New sufficient conditions for Hamiltonian, pancyclic and edge-Hamilton graphs

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ABSTRACT: The decycling number $\nabla(G)$ of a graph $G$ is the smallest number of vertices whose deletion yields a forest. Bau and Beineke proved that $\kappa(G) \leq \nabla(G) + 1$ for every graph $G$, where $\kappa(G)$ is the connectivity of $G$ (Australas J Combin, 25:285-298, 2002). In this paper, we consider graphs with $\kappa(G) = \nabla(G) + 1$ and establish sufficient conditions for such graphs to be Hamiltonian, pancyclic and edge-Hamilton, respectively. To our knowledge, this is the first result studying Hamilton problem in terms of decycling number. It is well-known that determining the decycling number of a graph is equivalent to finding the greatest order of an induced forest and some sufficient conditions for Hamiltonian graphs are also sufficient for the existence of completely independent spanning trees. This paper may provide a new condition implying completely independent spanning trees.

KEYWORDS: Hamilton cycle, pancyclic, edge-Hamilton, decycling number, connectivity

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INTRODUCTION

Graphs considered in this paper are finite, simple and connected. For general theoretic notations, we follow Bondy and Murty [1]. Throughout the paper, the letters $G$ denotes a graph. $\kappa(G)$ and $\alpha(G)$ denote the connectivity and independence number of $G$, respectively.

A cycle passing through all the vertices of a graph is called a Hamilton cycle. A graph is said to be Hamiltonian if it has a Hamilton cycle. We say that a graph is pancyclic if it contains cycles of all possible length from three up to the order of the graph. A graph is called an edge-Hamilton graph if every edge of the graph lies in a Hamilton cycle. Edge-Hamilton graphs and pancyclic graphs are generalizations of Hamiltonian graphs.

The decision problems that whether a graph contains a Hamilton cycle is one of the most famous NP-complete problems, and so it is unlikely that there exist good characterizations of such graphs. Although the Hamilton problem has been widely studied, researchers only went an initial step towards the sufficient and necessary conditions which ensure the existence of a Hamilton cycle. For this reason, it is natural to ask for sufficient or necessary conditions. The first sufficient condition for a graph to be Hamiltonian is due to Dirac in 1952 [2].

**Theorem 1** Every graph with $n \geq 3$ vertices and minimum degree at least $n/2$ is Hamiltonian.

In 1971, Bondy [3] raised a sufficient condition for a graph to be pancyclic.

**Theorem 2** Let $G$ be a Hamiltonian graph on $n$ vertices and $m$ edges. If $m \geq n^2/4$, then $G$ is either pancyclic or else is $K_{n/2, n/2}$.

Since then, many other interesting sufficient or necessary conditions for a graph to be Hamiltonian have been obtained, see [4, 5]. In particular, Chvátal and Erdös [6] proved that every graph $G$ on at least three vertices and $\alpha(G) \leq \kappa(G)$ has a Hamilton cycle. In other words, forbidding small connectivity admits a Hamilton cycle. It is interesting that many sufficient conditions for Hamiltonianities on classical graph properties can be naturally extended to the random graphs, see [7, 8]. Shang gave a sufficient condition for subgraphs random bipartite graph [9]. Furthermore, He studied the bipancyclicity of random bipartite graphs [10].

If $S \subseteq V(G)$ and $G - S$ is acyclic, then $S$ is said to be a decycling set of $G$ (also known as feedback vertex set). The smallest size of a decycling set of $G$ is said to be decycling number of $G$ and is denoted by $\nabla(G)$. A decycling set of this cardinality is called a $\nabla$-set.

In theory, determining the decycling number $\nabla(G)$ of graph is equivalent to finding the order of a greatest induced forest. The decycling number problem has a long and rich history and classical question concern its computation. However, it has been shown that determining the decycling number of graphs is NP-hard [11]. Indeed, only a few of graphs are available, such as cubic graphs [12]. It is worth noting that Bau and Beineke [13] considered the relation between decycling number and connectivity of graphs.

**Theorem 3** For every graph $G$, $\kappa(G) \leq \nabla(G) + 1$. 

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Motivated by the results above, in this paper we will characterize the Hamiltonian, pancyclic and edge-Hamilton properties in graphs with $\kappa(G) = \nabla(G) + 1$. Here, the purpose that $\kappa(G) = \nabla(G) + 1$ is to forbid small connectivity. Note that $G - S$ is a tree for every $\nabla$-set $S$ in such graphs. From now on, we use $t$ to denote the number of leaves in $G - S$.

The main results of this paper are presented as follows.

**Theorem 4** Let $G$ be a graph with $\kappa(G) = \nabla(G) + 1$. If there exists a $\nabla$-set $S$ of $G$ such that $\nabla(G) \geq t - 1$, then $G$ is Hamiltonian.

The condition that $\nabla(G) \geq t - 1$ could not be weakened, due to the well-known 1-tough property [14].

**Proposition 1** If a graph $G$ has a Hamilton cycle, then for each nonempty set $S \subseteq V(G)$, the graph $G - S$ has at most $|S|$ components.

