(X)$-coefficients and with supports in Φ. Moreover, we prove that a sufficient condition for CR L^2_{loc}-functions, defined on the boundary ∂D of D, being extended holomorphically to $\overline{D}$ is that the L^2-$\bar{\partial}$-cohomology groups must vanish. Similar results are given in the $\mathscr{C}^k$ and L^p-categories.

KEYWORDS: *CR*-forms, paracompactifying family of supports, vanishing theorem

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INTRODUCTION

Holomorphic extension phenomenon of *CR*-objects is one of the fundamental problems in the theory of several complex variables. It enriched the topic with valuable results and various quantitative applications. There are many significant contributions on this phenomenon, see [1] for the case of bounded domains and [2] for unbounded domains. This paper is partly around a work of Lupacciolo [3]. His main result relates to the holomorphic extension of *CR*-functions from the boundary of a domain $D \subset X$ which do not need to be relatively compact, but closure of which is supposed to belong to some paracompactifying family Φ of closed subsets of X (not the whole family of closed sets). The result of Lupacciolu is a generalization of the well-known global extension theorem which ensures the existence and uniqueness of a holomorphic extension on an open $D \subset\subset X$ for a *CR*-function defined on the boundary of D, under the assumption $H^1_c(X, \mathscr{O}) = 0$. As usual, $H^{0,1}(X)$ stands for the $\bar{\partial}$-cohomology groups of $(0,1)$-forms of class $\mathscr{C}^\infty$ on X. We denote also by $H^{0,1}_{\Phi,L^2_{loc}}(X)$ (resp., $H^{0,1}_{\Phi,\text{cur}}(X)$) the $\bar{\partial}$-cohomology groups of $(0,1)$ locally L^2-integrable forms (resp., $(0,1)$-currents) on X whose supports are in Φ.

In the current paper we focus on L^2_{loc} version of our previous work [4] where we give some geometric conditions for the holomorphic extension. More precisely, we will prove firstly that the natural map from $H^{0,1}_{\Phi,L^2_{\text{loc}}}(X)$ to $H^{0,1}_{\Phi,\text{cur}}(X)$ is an isomorphism. Then we show that there is an equivalence between the L^2_{loc} Hartogs-Bochner extension phenomenon and vanishing of the $\bar{\partial}$-cohomology group $H^{0,1}_\Phi(X)$. Examples on which the cohomology group $H^{0,1}_\Phi(X)$ is vanishing are given. We finally introduce a sufficient condition for which *CR* locally L^2 integrable function on ∂D can be extended holomorphically to locally L^2 integrable function on $\overline{D}$.

Definition 1 Let Σ be an oriented real hypersurface without boundary in $\mathbb{C}^n$. Recall that a function $f \in L^2_{loc}(\Sigma)$ is called *CR*-function if it satisfies the tangential Cauchy-Riemann equation in the weak sense, i.e.

$$\int_\Sigma f\,\bar{\partial}\lambda = 0$$

for any $(n, n-2)$-form λ of class $\mathscr{C}^1$ on an open neighborhood of Σ such that $\Sigma \cap \operatorname{supp}\lambda$ is compact. This is equivalent to $\bar{\partial}_b f = 0$ on Σ, where $\bar{\partial}_b$ is defined in sense of distributions. In this case, $f \wedge [\Sigma]^{0,1}$ is a $(0,1)$-current on $\mathbb{C}^n$ with support in Σ, and $\bar{\partial}(f \wedge [\Sigma]^{0,1}) = 0$, where $[\Sigma]^{0,1}$ is the bidegree $(0,1)$-part of the integration current on Σ (see [5]).

Definition 2 ([6]) A family Φ of closed subsets of a topological space M is a family of supports if it satisfies the following two conditions:
(i) If $A \in \Phi$, $A_1 \subset A$ and A_1 is closed in M, then $A_1 \in \Phi$.
(ii) If $A_1, A_2 \in \Phi$, then $A_1 \cup A_2 \in \Phi$.
A family Φ of supports is paracompactifying if it satisfies, besides conditions (i) and (ii), the following two conditions:
(iii) If $A \in \Phi$, then A is paracompact.

