RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2023.054

ScienceAsia 49 (2023): **1–8**

Proper single splittings over proper cones of rectangular matrices

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Received 22 Feb 2022, Accepted 16 May 2023
Available online

ABSTRACT: In this article, we introduce two new splittings for rectangular matrices, which are called proper single regular and weak regular splittings over proper cone. Convergence results for the proper single regular splitting over proper cones of a rectangular matrix are established. Meanwhile, comparison theorems between the spectral radii of matrices arising from proper single regular and/or weak regular splittings over proper cones of the same rectangular matrix or different rectangular matrices are presented. The work here extends the applicability of the splitting results over field of rectangular matrices.

KEYWORDS: rectangular matrix, proper single regular splitting over proper cone, proper single weak regular splitting over proper cone, convergence, comparison theorems

MSC2020: 15A09 65F15

INTRODUCTION

The linear system

$$A\mathbf{x} = \mathbf{b} \tag{1}$$

arises by applying finite difference methods to partial differential equations such as the Neumann Problem and Poisson's equation, where $A \in \mathbb{R}^{m\times n}$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^m$. In addition, the discretization of Fredholm integral equations of the first kind can also form this linear system [1]. In practice, the system (1) appears in several branches of science and engineering such as noisy image restorations [2], computer tomography and inverse problems within electromagnetic [3]. The structure of A in practical problems is different, so it becomes difficult to determine the exact solution even if the exact solution exists in most cases [4]. In order to avoid this problem, we give the following iteration method [5] to solve the rectangular linear system (1).

Splitting the matrix A into

$$A = U - V \tag{2}$$

with $U, V \in \mathbb{R}^{m\times n}$, then the splitting (2) is called a proper single splitting if $R(A) = R(U)$ and $N(A) = N(U)$ [5], where $R(\cdot)$ and $N(\cdot)$ denote the range space and the null space of a given matrix, respectively. Notice that the uniqueness of proper single splittings has been provided in [6]. If the splitting $A = U - V$ is a proper single splitting, then the iteration scheme

$$\mathbf{x_{k+1}} = U^{\dagger}V\mathbf{x_k} + U^{\dagger}\mathbf{b} \tag{3}$$

converges to $A^{\dagger}\mathbf{b}$, the least squares solution of minimum norm for any initial vector x_0 if and only if $\rho(U^{\dagger}V) < 1$ (see [5, Corollary 1]), where $U^{\dagger}$ is the Moore-Penrose inverse of U [7, 8], the matrix $U^{\dagger}V$ is called the iteration matrix of the scheme (3) and $\rho(U^{\dagger}V)$ is the spectral radius of the real square matrix $U^{\dagger}V$. According to [5, 9], if $A = U - V$ is not a proper single splitting, the iteration scheme (3) may not converge to $A^{\dagger}\mathbf{b}$ for any initial vector $\mathbf{x_0}$ even if $\rho(U^{\dagger}V) < 1$. For proper single splittings of rectangular matrices, convergence [9, 10] and comparison results [11, 12] have been comprehensively and systematically studied. It should be emphasized that we can construct U which is easy to compute $U^{\dagger}$ and $\rho(U^{\dagger}V)$ to reduce the complexity of calculations in the practical application.

In this paper, all the entries of $C \in \mathbb{R}^{m\times n}$ are nonnegative which means that C is nonnegative, i.e., $C \geqslant 0$, it also holds when $m = n$. The matrix $C \in \mathbb{R}^{m\times n}$ is said to be semimonotone if $C^{\dagger} \geqslant 0$, see [5]. The authors of [10] showed that the splitting $A = U - V$ is a proper single regular splitting if $R(A) = R(U)$, $N(A) = N(U)$, $U^{\dagger} \geqslant 0$ and $V \geqslant 0$, and is a proper single weak regular splitting if $R(A) = R(U)$, $N(A) = N(U)$, $U^{\dagger} \geqslant 0$ and $U^{\dagger}V \geqslant 0$. For the proper single regular splitting $A = U - V$, if $A^{\dagger} \geqslant 0$, it follows from [10, Theorem 3.1] that $\rho(U^{\dagger}V) < 1$. With the in-depth study of proper single regular splittings of rectangular matrices, we note that although certain conditions of [10, Theorem 3.1] are not satisfied, the convergence result still holds, see the following example.