For example, the graph $G$ as shown in Fig. 1 satisfies $\kappa(G) = 2$, $\nabla(G) = 1$ and $t = 3$ ($\nabla(G) < t - 1$). However, $G - \{s, u\}$ has 3 components, i.e., it fails the necessary condition of Proposition 1. Hence it is not Hamiltonian.

![Fig. 1](image)

In the rest of this paper, we will prove our main results. A brief word about our notation. For $W \subseteq V(G)$, by $G - W$ and $G[W]$ we mean the subgraphs induced by $V(G) - W$ and $G[W]$, respectively. For a vertex $v \in V(G)$, we denote by $d(v)$ the degree of $v$, and by $N(v)$ the neighborhood of $v$. Given a subgraph $H$ of $G$, we let $N_H(v) = N(v) \cap V(H)$, and $d_H(v) = |N_H(v)|$. If $S \subseteq V(G)$, we define $N_H(S) = N(S) \cap V(H)$ and $\delta(S) = |N_H(S)|$. We call a vertex of degree $i$ a $i$-vertex. Given a tree $T$ and a path $P = v_1v_2\cdots v_q$ ($q \geq 1$) on $T$, where $v_1$ is a leaf of $T$, then $P$ is called a pendant path of $T$ if it is a maximal path with no vertex of degree $\geq 3$ (see Fig. 2).

![Fig. 2](image)

**PROOF OF THEOREMS**

In order to prove Theorem 4, we need the following lemmas.

**Lemma 1** If $G$ has a bipartition $V(G) = S + T$ such that:

(a) $G[T]$ is a tree, (b) every $i$-vertex of $G[T]$ is adjacent to at least $|S| + 1 - i$ vertices of $S$, and (c) $|T| - 2 \geq |S| \geq t - 1$, where $t$ is number of leaves of $G[T]$, then $G$ is Hamiltonian.

**Lemma 2** Let $G$ be a graph with $\kappa(G) = \nabla(G) + 1$. If $\nabla(G) \geq |V(G)| - \nabla(G) - 1$, then $G$ is Hamiltonian.

**Proof:** If $\nabla(G) \geq |V(G)| - \nabla(G) - 1$, then $\nabla(G) + 1 \geq |V(G)|/2$. It follows that,

$$\delta(G) \geq \kappa(G) = \nabla(G) + 1 \geq \frac{|V(G)|}{2}.$$

According to Theorem 1, $G$ is Hamiltonian. □

Combining Lemma 1 and Lemma 2, one can easily prove Theorem 4.

**Proof of Theorem 4**

Let $S$ be a $\nabla$-set of $G$ such that $|S| \geq t - 1$. Define $T = G - S$. Then every $i$-vertex of $T$ is adjacent to at least $|S| + 1 - i$ vertices of $S$, since $\delta(G) \geq \kappa(G) = |S| + 1$. If $|T| - 2 \geq |S|$, then the theorem is true according to Lemma 1. Otherwise, $|S| \geq |T| - 1$, then $\nabla(G) \geq |V(G)|/2$.
$|V(G)| - \nabla(G) - 1$. So, the theorem holds in this case, due to Lemma 2.

Now, our goal is to prove Lemma 1. We first give a result which is a weaker version of Lemma 1.

**Lemma 3** If $G$ has a bipartition $V(G) = S + T$ such that: (1) $G(T)$ is a tree, (2) every $i$-vertex of $G[T]$ is adjacent to at least $|S| + 1 - i$ vertices of $S$, and (3) $|S| = t - 1$, where $t$ is number of leaves of $G(T)$, then $G$ is Hamiltonian.

**Proof:** We will write $T$ rather than the more customary $G[T]$. It suffices to prove the lemma in the case that every $i$-vertex of $T$ is adjacent to $|S| + 1 - i$ vertices of $S$, i.e., every vertex of $T$ has degree $|S| + 1$ in $G$. Note that every leaf of $T$ is adjacent to all vertices of $S$.

We apply induction on $t$. For $t = 2$, put $S = \{s_1\}$. In this case, $T$ is a path. Suppose that $G = v_1v_2 \cdots v_n$, $n \geq 2$. Then $s_1v_1v_2 \cdots v_nv_s1$ is a Hamilton cycle. Now, assume that $t = k + 1$ with $k \geq 2$ and set $S = \{s_1, s_2, \ldots, s_k\}$. Choose a pendant path $P = u_1u_2 \cdots u_q$ with $q \geq 1$, where $u_i$ is a leaf of $T$. Since $u_q$ is a 2-vertex of $T$, $u_q$ is adjacent to $|S| - 1$ vertices of $S$. Without loss of generality, let $N(u_q) = \{s_1, s_2, \ldots, s_{k-1}\}$ and let $u$ be a vertex (other than $u_{q-1}$) adjacent to $u_q$ on $T$. Then we have $d_s(u) \leq k - 2$, since $d_T(u) \geq 3$. So, we assume that $s_1 \notin N(u)$. Let $G_1 = G - \{P \cup \{s_1\}\}, T_1 = T - V(P)$ and $S_1 = S - \{s_1\}$. Then, $G_1$ has a bipartition $V(G_1) = T_1 + S_1$ satisfies (1), (2), and (3). Based on the induction hypothesis, $G_1$ is Hamiltonian. For any leaf $l_1$ of $T_1$ (in $G_1$), there exists a Hamilton cycle $C_1$ of $G_1$ passes through $l_1$. Since $l_1$ has only one neighbour in $T_1$, $C_1$ passes through an edge $s_1l_1$, where, $s_1$ is a vertex of $S_1$. We get a Hamilton cycle of $G$ by replacing the edge $s_1l_1$ of $C_1$ with the path $s_1u_1u_2 \cdots u_q s_1l_1$. The proof is completed. □