(iv) If $A \in \Phi$, then A has a closed neighborhood belonging to Φ.

A paracompactifying family Φ of supports is said to be cofinal if there is a sequence $\{C_j\}_{j\in\mathbb{N}}$ of sets $C_j \in \Phi$ such that each $C \in \Phi$ is contained in certain C_j. By virtue of Definition 2 (iv), the sequence $\{C_j\}_{j\in\mathbb{N}}$ can be chosen so that each C_j is contained in the interior of C_{j+1}.

Throughout the paper, we will fix a cofinal paracompactifying family of supports denoted also by Φ in a complex manifold X of complex dimension n.

RESULTS

The following theorems are the main results.

Theorem 1 *Let X and Φ be as above. Then we have the isomorphisms:*

$$H^{0,1}_{\Phi,L^2_{\mathrm{loc}}}(X) \cong H^{0,1}_{\Phi,\mathrm{cur}}(X) \cong H^{0,1}_{\Phi,\mathscr{C}^k}(X).$$

Recall that if K is a compact set in a complex manifold M and f is an arbitrary function holomorphic on the connected set $M\backslash K$ so that each such f has a holomorphic extension to all of M, then we say that the Hartogs phenomenon holds in M. The L^2_{loc}-Hartogs-Bochner phenomenon holds for a domain $U \Subset M$ if any CR (p,q)-form with L^2_{loc}-coefficients defined on the boundary ∂U can be extended to a (p,q)-form F with coefficients in $L^2_{\mathrm{loc}}(\overline{U})$, $\bar\partial F = 0$ in U, and $F_{|\partial U} = f$. The extension phenomena are closely related to the cohomology groups with support condition.

Theorem 2 *Let X be a Stein manifold of complex dimension $n \geqslant 2$ and let Φ be a paracompactifying family of closed subsets of X such that Φ does not contain X, and for all $K \in \Phi$, we have X is a 1-concave extension of $X\backslash K$. Then the Hartogs-Bochner phenomenon holds in the $L^2_{\mathrm{loc}}(X)$-setting.*

The following example shows that if X is a Stein manifold of complex dimension $n \geqslant 2$, we can find a paracompactifying family of closed subsets Φ of X that does not contain X for which $H^{0,1}_{\Phi}(X) = 0$.

Example 1 Let X be an n dimensional Stein manifold with $n \geqslant 2$ and ϕ a strictly plurisubharmonic exhaustive function on X. Let $D \subset X$ be a domain which don't need to be relatively compact in X such that $\overline{D} \neq X$. Take Φ a family of closed subsets containing $\overline{D \cup \{\phi < c\}_{c<\infty}}$. Then $D_c = D \cup \{\phi < c\}_{c<\infty}$ is a family of non-relatively compact domains with connected boundaries such that X is a generalized 1-concave extension of $X\backslash D_c$. In particular, if $X = \mathbb{C}^n$, $n \geqslant 2$, take $\phi(z) = |z|^2$, $D = \{z \in \mathbb{C}^n, Im(z_n) > 0\}$, $D_c = D \cup \{z \in \mathbb{C}^n, \phi(z) < c\}$. It is clear that $\cup_{c\in\mathbb{R}^+_*} D_c = \mathbb{C}^n$ and $\mathbb{C}^n$ is a 1-concave generalized extension of $\mathbb{C}^n\backslash D_c$. Consider Φ a paracompactifying family of closed subsets containing $\overline{D \cup \{\phi(z) < c\}}$ for all $c < \infty$.

Thus the following theorem makes sense.