Example 1 Let

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{4} & \frac{7}{2} & 0 \end{pmatrix}$$

be splitted as

$$A = U - V \tag{4}$$

with

$$U = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} \frac{3}{2} & 0 & 0 \\ \frac{1}{4} & -\frac{1}{2} & 0 \end{pmatrix}.$$

It is easy to get $A^{\dagger} \geqslant 0$, but the splitting (4) is not a proper single regular splitting. However, we still have $\rho(U^{\dagger}V) = 0.75 < 1$.

The above example motivates us to study other conditions which convergence and comparison results of proper single splittings over field of rectangular matrices hold. At the same time, we noticed that the results of splittings of nonsingular matrices can be extended to the splittings over proper cones of nonsingular matrices [8, 13]. Beside, we know from [8, 14] that the results that do not satisfy the conditions of splittings over field may satisfy the conditions of splittings over proper cones. Based on these, we mainly consider proper single regular splittings over proper cones (see Definition 5) and proper single weak regular splittings over proper cones (see Definition 6) of rectangular matrices in this paper. The authors of [5, 15] have studied the convergence of proper single weak regular splittings over proper cones of rectangular matrices, but comparison results for proper single splittings over proper cones have hardly been studied.

PRELIMINARIES

In this section, we will list some definitions and notations that are used throughout the paper.

Firstly, let us recall that a nonempty convex set $K \subseteq \mathbb{R}^n$ is said to be a cone if $\alpha K \subseteq K$ for all $0 \leqslant \alpha$. Moreover, the cone K is called proper if it is closed, pointed ($K \cap -K = \{0\}$) and has nonempty interior (usually denotes by $\operatorname{int} K$) [16]. It should be noted that both the nonnegative cone $\mathbb{R}^n_+$ and the ice cream cone $\{\mathbf{x} \in \mathbb{R}^n | (x_2^2 + x_3^2 + \cdots + x_n^2)^{\frac{1}{2}} \leqslant x_1\}$ are particular proper cones.

Secondly, we will review some concepts about the nonnegativity over proper cones [13, 16].

Definition 1 Let K be a proper cone in $\mathbb{R}^n$, a vector $\mathbf{x} \in \mathbb{R}^n$ is called nonnegative (respectively, positive) over the proper cone K if $\mathbf{x}$ belongs to K (respectively, $\mathbf{x}$ belongs to $\operatorname{int} K$) and is denoted as $\mathbf{x} \geqslant_K 0$ (respectively, $\mathbf{x} >_K 0$). If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ satisfy $\mathbf{x} - \mathbf{y} \geqslant_K 0$ (respectively, $\mathbf{x} - \mathbf{y} >_K 0$), which is denoted as $\mathbf{x} \geqslant_K \mathbf{y}$ (respectively, $\mathbf{x} >_K \mathbf{y}$).

Definition 2 Let K be a proper cone in $\mathbb{R}^n$, one matrix $A \in \mathbb{R}^{n\times n}$ is called nonnegative (respectively, positive) over the proper cone K if $AK \subseteq K$ (respectively, $A(K - \{0\}) \subseteq \operatorname{int} K$) and is denoted as $A \geqslant_K 0$ (respectively, $A >_K 0$). For $A, B \in \mathbb{R}^{n\times n}$, $A \geqslant_K B$ (respectively, $A >_K B$) means $A - B \geqslant_K 0$ (respectively, $A - B >_K 0$).

Let $\pi(K)$ denote the set of matrices $A \in \mathbb{R}^{n\times n}$ for which $AK \subseteq K$, then $A \in \mathbb{R}^{n\times n}$ is nonnegative over the proper cone K is equivalent to $A \in \pi(K)$, see [16]. Moreover, $A \in \mathbb{R}^{n\times n}$ is a monotone matrix over the proper cone K if $A^{-1} \geqslant_K 0$, i.e., $A^{-1} \in \pi(K)$, see [13]. The properties of nonnegative matrices over the proper cone K are similar to that of nonnegative matrices [13, 16].

Lastly, we will give some concepts related to rectangular matrices.