For example, Fig. 3 shows a construction of a Hamilton cycle.

**Remark 1** Remark that we restrict the number of edges between $S$ and $T$ (every $i$-vertex of $T$ is adjacent to $|S| + 1 - i$ vertices of $S$) in the proof of Lemma 3. In the proofs of Lemma 1, Lemma 4, Lemma 6, Lemma 7 and Lemma 9, we will keep this restriction.

By refining slightly the proof of Lemma 3, one can obtain Lemma 1.

**Proof of Lemma 1**

We finish this lemma by applying double induction on $t$ and $|T|$. For $t = 2$, we prove it by using induction on $|T|$. If $|T| = 3$, then $|S| = 1$. Based on Lemma 3, the statement is true. Assume that $T = v_1v_2 \cdots v_k$, $k \geq 4$ and set $S = \{s_1, s_2, \ldots, s_{k-1}\}$. We get a Hamilton cycle of $G_1$ of $G$ by replacing the edge $s_1v_k$ by the path $s_1v_1s_kv_2$. Then, $G_0$ has a bipartition $V(G_0) = T_0 + S_0$ satisfies (a), (b), and (c).

For the induction hypothesis on $|T|$, $G_0$ has a Hamilton cycle, say $C_0$. It follows that there exists a vertex $s_k \in S_0 - \{s_1\}$ such that $C_0$ passes through the edge $s_1v_k$. Replacing the edge $s_1v_k$ by the path $s_1v_1s_kv_2$ turns into a Hamilton cycle of $G$.

Now, assume that $t = m > 3$. If $|T| = m + 1$, then $|S| = m - 1$. Based on Lemma 3, the statement is true. Now, assume that $|T| = k$ with $k \geq m + 2$ and let $S = \{s_1, s_2, \ldots, s_l\}$, where, $m + 1 \leq l \leq k - 2$. We distinguish two cases.

**Case 1.** There is a pendant path $P_1$ on $T$ with $|P_1| = 1$.

In this case, suppose that $P_1 = u_1$. Let $u$ be a vertex adjacent to $u_1$ on $T$ and assume that $N_G(u) \subseteq \{s_1, s_2, \ldots, s_{l-1}\}$. Denote $G_1 = G - \{u_1, s_1\}$. It is easy to check that $G_1$ satisfies (a), (b), and (c). By the induction hypothesis on $t$, $G_1$ has a Hamilton cycle, say $C_1$. For a leaf $l_1$ of $T - \{u_1\}$, there exists a vertex in $S - \{s_1\}$ such that $C_1$ passes through the edge $s_1l_1$. Replacing the edge $s_1l_1$ by the path $s_1u_1s_1l_1$ makes up a Hamilton cycle of $G$.

**Case 2.** Every pendant path on $T$ contains at least two vertices. Under this case, we use induction on $|T|$. First, choose a pendant path $P_2 = w_1w_2 \cdots w_q$ of $T$, $q \geq 2$. Assume, without loss of generality, that $N_G(w_1) \subseteq \{s_1, s_2, \ldots, s_{l-1}\}$. Let $G_2 = G - \{w_1\}$. Then, $G_2$ satisfies (a), (b), and (c). By the induction hypothesis on $|T|$,
$G_2$ contains a Hamilton cycle $C_2$. For a leaf $l_2$ of $T - P$, there exists a vertex $s_i \in S - \{s_k\}$ such that $C_2$ passes through the edge $s_i l_2$. Replacing the edge $s_i l_2$ by the path $s_i w_1 s_1 l_2$ turns into a Hamilton cycle of $G$.

We are now ready to prove Theorem 5 employing the ideas in Theorem 4. First, we also give two lemmas.

**Lemma 4** If $G$ has a bipartition $V(G) = S + T$ such that:
(a) $G[T]$ is a tree, (b) every i-vertex of $G[T]$ is adjacent to at least $|S| + 1 - i$ vertices of $S$, and (c) $|T| - 1 \geq |S| \geq |T| - 2$, then $G$ is pancyclic.

**Proof:** If $V(G) \geq |V(G)| - \nabla(G) - 1$, then $\nabla(G) + 1 > |V(G)|/2$. Thereby, $\delta(G) > |V(G)|/2$. Based on Theorem 2, $G$ is a pancyclic.