Theorem 3 *Let $D \subset X$ be a domain with smooth connected boundary in a complex manifold X of complex dimension $n \geqslant 2$. Let Φ be a paracompactifying family of closed subsets of X. Assume that X does not belong to Φ and that $H^{0,1}_{\Phi}(X) = 0$. Then for every CR function $f \in L^2_{\mathrm{loc}}(\partial D)$, there exists a function $F \in Ł^2_{\mathrm{loc}}(\overline{D}) \cap \mathscr{O}(D)$ such that $F|_{\partial D} = f$.*

Remark 1 As in the same way as for the L^2_{loc}-case, the above results of Theorem 2 and Theorem 3 hold true for L^p_{loc} and $\mathscr{C}^k$-topologies.

Proof of Theorem 1

We first fix the following notation. Let X, D, and Φ be as above. Denote by $\Lambda^{p,q}(X)$ the space of $\mathscr{C}^\infty$ smooth (p,q)-forms on X. The associated $\bar\partial$-cohomology group is denoted as above by $H^{p,q}(X)$. Denote by $\Lambda^{p,q}_{\Phi}(X)$ the space of $\mathscr{C}^\infty$ smooth (p,q)-forms on X with support in Φ. If $\phi \in \Phi$, we denote by $\Lambda^{p,q}_{\phi}(X)$ the subspace of $\Lambda^{p,q}(X)$ consisting of (p,q)-forms supported in ϕ. Then

$$\Lambda^{p,q}_{\Phi}(X) = \bigcup_{\phi\in\Phi} \Lambda^{p,q}_{\phi}(X).$$

Each of the spaces $\Lambda^{p,q}_{\phi}(X)$ is closed in $\Lambda^{p,q}(X)$, they are therefore Fréchet spaces and the topology on $\Lambda^{p,q}_{\Phi}(X)$ is the finest topology for which the inclusions $\Lambda^{p,q}_{\phi_j}(X) \hookrightarrow \Lambda^{p,q}_{\Phi}(X)$ are all continuous. In this way, $\Lambda^{p,q}_{\Phi}(X)$ is exhibited as the strict inductive limit of the sequence $\{\Lambda^{p,q}_{\phi_j}(X)\}_{j=1}^{\infty}$ of Fréchet spaces. The space $\Lambda^{p,q}_{\Phi}(X)$ is a locally convex, Hausdorff, complete topological vector space. In general, it is not metrizable. The $\bar\partial$ operator maps $\Lambda^{p,q}_{\Phi}(X)$ continuously to $\Lambda^{p,q+1}_{\Phi}(X)$. We denote by $\mathscr{Z}^{p,q}_{\Phi}(X)$ the subspace of all $\bar\partial$-closed forms in $\Lambda^{p,q}_{\Phi}(X)$. The $\bar\partial$-cohomology group with support in Φ is then defined as the quotient space

$$H^{p,q}_{\Phi}(X) = \mathscr{Z}^{p,q}_{\Phi}(X)/\bar\partial\Lambda^{p,q-1}_{\Phi}(X).$$

This group is equipped with the quotient topology which in general is not Hausdorff. Cohomology groups for $\mathscr{C}^k$ and L^2_{loc} denoted $H^{p,q}_{\Phi,\mathscr{C}^k}(X)$ and $H^{p,q}_{\Phi,L^2_{\mathrm{loc}}}(X)$ are defined similarly. If $\overline{D} \in \Phi$, the cohomology groups of (p,q)-forms with coefficients in $\mathscr{C}^\infty(X)$ (resp. $\mathscr{C}^k(X)$ and $L^2_{\mathrm{loc}}(X)$) and with supports in $\overline{D}$ are denoted by $H^{p,q}_{\overline{D}}(X)$ (resp. $H^{p,q}_{\overline{D},\mathscr{C}^k}(X)$ and $H^{p,q}_{\overline{D},L^2_{\mathrm{loc}}}(X)$). In particular, if Φ is the family of compact subset of X such cohomology groups are called cohomology groups with compact supports and denoted respectively by $H^{p,q}_c(X)$, $H^{p,q}_{c,\mathscr{C}^k}(X)$, and $H^{p,q}_{c,L^2_{\mathrm{loc}}}(X)$. These groups encode the obstruction for solving the $\bar\partial$-equation in such classes of forms with supports in Φ.