Definition 3 Let K_1 and K_2 be proper cones, in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. A matrix $A \in \mathbb{R}^{m\times n}$ is called

(i) nonnegative over proper cones if $AK_1 \subseteq K_2$;

(ii) positive over proper cones if $A(K_1 - \{0\}) \subseteq \operatorname{int} K_2$.

Similarly, denote by $\pi(K_1, K_2)$ the set of matrices $A \in \mathbb{R}^{m\times n}$ for which $AK_1 \subseteq K_2$. Furthermore, according to the concept of the monotonicity over the proper cone K of square matrices, we can give the following definition.

Definition 4 Let K_1 and K_2 be proper cones, in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. A real rectangular matrix $A \in \mathbb{R}^{m\times n}$ is called a semimonotone matrix over proper cones if $A^{\dagger} \in \pi(K_2, K_1)$.

In the following, we extend the concepts of the different types of proper single splittings that appear in [10] for the particular case $K = \mathbb{R}^n_+$ to general proper cones.

Definition 5 Let K_1 and K_2 be proper cones, in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. For $A \in \mathbb{R}^{m\times n}$, the splitting $A = U - V$ is called a proper single regular splitting over proper cones if it is a proper single splitting such that $U^{\dagger} \in \pi(K_2, K_1)$ and $V \in \pi(K_1, K_2)$.

Definition 6 Let K_1 and K_2 be proper cones, in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. For $A \in \mathbb{R}^{m\times n}$, the splitting $A = U - V$ is called a proper single weak regular splitting over proper cones if it is a proper single splitting such that $U^{\dagger} \in \pi(K_2, K_1)$ and $U^{\dagger}V \geqslant_{K_1} 0$.

Combining Definition 5 and Definition 6, we can obtain the following result.

Remark 1 Let K_1 and K_2 be proper cones, in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. If $A = U - V$ is a proper single regular splitting over proper cones of $A \in \mathbb{R}^{m\times n}$, then $A = U - V$ is also a proper single weak regular splitting over proper cones of $A \in \mathbb{R}^{m\times n}$.

Proof: since $A = U - V$ is a proper single regular splitting over proper cones of $A \in \mathbb{R}^{m\times n}$, so $A = U - V$ is a proper splitting with $U^{\dagger} \in \pi(K_2, K_1)$ and $V \in \pi(K_1, K_2)$. Then we have $U^{\dagger}VK_1 \subseteq U^{\dagger}K_2 \subseteq K_1$, i.e., $U^{\dagger}V \geqslant_{K_1} 0$. Therefore, any proper single regular splitting over proper cones of $A \in \mathbb{R}^{m\times n}$ is a proper single weak regular splitting over proper cones of $A \in \mathbb{R}^{m\times n}$. □

The following example shows that the converse of Remark 1 is not true.

Example 2 Consider proper cones $K_1=\{\mathbf{x}\in\mathbb{R}^3|(x_2^2+x_3^2)^{\frac{1}{2}}\leqslant x_1\}$ and $K_2=\{\mathbf{x}\in\mathbb{R}^2|(x_2^2)^{\frac{1}{2}}\leqslant x_1\}$.

Assume that

$$A=\begin{pmatrix}\frac{1}{6} & 0 & 0\\ \frac{1}{10} & \frac{5}{2} & 2\end{pmatrix}$$

Let A be splitted as $A=U_1-V_1=U_2-V_2$ with

$$U_1=\begin{pmatrix}1 & 0 & 0\\ \frac{2}{5} & \frac{5}{2} & 2\end{pmatrix},\quad V_1=\begin{pmatrix}\frac{5}{6} & 0 & 0\\ \frac{3}{10} & 0 & 0\end{pmatrix}$$

and

$$U_2=\begin{pmatrix}\frac{1}{2} & 0 & 0\\ \frac{11}{10} & \frac{5}{2} & 2\end{pmatrix},\quad V_2=\begin{pmatrix}\frac{1}{3} & 0 & 0\\ 1 & 0 & 0\end{pmatrix}.$$