**Proof of Theorem 5**

Let $S$ be a $\nabla$-set of $G$ such that $|S| \geq t$. Define $T = G - S$. Then every i-vertex of $T$ is adjacent to at least $|S| + 1 - i$ vertices of $S$, since $\delta(G) \geq |S| + 1$. If $|T| - 2 \geq |S| \geq t$, then our theorem is true according to Lemma 4. Otherwise, $|S| \geq |T| - 1$, i.e., $\nabla(G) \geq |V(G)| - \nabla(G) - 1$. By Lemma 5, the theorem holds.

To prove Lemma 4, we raise the following result.

**Lemma 6** If $G$ has a bipartition $V(G) = S + T$ such that:
(a) $G[T]$ is a tree, (b) every i-vertex of $G[T]$ is adjacent to at least $|S| + 1 - i$ vertices of $S$, and (c) $|T| - 2 \geq |S|$, then our theorem is true according to Lemma 4.

**Proof:** Let $n$ denote the number of vertices in $G$. We establish the lemma in the same way as we did in Lemma 3. We use induction on $t$.

For $t = 2$, let $S = \{s_1, s_2\}$. For $|T| = 2$, there is nothing to prove, so assume that $T = v_1 v_2 \ldots v_m$, where $m \geq 3$ and $m + 2 = n$. Notice that for a given integer $i$ with $1 \leq i \leq m$, $v_i$ is adjacent to at least one vertex of $S$. Without loss of generality, assume that $v_i$ is adjacent to $s_j$. Then $s_1 v_1 v_2 \ldots v_i s_j$ is a cycle of length $i + 1$. In addition, by Lemma 3, $G$ has a Hamilton cycle, i.e., a cycle of length $n$. Thereby, the lemma holds for $t = 2$.

Let us now consider the case $t = k \geq 3$. We choose a pendant path $P = u_1 u_2 \ldots u_q$ with $q > 1$ on $T$ and a vertex $u$ adjacent to $u_q$. We may as well suppose that $N_G(u) \subseteq \{s_1, s_2, \ldots, s_{k-2}\}$.

These cycles can be constructed as follows.

Let $G_1 = G - (V(P) \cup \{s_k\})$. Then $G_1$ satisfies (1), (2) and (3). It follows that $G_1$ is pancyclic, in other words, $G_1$ has cycles of length from $3$ up to $n - q - 1$. Pick a cycle $C_1$ of length $n - q - 1$ in $G_1$ and a leaf $l_1$ of $T - P$. Then there exists a vertex $s_j \in S - \{s_k\}$ such that $C_1$ passes through the edge $s_j l_1$. For each $1 \leq j \leq q$, if $u_j$ is adjacent to $s_j$, then we complete a cycle of length $n - q + j$ by replacing the edge $s_j l_1$ with path $s_i u_1 u_2 \ldots u_j s_j l_1$; otherwise, $u_j$ is adjacent to every one of $S - \{s_k\}$, then we complete a cycle of length $n - q + j$ by replacing the edge $s_j l_1$ with path $s_i u_1 u_2 \ldots u_j s_j l_1$.

In addition, $G_2$ also has a cycle $C_2$ of length $n - q - 2$. What's more, the cycle $C_2$ must contain a leaf $l_2$ of $T - P$, since $T - P$ has at least two leaves. Thereby, there exists a vertex $s_i \in S - \{s_k\}$ such that $C_2$ passes through the edge $s_i l_2$. We get a cycle of length $n - q$ by replacing $s_i l_2$ with path $s_i u_1 u_2 \ldots u_j s_i l_2$. This builds the lemma.

**Proof of Lemma 4**

Let $n$ denote the number of vertices in $G$. We achieve the lemma by applying induction on $|S|$. According to Lemma 6, the lemma holds for $|S| = t$. Assume that $|S| = k$, where $|T| - 2 > k > t$. Put $S = \{s_1, s_2, \ldots, s_k\}$. Let $T_1 = T$, $S_1 = S - \{s_k\}$ and $G_1 = G - \{s_k\}$. Since each i-vertex $u$ in $T$ satisfies $d_G(u) \geq d_G(u) - 1 = |S| - |S| - 1 = |S_i| - 1 + 1$, $G_1$ has a bipartition $V(G_1) = T_1 + S_1$ satisfies (a), (b), and (c). By the induction hypothesis, $G_1$ is a pancyclic graph, i.e., $G_1$ has cycles of length $j$ for all $3 < j < n - 1$. Therefore, $G$ has cycles of length $j$ for all $3 < j < n - 1$. Further, according to Lemma 1, $G$ is Hamiltonian. That is to say $G$ has a cycle of length $n$. It follows that $G$ is a pancyclic graph.

In the remainder of this paper, we will finish the proof of Theorem 6. The actual proof will be preceded by two lemmas.