The space of currents of bidegree (p,q) on X is denoted as usual by $\mathscr{D}'^{p,q}(X)$. We denote by $\mathscr{D}'^{p,q}_\Phi(X)$ the subspace of $\mathscr{D}'^{p,q}(X)$ consisting of currents whose supports belong to Φ. By duality $\bar\partial$ extends to a map from $\mathscr{D}'^{p,q}_\Phi(X)$ to $\mathscr{D}'^{p,q+1}_\Phi(X)$. Cohomology groups for currents with supports in Φ is defined as the quotient space

$$H^{p,q}_{\Phi,\mathrm{cur}}(X) = \mathscr{Z}'^{p,q}_\Phi(X)/\bar\partial\mathscr{D}'^{p,q-1}_\Phi(X).$$

Proof Theorem 1: Let us prove the first isomorphism. This will be achieved by showing that the natural map

$$i : H^{0,1}_{\Phi,L^2_{\mathrm{loc}}}(X) \to H^{0,1}_{\Phi,\mathrm{cur}}(X)$$

is bijective.

(a) Injectivity: It follows from [7] that for all $q \geqslant 1$ and every $\varepsilon > 0$ there are linear operators

$$R_\varepsilon : \mathscr{D}'^{0,q}(X) \to \Lambda^{0,q}(X)$$

and

$$A_\varepsilon : \mathscr{D}'^{0,q}(X) \to \mathscr{D}'^{0,q-1}(X)$$

such that the operator A_ε is also continuous from $(L^2_{\mathrm{loc}}(X))^{0,q}$ to $(L^2_{\mathrm{loc}}(X))^{0,q-1}$ and from $\Lambda^{0,q}(X)$ to $\Lambda^{0,q-1}(X)$. In addition, For all $T \in \mathscr{D}'^{0,q}(X)$, the supports of $R_\varepsilon T$ and $A_\varepsilon T$ are contained in an ε-neighborhood of T, and we have

$$T = R_\varepsilon T + \bar\partial A_\varepsilon T + A_\varepsilon\bar\partial T. \tag{1}$$

Now consider a class $[f]$ in $H^{0,1}_{\Phi,L^2_{loc}}(X)$ such that $i[f] = [0]$ in $H^{0,1}_{\Phi,\mathrm{cur}}(X)$. Then there exists a distribution S with support in Φ such that $\bar\partial S = f$ in X. By (1) and the continuity of A_ε on $(L^2_{\mathrm{loc}}(X))^*$, we get $f = \bar\partial(R_\varepsilon S + A_\varepsilon f)$ and $(R_\varepsilon S + A_\varepsilon f) \in L^2_{\mathrm{loc}}(X)$ with support in ε-neighborhood of the support of f. For $\varepsilon > 0$ small enough, by Definition 2 (iv), the support of $(R_\varepsilon S + A_\varepsilon f)$ is an element of Φ. Thus $[f] = [0]$ in $H^{0,1}_{\Phi,L^2_{\mathrm{loc}}(X)}(X)$. This proves the injectivity of i.

(b) Surjectivity: Let $[f]$ be a class in $H^{0,1}_{\Phi,\mathrm{cur}}(X)$ with $\bar\partial f = 0$. From (1), we get $f - R_\varepsilon f = \bar\partial A_\varepsilon f$, where $R_\varepsilon f \in \mathscr{C}^\infty(X) \subset L^2_{\mathrm{loc}}(X)$, $A_\varepsilon f \in \mathscr{D}'(X)$, and their supports are in some ε-neighborhood of the support of f, for $\varepsilon > 0$ small enough, the supports of $R_\varepsilon f$ and $A_\varepsilon f$ are in Φ. Then $[f] = [R_\varepsilon f]$ in $H^{0,1}_{\Phi,\mathrm{cur}}(X)$. This shows that i is surjective. The second isomorphism is proved in a similar fashion by using the $\mathscr{C}^k$-estimates for $\bar\partial$. □

We can also invoke the following consequence of Proposition 2.2 in [8].