By calculation, we have

$$U_1^{\dagger}=\begin{pmatrix}1 & 0\\ -0.0976 & 0.2439\\ -0.0780 & 0.1951\end{pmatrix}$$

and

$$U_2^{\dagger}=\begin{pmatrix}2 & 0\\ -0.5366 & 0.2439\\ -0.4293 & 0.1951\end{pmatrix}.$$

It is easy to see that

$$U_1^{\dagger}V_1=\begin{pmatrix}0.8333 & 0 & 0\\ -0.0081 & 0 & 0\\ -0.0065 & 0 & 0\end{pmatrix}$$

and

$$U_2^{\dagger}V_2=\begin{pmatrix}0.6667 & 0 & 0\\ 0.0650 & 0 & 0\\ 0.0520 & 0 & 0\end{pmatrix}.$$

Obviously, $A=U_1-V_1$ is not only a proper single regular splitting over proper cones, but also a proper single weak regular splitting over proper cones. However, $A=U_2-V_2$ is only a proper single weak regular splitting over proper cones.

CONVERGENCE RESULTS FOR PROPER SINGLE REGULAR SPLITTINGS OVER PROPER CONES

Convergence results for the proper single weak regular splitting over proper cones of a rectangular matrix are studied extensively in [5, 15]. In what follows of this section, we will propose convergence results for the proper single regular splitting over proper cones of a rectangular matrix.

Theorem 1 *Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. If $A=U-V$ is a proper single regular splitting over proper cones of $A\in\mathbb{R}^{m\times n}$, then $A^{\dagger}V\geqslant_{K_1}0$ if and only if $\rho(U^{\dagger}V)=\frac{\rho(A^{\dagger}V)}{1+\rho(A^{\dagger}V)}<1$.*

Proof: By the assumption, we know that $A=U-V$ is a proper splitting with $U^{\dagger}\in\pi(K_2,K_1)$ and $V\in\pi(K_1,K_2)$. Then we have $U^{\dagger}V\geqslant_{K_1}0$.

Suppose that $A^{\dagger}V\geqslant_{K_1}0$, it follows from [16, Theorem 3.2] that $\rho(A^{\dagger}V)$ is an eigenvalue of $A^{\dagger}V$. Similarly, we can get that $\rho(U^{\dagger}V)$ is an eigenvalue of $U^{\dagger}V$. Let λ and μ be any eigenvalue of $A^{\dagger}V$ and $U^{\dagger}V$, respectively. If $f(\lambda)=\frac{\lambda}{1+\lambda}$ and $\lambda\geqslant 0$, then we get $\mu=\frac{\lambda}{1+\lambda}$ by [17, Lemma 2.6]. So μ attains its maximum when λ is maximum. Here, λ is maximum when $\lambda=\rho(A^{\dagger}V)$. As a result, the maximum value of μ is $\rho(U^{\dagger}V)$. Therefore, $\rho(U^{\dagger}V)=\frac{\rho(A^{\dagger}V)}{1+\rho(A^{\dagger}V)}<1$.

Conversely, if $\rho(U^{\dagger}V)<1$, from [8, Lemma 2] we obtain $(I-U^{\dagger}V)^{-1}\geqslant_{K_1}0$. Moreover, [5, Theorem 1] implies $A^{\dagger}=(I-U^{\dagger}V)^{-1}U^{\dagger}$. Hence, $A^{\dagger}VK_1=(I-U^{\dagger}V)^{-1}U^{\dagger}VK_1\subseteq(I-U^{\dagger}V)^{-1}K_1\subseteq K_1$, i.e., $A^{\dagger}V\geqslant_{K_1}0$. □

For the proper single regular splitting over proper cones $A=U-V$, if $\rho(U^{\dagger}V)<1$, then we say that $A=U-V$ is convergent. When we consider the convergent proper single regular splitting over proper cones, we have the following theorem.

Theorem 2 *Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. If $A=U-V$ is a convergent proper single regular splitting over proper cones of $A\in\mathbb{R}^{m\times n}$, then*

(i) $(I-U^{\dagger}V)^{-1}\geqslant_{K_1}0$;

(ii) $A^{\dagger}V\geqslant_{K_1}U^{\dagger}V\geqslant_{K_1}0$.