**Lemma 7** If $G$ has a bipartition $V(G) = S + T$ such that:
(a) $G[T]$ is a tree, (b) every i-vertex of $G[T]$ is adjacent to at least $|S| + 1 - i$ vertices of $S$, and (c) $|T| - 1 \geq |S|$, then $G$ is an edge-Hamilton graph.

**Lemma 8** Let $G$ be a graph with $\kappa(G) = \nabla(G) + 1$. If $|V(G)| - \nabla(G) - 1$, then $G$ is Edge-Hamilton.

In order to prove Lemma 8, we should introduce another concept. A graph $G$ is called a Hamilton-connected graph if every two vertices of $G$ are connected by a Hamilton path. Surely, all Hamilton-connected graphs are edge-Hamilton. Benhocine and Wojda [15] have shown the following result.

**Theorem 7** Let $G$ be a 2-connected graph on $n \geq 3$ vertices. If

$$d_G(u,v) = 2 \implies \max(d_G(u),d_G(v)) \geq \frac{n+1}{2}$$

for every pair of vertices $u$ and $v$ in $G$, then $G$ is Hamilton-connected.

**Proof of Lemma 8**

Combining the conditions that $\kappa(G) = \nabla(G) + 1$ and $|V(G)| - \nabla(G) - 1$, we deduce that $G$ is 2-connected and $2|V(G)| + 2 \geq |V(G)| + 1$. Hence

$$\delta(G) \geq \kappa(G) = \nabla(G) + 1 \geq \frac{|V(G)| + 1}{2}.$$

Based on Theorem 7, $G$ is Hamilton-connected. It follows that $G$ is edge-Hamilton.
Theorem 6 follows from Lemma 7 and Lemma 8. Its proof is similar to that of Theorem 5. So, we omit it here.

In the following, we will prove Lemma 7. First, we also provide a weaker version.

Lemma 9 If G has a bipartition V(G) = S + T such that:
(1) G[T] is a tree,
(2) every i-vertex of G[T] is adjacent to at least |S| + 1 − i vertices of S, and
(3) |S| = t, where t is the number of leaves of G[T], then G is edge-Hamilton.

Proof: We complete this lemma by applying induction on t as well. For t = 2, let S = {s1, s2}. It is easy to check that the lemma holds for |T| = 2, so suppose that T = v1v2 ···vm with m ≥ 3. Considering an edge sv, where s ∈ S and v ∈ T, we only refer to the edge s1v2, since the other cases resemble it. If v is adjacent to s1, then s1v2v1v2vm−1 ···v2s1 is a Hamilton cycle with the edge s1v2. Otherwise, v is adjacent to s2, then v2s1v2vm−1 ···s2v2v1 is a Hamilton cycle with s1v2. Notice that from procedure of finding Hamilton cycle passing through s1v2, it is easy to find a Hamilton cycle passing through any given edge on T. In addition, v1v2 ···vqs1s2v1 is a Hamilton cycle containing the edge s1s2. Hence, the statement is true for t = 2.

Let us consider the case t = k ≥ 3 and set S = {s1, s2, . . . , sk}.

(I) Edge xy on T.

We first choose a pendant path P = v1v2 ···vq, q ≥ 1, such that x, y ∉ P. Let further v (other than vq−1) be a vertex adjacent to vq on T. We suppose that Nq(v) ⊆ {s1, s2, . . . , sk−2}. Let G1 = G − (V(P) ∪ {s1}). Then G1 satisfies (1), (2), and (3), which implies that G1 has a Hamilton cycle C1 containing xy. For a leaf l1 of T − P, there exists a vertex si ∈ S − {s1} such that C1 passes through the edge sis1. If v is adjacent to s1, then we make up a Hamilton cycle of G containing the edge xy by replacing s1l1 with the path s1v1v2 ···vqs1l1. Otherwise, v is adjacent to every vertex of S − {s1}, then we complete a Hamilton cycle of G containing the edge xy by replacing s1l1 with the path s1v1vq−1 ···vqs1l1.

(II) Edges sz, where s ∈ S and z ∈ T.

Here, we only refer to the edge s1z, since the other cases resemble it. Pick a pendant path P = u1u2 ···uq with q ≥ 1, such that z belongs to T − P. Choose a vertex u (other than uz−1) adjacent to uz on T.

Case 1. u is adjacent to s1.

Assume that Nq(u) ⊆ {s1, s2, . . . , sk−2}. Denote then G2 = G − (V(P) ∪ {s1}). Then G2 satisfies (1), (2) and (3), which yields that G2 has a Hamilton cycle C2 passing through the edge s1z. Since T − P contains at least two leaves, there is an edge s2l2 on C2, where s2 ∈ S − {s1} and l2 is a leaf of T − (V(P) ∪ {z}). If u is adjacent to s2, then we form a Hamilton cycle of G passing through the edge s1z by replacing s1l1 with the path s1u1u2 ···us2l2. Otherwise, u is adjacent to every vertex of S − {s1}, then we complete a Hamilton cycle of G passing through the edge s1z by replacing s1l1 with the path s1uquq−1 ···us2l2.

Case 2. u and s1 are non-adjacent.