Corollary 1 *If D is a domain in X with $\overline{D} \in \Phi$ and $H^{0,1}_\Phi(X) = 0$, then $H^{0,1}_{\overline{D}}(X) = 0$, $H^{0,1}_{\overline{D},\mathscr{C}^k}(X) = 0$, and $H^{0,1}_{\overline{D},L^2_{\mathrm{loc}}}(X) = 0$.*

Proof: We will prove $H^{0,1}_{\overline{D}}(X) = 0$ provided that $H^{0,1}_\Phi(X) = 0$. Let $f \in \mathscr{Z}^{0,1}_{\overline{D}}(X)$. As $H^{0,1}_\Phi(X) = 0$, there exists a function g in $\mathscr{C}^\infty(X)$ with support in Φ such that $\bar\partial g = f$. So $\mathrm{supp}(f) \subset \mathrm{supp}(g)$ and g is holomorphic on $X\backslash\mathrm{supp}(g)$. If $\overline{D} \supseteq \mathrm{supp}(g)$, we have a solution with exact support. If not, we may assume $\overline{D} \subset \mathrm{supp}(g)$, then $X\backslash\mathrm{supp}(g) \subset X\backslash\overline{D}$. Since $g = 0$ in $X\backslash\mathrm{supp}(g) \subset X\backslash\overline{D}$, we then have $g = 0$ in $X\backslash\overline{D}$, by analytic continuation. Therefore $\mathrm{supp}(g) \subset \overline{D}$. The other two cases can be proved similarly. □

Proof of Theorem 2 and Theorem 3

Proof Theorem 2: Under the hypotheses, it follows from Proposition 1.2 in [9] that $H^{0,1}_\Phi(X)$ is isomorphic to $H^{0,1}_{\Phi,L^2_{loc}}(X)$ and since $H^{0,1}_\Phi(X) = 0$, by Theorem 3.2 in [4], we have $H^{0,1}_{\Phi,L^2_{\mathrm{loc}}}(X) = 0$. Let D be a domain with connected boundary ∂D in X and let $f \in L^2_{\mathrm{loc}}(\partial D)$ be a CR function. Let $\tilde f$ be the extension of f by zero to $\overline{D}$. Then $\tilde f \in L^2_{\mathrm{loc}}(\overline{D})$ and $\bar\partial\tilde f \in (L^2_{\mathrm{loc}}(X))^{0,1}$. Let $\chi_{\overline{D}}$ be the characteristic function of $\overline{D}$, we then have $\tilde h = \chi_{\overline{D}}\bar\partial\tilde f \in (L^2_{\mathrm{loc}}(X))^{0,1}$, with support in $\overline{D} \in \Phi$, and $\bar\partial\tilde h = 0$. As $H^{0,1}_{\Phi,L^2_{\mathrm{loc}}}(X) = 0$, there is an $L^2_{\mathrm{loc}}(X)$-function λ with support in Φ such that $\bar\partial\lambda = \tilde h$. Since $X \notin \Phi$, we have $X\backslash\overline{D} \neq \varnothing$, $\overline{D} \subset \mathrm{supp}(\lambda)$, and $\bar\partial\lambda_{|_{X\backslash\mathrm{supp}(\lambda)}} = 0$. By Proposition 1.1 in [9], the space $H^{0,0}(X)$ is isomorphic to $H^{0,0}_{L^2_{\mathrm{loc}}}(X)$ and hence $H^{0,0}(X\backslash\mathrm{supp}(\lambda))$ is isomorphic to $H^{0,0}_{L^2_{\mathrm{loc}}}(X\backslash\mathrm{supp}(\lambda))$. Since X is a 1-concave extension of $X\backslash\mathrm{supp}(\lambda)$, then the restriction map $H^{0,0}_{L^2_{\mathrm{loc}}}(X) \to H^{0,0}_{L^2_{\mathrm{loc}}}(X\backslash\mathrm{supp}(\lambda))$ is an isomorphism. Thus λ has an L^2_{loc}-holomorphic extension $\tilde\lambda$ to X. Set $F = \tilde f - (\lambda - \tilde\lambda)|_{\overline{D}}$. It is clear that this function is in $L^2_{\mathrm{loc}}(\overline{D})$, holomorphic in D, and $F|_{\partial D} = f$. This shows that the Hartogs-Bochner phenomenon holds in L^2_{loc}-category. □