Proof: By Definition 5, we know that $A=U-V$ is a proper splitting with $U^{\dagger}\in\pi(K_2,K_1)$ and $V\in\pi(K_1,K_2)$. Then we have $U^{\dagger}V\geqslant_{K_1}0$. In addition, we can obtain that $\rho(U^{\dagger}V)<1$ by the convergence of the splitting $A=U-V$.

(i) Clearly, [8, Lemma 2] can show that $(I-U^{\dagger}V)^{-1}\geqslant_{K_1}0$.

(ii) [5, Theorem 1] implies

$$A^{\dagger}=(I-U^{\dagger}V)^{-1}U^{\dagger}.$$

Moreover, Theorem 1 shows $A^{\dagger}V\geqslant_{K_1}0$. Thus, we have

$$(A^{\dagger}V-U^{\dagger}V)K_1=U^{\dagger}VA^{\dagger}VK_1\subseteq U^{\dagger}VK_1\subseteq K_1,$$

i.e., $A^{\dagger}V\geqslant_{K_1}U^{\dagger}V$. Consequently, $A^{\dagger}V\geqslant_{K_1}U^{\dagger}V\geqslant_{K_1}0$.

□

If we consider the proper single regular splitting over proper cones of a semimonotone matrix $A\in\mathbb{R}^{m\times n}$ over proper cones, from Theorem 1, we can get the following corollary.

Corollary 1 *Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. If $A=U-V$ is a proper single regular splitting over proper cones of a semimonotone matrix $A\in\mathbb{R}^{m\times n}$ over proper cones, then $\rho(U^\dagger V)=\frac{\rho(A^\dagger V)}{1+\rho(A^\dagger V)}<1$.*

Proof: The fact that $A=U-V$ is a proper single regular splitting over proper cones of a semimonotone matrix $A\in\mathbb{R}^{m\times n}$ over proper cones yields $A^\dagger\in\pi(K_2,K_1)$ and $V\in\pi(K_1,K_2)$. It is easy to show that $A^\dagger VK_1\subseteq A^\dagger K_2\subseteq K_1$, i.e., $A^\dagger V\geqslant_{K_1}0$. Therefore, Theorem 1 implies $\rho(U^\dagger V)=\frac{\rho(A^\dagger V)}{1+\rho(A^\dagger V)}<1$. □

It should be noted that Corollary 1 is a generalization of [10, Theorem 3.1 (c)].

Remark 2 For Example 1, if we let $K_1=\{\mathbf{x}\in\mathbb{R}^3|(x_2^2+x_3^2)^{\frac{1}{2}}\leqslant x_1\}$ and $K_2=\{\mathbf{x}\in\mathbb{R}^2|(x_2^2)^{\frac{1}{2}}\leqslant x_1\}$, we can see that although $V\not\geqslant 0$, $V\in\pi(K_1,K_2)$.

By calculating, we have

$$U^\dagger=\begin{pmatrix}0.5 & 0\\ 0 & 0.3333\\ 0 & 0\end{pmatrix} \text{ and } A^\dagger=\begin{pmatrix}2 & 0\\ 0.1429 & 0.2857\\ 0 & 0\end{pmatrix}.$$

It is easy to verify that the assumptions of Corollary 1 are satisfied, so $\rho(U^\dagger V)=0.75<1$.

We then present certain properties of the proper single regular splitting over proper cones of a semimonotone matrix $A\in\mathbb{R}^{m\times n}$ over proper cones.

Corollary 2 *Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Assume that $A=U-V$ is a proper single regular splitting over proper cones of a semimonotone matrix $A\in\mathbb{R}^{m\times n}$ over proper cones, then*

(i) $(I-U^\dagger V)^{-1}\geqslant_{K_1}0$;

(ii) $(A^\dagger-U^\dagger)\in\pi(K_1,K_2)$;

(iii) $A^\dagger V\geqslant_{K_1}U^\dagger V\geqslant_{K_1}0$.

Proof: Given that $A=U-V$ is a proper single regular splitting over proper cones of a semimonotone matrix $A\in\mathbb{R}^{m\times n}$ over proper cones, so $U^\dagger\in\pi(K_2,K_1)$, $V\in\pi(K_1,K_2)$ and $A^\dagger\in\pi(K_2,K_1)$. In the meantime, [5, Theorem 1] implies $A^\dagger=(I-U^\dagger V)^{-1}U^\dagger$.