We may assume that Nq(u) ⊆ {s3, s4, . . . , sk}, as well. Let G3 = G − (V(P) ∪ {s1}). Then we can find a Hamilton cycle C3 of G3 containing the edge s1z. Let l2 be a leaf of T − (P ∪ {z}). Then there is a vertex si ∈ S − {s2} such that the edge si1l2 lies in C3. If u is adjacent to s2, then we make up a Hamilton cycle of G passing through the edge s1z by replacing si1l2 with the path si1u1u2 ···us2l2. Otherwise, u is adjacent to every vertex s ∈ S − {s2}, then we finish a Hamilton cycle G with the edge s1z by replacing si1l2 with the path s1uquq−1 ···us2l2.

(III) Edge sjsi with 1 ≤ i < j ≤ k.

We only refer to the edge sjsk. Choose a pendant path P = w1w2 ···wq with q ≥ 1 and a vertex w (other than wq−1) adjacent to wq on T.

Case 1. w is adjacent to both s1 and s2.

Without loss of generality, suppose that Nq(w) ⊆ {s1, s2, . . . , sk−2}. Denote G1 = G − (V(P) ∪ {s1}). Then G1 has a Hamilton cycle C1 passing through the edge s1s2. Furthermore there is an edge s1l1 in C1, where s1 ∈ S − {s2} and l1 is a leaf of T − P. If w is adjacent to s1, then we complete a Hamilton cycle of G with the edge s1s2 by replacing s1l1 with path w1w2 ···ws1l1. Otherwise, w is adjacent to every vertex S − {s1}, then we complete a Hamilton cycle G with the edge s1s2 by replacing s1l1 with path s1w1w2 ···ws1s2.

Case 2. w is adjacent to neither s1 nor s2.

Under this case, Nq(w) ⊆ {s3, s4, . . . , sk}. Note that w is adjacent to at least one of {s1, s2}. Suppose that w is adjacent to s1 and let G2 = G − (V(P) ∪ {s1}). By Lemma 4, G2 has a Hamilton cycle C2. Pick a leaf l2 of T − P. Then there exists a vertex s2 ∈ S − {s1, s2} such that C2 passes through the edge s2l2. Consequently, we complete a Hamilton cycle of G with the edge s1s2 by replacing the edge s1l2 with path s1w1 ···wqs1s2.

Case 3. w is adjacent to only one of {s1, s2}.

Assume that w is adjacent to s2 and Nq(w) ⊆ {s2, s3, . . . , sk−1}. Denote G3 = G − (V(P) ∪ {s1}). Then G3 has a Hamilton cycle C3 passing through the edge s2w. We replace s2w with s3w1w2 ···wqs1s2, forming a Hamilton cycle of G with the edge s1s2.

Proof of Lemma 7

Similarly, we use double induction method on t and T.

(I) Edge xy on T.

For t = 2, we prove our statement by using induction on |T|. When |T| = 4, we have |S| = 2. As we discussed in Lemma 9, the statement is true. So, assume that T = v1v2 ···vn, n ≥ 5 and let S = {s1, s2, . . . , sk}, 2 ≤ i ≤ n − 2. Without loss of generality, assume that v1 ̸= x, y and Nq(v2) = {s1, s2, . . . , sk−1}. Define G0 = G − {v1, s1}. Then G0 satisfies (a), (b), and
(c). It follows that there is a Hamilton cycle \( C_0 \) of \( G_0 \)
passing through the edge \( xy \). Further, there exists a vertex \( s_{i} \) in \( S - \{s\} \) such that the edge \( s_{i} y \) belongs to \( C_0 \). Replacing the edge \( s_{i} y \) by the path \( s_{i} v_{3} s_{i} y \) turns into a Hamilton cycle of \( G \) containing the edge \( xy \).

For \( t = m \geq 3 \), we prove by using induction on \( |T| \). If \( |T| = m + 2 \), then \( |S| = m \). As we proved in Lemma 9, the statement is true. Let \( |T| = k \geq m + 3 \) and \( S = \{s_1, s_2, \ldots, s_i\}, m < i < k - 2 \). We deal with the following cases.

Case 1. \( T \) contains at least two pendant paths consisting of only one vertex.

In this case, we choose a pendant path \( P_1 = u_1 \), such that \( u_1 \neq x, y \). Let \( u \) be a vertex adjacent to \( u_1 \) on \( T \). We suppose that \( N(u) \subseteq \{s_1, s_2, \ldots, s_{i-2}\} \). Let \( G_1 = G - \{u_1, s_i\} \). It is easy to check that \( G_1 \) satisfies (a), (b), and (c). By the induction hypothesis on \( t \), \( G_1 \) is edge-Hamilton, in other words there is a Hamilton cycle \( C_1 \) of \( G_1 \) passing through the edge \( xy \). Pick a leaf \( l_1 \) of \( T - \{u_1\} \). Then there exists \( i_1 \in \{1, 2, \ldots, i-1\} \) such that \( C_1 \) passes through the edge \( s_{i_1} l_1 \). Replacing the edge \( s_{i_1} l_1 \) by the path \( s_{i_1} u_1 l_1 \) turns into a Hamilton cycle of \( G \) containing the edge \( xy \).