Proof Theorem 3: Suppose $f \in L^2_{\mathrm{loc}}(\partial D)$ is a CR function. Let us extend f by zero to a function $\tilde f \in L^2_{\mathrm{loc}}(\overline{D})$ so that $\bar\partial\tilde f$ is a $(0,1)$-form with $L^2_{\mathrm{loc}}(\overline{D})$-coefficients. As above, $\tilde h = \chi_{\overline{D}}\bar\partial\tilde f$ is a $\bar\partial$-closed $(0,1)$-form with $L^2_{\mathrm{loc}}(X)$-coefficients. Since $\mathrm{supp}(\tilde h) \subset \overline{D}$, then $\mathrm{supp}(\tilde h) \in \Phi$. Since $H^{0,1}_{\Phi,L^2_{\mathrm{loc}}}(X) = 0$, then $\tilde h = \bar\partial\lambda$ with $\lambda \in L^2_{\mathrm{loc}}(X)$ and $\mathrm{supp}(\lambda) \in \Phi$. Further, $\bar\partial\lambda|_{X\backslash\overline{D}} = 0$ and $\lambda \equiv 0$ in $X\backslash\mathrm{supp}(\tilde h)$, so $\lambda \equiv 0$ in $X\backslash\overline{D}$ by the principle of analytic continuation.

Set

$$\tilde\lambda = \begin{cases} \lambda, & \text{in } D; \\ 0, & \text{in } X\backslash D. \end{cases}$$

We have $\partial\tilde\lambda = \bar\partial\tilde f$ in D. Set $F = \tilde f - \tilde\lambda$. We have $\bar\partial F = 0$ in D and $F = f$ on ∂D, so F is the desired L^2_{loc}-holomorphic extension of f in D. □

CONCLUSION

This paper dealt with the Hartogs-Bochner extension phenomenon for *CR* $(0,1)$-forms in $L^2_{\rm loc}$-setting. Geometric characterizations of the phenomenon related to the vanishing of certain cohomology groups are given.

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REFERENCES

1. Laurent-Thiébaut C (1989) On the Hartogs-Bochner extension phenomenon for differential forms. *Math Ann* **284**, 103–119.
2. Merker J, Porten E (2006) Holomorphic extension of *CR* functions, envelopes of holomorphy and removable singularities. *Int Math Res Surv* **2006**, 28925.
3. Lupacciolu G (1990) Some global results on extension of *CR* objects in complex manifolds. *Trans Am Math Soc* **321**, 761–774.
4. Khidr S, Sambou S (2021) Generalization of the Hartogs-Bochner theorem to an unbounded domain. [Submitted]
5. Kohn JJ, Rossi H (1965) On the extension of holomorphic functions from the boundary of a complex manifold. *Ann Math* **81**, 451–472.
6. Morimoto M (1993) *An Introduction to Sato's Hyperfunctions*, Trans Math Monographs, Vol **129**, AMS, Providence, Rhode Island.
7. Chirka EM (1979) Regularization and $\bar{\partial}$-homotopy on a complex manifold. *Dokl Akad Nauk SSSR* **244**, 300–303.
8. Laurent-Thiébaut C, Shaw MC (2019) Solving $\bar{\partial}$ with prescribed support on Hartogs triangles in $\mathbb{C}^2$ and $\mathbb{CP}^2$. *Trans Amer Math Soc* **371**, 6531–6546.
9. Laurent-Thiébaut C (2015) Théorie L^p et dualité de Serre pour l' équation de Cauchy-Riemann. *Ann Fac Sci Toulouse Math* **24**, 251–279.