(i) Then we have $U^\dagger V\geqslant_{K_1}0$. Furthermore, we can obtain $\rho(U^\dagger V)<1$ from Corollary 1. Therefore, [8, Lemma 2] yields $(I-U^\dagger V)^{-1}\geqslant_{K_1}0$.

(ii) It is easy to show that $(A^\dagger-U^\dagger)K_2=U^\dagger VA^\dagger K_2\subseteq U^\dagger VK_1\subseteq K_1$, i.e., $(A^\dagger-U^\dagger)\in\pi(K_2,K_1)$.

(iii) Clearly, we have

$$(A^\dagger-U^\dagger)VK_1=U^\dagger VA^\dagger VK_1\subseteq U^\dagger VA^\dagger K_2\subseteq U^\dagger VK_1\subseteq K_1,$$

i.e., $A^\dagger V\geqslant_{K_1}U^\dagger V$. Consequently, $A^\dagger V\geqslant_{K_1}U^\dagger V\geqslant_{K_1}0$. □

In what follows, we give another result for the proper single regular splitting over proper cones of $A\in\mathbb{R}^{m\times n}$.

Theorem 3 *Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $A=U-V$ be a proper single regular splitting over proper cones of $A\in\mathbb{R}^{m\times n}$. If $A^\dagger U\geqslant_{K_1}0$, then*

(i) $A^\dagger U\geqslant_{K_1}U^\dagger U$;

(ii) $\rho(U^\dagger V)=\frac{\rho(A^\dagger U)-1}{\rho(A^\dagger U)}<1$.

Proof: From Definition 5, we have $U^\dagger\in\pi(K_2,K_1)$ and $V\in\pi(K_1,K_2)$, so $U^\dagger V\geqslant_{K_1}0$. In addition, we can obtain that $A^\dagger=(I-U^\dagger V)^{-1}U^\dagger$.

(i) Clearly, we then have $A^\dagger U-U^\dagger U=U^\dagger VA^\dagger U$. The facts $A^\dagger U\geqslant_{K_1}0$ and $U^\dagger V\geqslant_{K_1}0$ imply $(A^\dagger U-U^\dagger U)K_1=U^\dagger VA^\dagger UK_1\subseteq U^\dagger VK_1\subseteq K_1$, i.e., $A^\dagger U\geqslant_{K_1}U^\dagger U$.

(ii) By [16, Theorem 3.2], there exists a nonzero vector $\mathbf{x}$ ($\mathbf{x}\in K_1$) such that $U^\dagger V\mathbf{x}=\rho(U^\dagger V)\mathbf{x}$. Hence $\mathbf{x}\in R(U^\dagger)=R(U^\top)=R(A^\top)$ so that $U^\dagger U\mathbf{x}=\mathbf{x}$. Moreover, we can get $A^\dagger U=(I-U^\dagger V)^{-1}U^\dagger U$. Then

$$\begin{aligned}(I-U^\dagger V)^{-1}U^\dagger U\mathbf{x}&=(I-U^\dagger V)^{-1}\mathbf{x}\\&=\frac{1}{1-\rho(U^\dagger V)}\mathbf{x}=A^\dagger U\mathbf{x},\end{aligned}$$

which shows that $\frac{1}{1-\rho(U^\dagger V)}$ is an eigenvalue of $A^\dagger U$. It is easy to see that $A^\dagger U\mathbf{x}\geqslant_{K_1}0$, so $\frac{1}{1-\rho(U^\dagger V)}\mathbf{x}\geqslant_{K_1}0$. It follows from the definition of the cone that $\frac{1}{1-\rho(U^\dagger V)}\geqslant 0$. Hence, $0\leqslant\frac{1}{1-\rho(U^\dagger V)}\leqslant\rho(A^\dagger U)$, i.e., $\rho(U^\dagger V)\leqslant\frac{\rho(A^\dagger U)-1}{\rho(A^\dagger U)}$.