Case 2. \( T \) contains at most one pendant path consisting of only one vertex.

Under this case, we choose a pendant path \( P_2 = w_1 w_2 \cdots w_q \), \( q \geq 2 \) on \( T \) such that \( x, y \in T - P \). Suppose first that \( N_2(w_q) = \{s_1, s_2, \ldots, s_{i-1}\} \). Let now \( G_2 = G - \{w_q, s_i\} \). By induction hypothesis on \( |T| \), there is a Hamilton cycle \( C_2 \) of \( G_2 \) passing through the edge \( xy \). Pick a leaf \( l_2 \) of \( T - P \). Then there exists \( i_2 \in \{1, 2, \ldots, i-1\} \) such that \( C_2 \) passes through the edge \( s_{i_2} l_2 \). Replacing the edge \( s_{i_2} l_2 \) by the path \( s_{i_2} w_1 l_2 \) turns into a Hamilton cycle of \( G \) containing the edge \( xy \).

(II). Edges \( sz \), where \( s \in S \) and \( z \in T \).

For \( t = 2 \), set \( T = v_1 v_2 \cdots v_q \). We only treat the edge \( s_1 v_2 \), since we could solve the other cases analogously. As before, we prove our statement by using induction on \( n \). When \( n = 4 \), we have \( |S| = 2 \). According to Lemma 9, the statement is true. Assume that \( n \geq 5 \), let \( S = \{s_1, s_2, \ldots, s_i\} \) with \( 2 < i \leq n - 2 \).

Case 1. \( v_1 \) is adjacent to \( s_1 \).

Let \( G_1 = G - \{s_1\} \). By (I), \( G_1 \) has a Hamilton cycle \( C_1 \) passing through the edge \( v_1 v_2 \). We replace the edge \( v_1 v_2 \) by path \( v_2 s_1 v_3 \), forming a Hamilton cycle of \( G \) with the edge \( s_1 v_2 \).

Case 2. \( v_1 \) is not adjacent to \( s_1 \).

Define \( G_2 = G - \{v_1, v_2, s_1\} \). Then \( G_2 \) satisfies (a), (b), and (c). By induction hypothesis, \( G_2 \) has a Hamilton cycle \( C_2 \) passing through \( v_1 v_2 \). Replacing the edge \( s_1 v_2 \) by the path \( s_1 v_2 s_1 v_3 \) turns into a Hamilton cycle of \( G \) with the edge \( s_1 v_2 \).

Suppose that the result is true for \( t < m-1 \). For \( t = m \geq 3 \), we prove it by induction on \( |T| \) and. Choose a vertex of \( S \) and a vertex of \( T \), say \( s_1 \) and \( z \), respectively. If \( |T| = m + 2 \), then \( |S| = m \). According to Lemma 9, the edge \( s_1 z \) lies in a Hamilton cycle. Assume that \( |T| = k \geq m + 3 \) and let \( S = \{s_1, s_2, \ldots, s_i\} \) with \( m < i < k - 2 \).

There are at least three pendant paths, we deal with the following cases.

Case 1. \( T \) contains at least two pendant paths consisting of only one vertex.

In this case, we choose a pendant which consist of only one vertex, say \( u_1 \), such that \( u_1 \neq z \). Let \( u \) be a vertex adjacent to \( u_1 \) on \( T \), then \( d_T(u) \geq 3 \). Thus, \( u \) is adjacent to at most \( i + 1 - 3 = i - 2 \) vertices of \( S \). If \( u \) is adjacent to \( s_1 \), we suppose that \( N(u) \subseteq \{s_1, s_2, \ldots, s_{i-2}\} \). Let \( G_3 = G - \{u_1, s_1\} \). It is easy to check that \( G_3 \) has a bipartition \( V(G_3) = (T - \{u_1\}) + (S - \{s_1\}) \) satisfies (a), (b), and (c). By the induction hypothesis on \( t \), \( G_3 \) is edge-Hamilton, in other words \( G_3 \) has a Hamilton cycle \( C_3 \) containing the edge \( s_1 z \). Pick a leaf \( l_3 \) of \( T - \{u_1, z\} \) (\( T \) has at least three leaves) such that \( C_3 \) passes through the edge \( s_1 l_3 \). Replacing the edge \( s_1 l_3 \) by the path \( s_1 u_1 l_3 \) turns into a Hamilton cycle of \( G \) containing the edge \( s_1 z \). Otherwise, suppose that \( N(u) \subseteq \{s_3, \ldots, s_i\} \). Let \( G_3 = G - \{u_1, s_1\} \). Then \( G_3 \) contains a Hamilton cycle \( C_3 \) passing through \( s_1 z \). Pick a leaf \( l_3 \) of \( T - \{u_1, z\} \). Then there exists \( i_3 \in \{1, \ldots, i-1\} \) such that \( C_3 \) passes through the edge \( s_1 l_3 \). Replacing the edge \( s_1 l_3 \) by the path \( s_1 w_1 l_3 \) turns into a Hamilton cycle of \( G \) containing the edge \( s_1 z \).