Again, the condition $A^\dagger U\geqslant_{K_1}0$ implies the existence of a nonzero vector $\mathbf{y}$ ($\mathbf{y}\in K_1$) such that $A^\dagger U\mathbf{y}=\rho(A^\dagger U)\mathbf{y}$. Then $\mathbf{y}\in R(A^\top)=R(U^\top)$. Therefore, we can obtain

$$(I-U^\dagger V)^{-1}U^\dagger U\mathbf{y}=(I-U^\dagger V)^{-1}\mathbf{y}=\rho(A^\dagger U)\mathbf{y}.$$

So, we have $\frac{1}{\rho(A^\dagger U)}\mathbf{y}=\mathbf{y}-U^\dagger V\mathbf{y}$, i.e., $U^\dagger V\mathbf{y}=\frac{\rho(A^\dagger U)-1}{\rho(A^\dagger U)}\mathbf{y}$. Thus, $\rho(U^\dagger V)\geqslant\frac{\rho(A^\dagger U)-1}{\rho(A^\dagger U)}$.

Consequently, we have $\rho(U^\dagger V)=\frac{\rho(A^\dagger U)-1}{\rho(A^\dagger U)}<1$. □

For the proper single regular splitting over proper cones of $A\in\mathbb{R}^{m\times n}$, if $A^\dagger\in\pi(K_2,K_1)$ and $U\in\pi(K_1,K_2)$, we can get the following corollary directly.

Corollary 3 *Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $A=U-V$ be a proper single regular splitting over proper cones of $A\in\mathbb{R}^{m\times n}$. If $A^\dagger\in\pi(K_2,K_1)$ and $U\in\pi(K_1,K_2)$, then*

(i) $A^\dagger U\geqslant_{K_1}U^\dagger U$;

(ii) $\rho(U^\dagger V)=\frac{\rho(A^\dagger U)-1}{\rho(A^\dagger U)}<1$.

Proof: The given conditions $A^\dagger\in\pi(K_2,K_1)$ and $U\in\pi(K_1,K_2)$ show

$$A^\dagger UK_1\subseteq A^\dagger K_2\subseteq K_1,$$

i.e., $A^{\dagger}U \geqslant_{K_1} 0$. Hence, the desired result can now be obtained by applying Theorem 3. □

The following example shows that the condition $A^{\dagger} \in \pi(K_2, K_1)$ cannot be dropped in Corollary 3.

Example 3 Consider proper cones $K_1 = \{\mathbf{x} \in \mathbb{R}^3 | (x_2^2 + x_3^2)^{\frac{1}{2}} \leqslant x_1\}$ and $K_2 = \{\mathbf{x} \in \mathbb{R}^2 | (x_2^2)^{\frac{1}{2}} \leqslant x_1\}$. Let

$$A = \begin{pmatrix} -1 & 0 & 0 \\ -\frac{1}{4} & \frac{3}{2} & 0 \end{pmatrix}$$

be splitted as $A = U - V$ with

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 2 & 0 & 0 \\ \frac{1}{4} & -\frac{1}{2} & 0 \end{pmatrix}.$$

Following the operations, we can get

$$A^{\dagger} = \begin{pmatrix} -1 & 0 \\ -0.1667 & 0.6667 \\ 0 & 0 \end{pmatrix} \text{ and } U^{\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Here $A = U - V$ is a proper single regular splitting over proper cones and $U \in \pi(K_1, K_2)$, but $A^{\dagger} \notin \pi(K_2, K_1)$.

Moreover, we have

$$A^{\dagger}U - U^{\dagger}U = \begin{pmatrix} -2 & 0 & 0 \\ -0.1667 & -0.3333 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

i.e., $A^{\dagger}U \ngeqslant_{K_1} U^{\dagger}U$. In fact, we also have $\rho(U^{\dagger}V) = 2 > 1$.

COMPARISON RESULTS FOR PROPER SINGLE SPLITTINGS OVER PROPER CONES

Comparison theorems between the spectral radii of iteration matrices are useful tools to analyze the convergent rate of iteration methods or to judge the effectiveness of preconditioners [18, 19]. In this section, we will give comparison results for proper single regular and/or weak regular splittings over proper cones of the same rectangular matrix or different rectangular matrices.