Case 2. \( T \) contains at most one pendant path consisting of only one vertex.

Under this case, \( T \) contains at least two pendant paths which consists of more than one vertex. We choose a pendant path \( P_2 = w_1 w_2 \cdots w_q \) on \( T \) such that \( z, T - P_2 \). If \( w_2 \) is adjacent to \( s_1 \), we suppose that \( N_2(w_2) = \{s_1, s_2, \ldots, s_{i-1}\} \). Let \( G_2 = G - \{w_1, s_1\} \), then \( G_2 \) has a bipartition \( V(G_2) = (T - \{w_1\}) + (S - \{s_1\}) \) satisfies (a), (b), and (c). By the induction hypothesis on \( |T| \), there is a Hamilton cycle \( C_4 \) of \( G_2 \) passing through \( s_1, z \). Pick a leaf \( l_3 \) of \( T - (V(P_2) \cup \{z\}) \). Then there exists \( i_3 \in \{1, \ldots, i-1\} \) such that \( C_4 \) passes through the edge \( s_1 l_3 \). Replacing the edge \( s_1 l_3 \) by the path \( s_1 w_1 l_3 \) turns into a Hamilton cycle of \( G \) containing the edge \( s_1 z \).

Subcase 2.1. There is a vertex \( s' \) in \( S - \{s_1\} \), such that \( s' \) is adjacent to \( z \).

Let \( G'_4 = G - \{w_1, s_1\} \). Then \( G'_4 \) has a bipartition \( V(G'_4) = (T - \{w_1\}) + (S - \{s_1\}) \) satisfies (a), (b), and (c). By the induction hypothesis on \( |T| \), \( G'_4 \) contains a Hamilton cycle \( C'_4 \) passing through \( s' \). We replace \( s z \) by the path \( s' w_1 s_1 z \), forming a Hamilton cycle containing \( s_1 z \).

Subcase 2.2. \( z \) is only adjacent \( s_1 \).
Since every \( j \)-vertex of \( T \) is adjacent to at least \( |S| + 1 - j \) vertices of \( S \), \( d_T(x) \geq |S| \). Therefore, \( T \) has at least \( |S| \geq 4 \) pendant paths. Under the case, \( T \) contains at most one pendant path consisting of one vertex, this implies that there is at least tree pendant paths consisting of at least two vertices. Assume that \( P_3 = a_1a_2 \cdots a_q, q \geq 2 \) is pendant path of \( T \). Then \(|T| \geq 1 + |S| + 1 + q - 1 \geq |S| + q + 2 \). Thereby, \( |T - P_3| \geq |S| \). By repeating the procedure of II in Lemma 9, one can get a Hamilton cycle with the edge \( s_1 \).

(III). Edges \( s_is_j \), where \( s_is_j \in S \).

Here, we only consider \( s_is_j \). The proof follows by induction on \( |S| \). If \( |S| = t \), then our statement is true and assume that \( |S| = k \) where \( t < k \leq |T| - 2 \), let \( G_1 = G - \{s_2\} \). According to part II, \( G_1 \) has a Hamilton cycle with the edge \( s_1l_1 \), where \( l_1 \) is a leaf of \( T \). Replacing \( s_1l_1 \) with \( s_1s_2l_1 \), forms a Hamilton cycle of \( G \) containing the edge \( s_1s_2 \).

CONCLUSION

In this paper, we discover the new applications of decycling number, which is giving sufficient conditions for a class of graphs to be Hamiltonian, pancyclic and edge-Hamilton, respectively. This opens a new perspective for the study of Hamilton problem. In the proofs, we mainly use the double induction on \( t \) and \( |T| \). Here, the pendant path plays a huge role. The difficulty in the proof is the construction of edge-Hamilton graphs. In addition, we try to solve the Hamilton problem in graphs with \( k(G) = \nabla(G) \), the proof is still incomplete.

Let \( T_1, T_2, \ldots, T_k \) be spanning trees in a graph \( G \).

For any two vertices \( u, v \) of \( G \), if the paths from \( u \) to \( v \) in these \( k \) trees are pairwise openly disjoint, then we say that \( T_1, T_2, \ldots, T_k \) are completely independent spanning trees in \( G \). By the definition, completely independent spanning trees are also edge-disjoint spanning trees. It is worth mentioning some sufficient conditions for Hamiltonicity also guarantees the existence of completely independent spanning trees [16]. It happened that our conditions is proposed in terms of decycling number. It is our hope that researchers alike will find in this work inspiration and ideas to further light on this fascinating topic, especially, explore a sufficient condition for the existence of completely independent spanning trees.

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REFERENCES

5. Su GE, Li ZH, Shi HC (2020) Some sufficient conditions for a graph to be \( k \)-edge-hamiltonian \( k \)-path-coverable, traceable and Hamilton-connected. Australas J Combin 77, 269–284.