Comparison results for proper single splittings over proper cones of one rectangular matrix

Let us first consider comparison results for proper single splittings over proper cones of one rectangular matrix. If $A = U_1 - V_1 = U_2 - V_2$ are proper single regular splittings over proper cones of $A \in \mathbb{R}^{m\times n}$, then our main results for comparing $\rho(U_1^{\dagger}V_1)$ with $\rho(U_2^{\dagger}V_2)$ are stated as the following.

Theorem 4 *Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $A = U_1 - V_1 = U_2 - V_2$ be proper single regular splittings over proper cones of $A \in \mathbb{R}^{m\times n}$. If any one of the following conditions*

(i) $A^{\dagger}V_2 \geqslant_{K_1} A^{\dagger}V_1 >_{K_1} 0$;

(ii) $A^{\dagger}U_2 \geqslant_{K_1} A^{\dagger}U_1 >_{K_1} 0$,

holds, then $\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2) < 1$.

Proof:

(i) As $A^{\dagger}V_2 \geqslant_{K_1} A^{\dagger}V_1 >_{K_1} 0$, it follows from Theorem 1 that

$$\rho(U_i^{\dagger}V_i) = \frac{\rho(A^{\dagger}V_i)}{1 + \rho(A^{\dagger}V_i)} < 1$$

for $i = 1, 2$. Next, we need to show that $\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2)$.

Under the assumption $A^{\dagger}V_2 \geqslant_{K_1} A^{\dagger}V_1 >_{K_1} 0$, [16, Corollary 3.29] and [13, Corollary 2.6] yield $\rho(A^{\dagger}V_1) < \rho(A^{\dagger}V_2)$. Let $f(\lambda) = \frac{\lambda}{1+\lambda}$, it is easy to prove that $f(\lambda)$ is a strictly increasing function for $\lambda \geqslant 0$. Hence the inequality $\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2)$ holds.

(ii) Since $A^{\dagger}U_2 \geqslant_{K_1} A^{\dagger}U_1 >_{K_1} 0$, from Theorem 3 we obtain that

$$\rho(U_i^{\dagger}V_i) = \frac{\rho(A^{\dagger}U_i) - 1}{\rho(A^{\dagger}U_i)} < 1$$

for $i = 1, 2$. Moreover, [16, Corollary 3.29] and [13, Corollary 2.6] imply $\rho(A^{\dagger}U_1) < \rho(A^{\dagger}U_2)$. Let $f(\lambda) = \frac{\lambda-1}{\lambda}$, then $f(\lambda)$ is a strictly increasing function for $\lambda > 0$. Therefore, the inequality $\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2)$ holds.

□

For proper single regular splittings over proper cones of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$ over proper cones, the above result reduces to the following corollary.

Corollary 4 *Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $A = U_1 - V_1 = U_2 - V_2$ be proper single regular splittings over proper cones of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$ over proper cones. If any one of the following conditions*

(i) $(V_2 - V_1) \in \pi(K_1, K_2)$ *and* $A^{\dagger}V_1 >_{K_1} 0$;

(ii) $(U_2 - U_1) \in \pi(K_1, K_2)$ *and* $A^{\dagger}U_1 >_{K_1} 0$,

holds, then $\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2) < 1$.

Proof:

(i) The semi-monotonicity of A over proper cones implies that $A^{\dagger} \in \pi(K_2, K_1)$, combining $(V_2 - V_1) \in \pi(K_1, K_2)$, we drive

$$A^{\dagger}(V_2 - V_1)K_1 \subseteq A^{\dagger}K_2 \subseteq K_1,$$

i.e., $A^{\dagger}V_2 \geqslant_{K_1} A^{\dagger}V_1$. As $A^{\dagger}V_1 >_{K_1} 0$, so

$$A^{\dagger}V_2 \geqslant_{K_1} A^{\dagger}V_1 >_{K_1} 0.$$

It follows from Theorem 4 that $\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2) < 1$.

(ii) Similar to the proof of (i), under the assumptions, we get that $A^{\dagger}U_2 \geqslant_{K_1} A^{\dagger}U_1 >_{K_1} 0$. Thus, we have $\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2) < 1$ by Theorem 4.

□

Remark 1 shows that any proper single regular splitting over proper cones of $A \in \mathbb{R}^{m\times n}$ is a proper single weak regular splitting over proper cones of $A \in \mathbb{R}^{m\times n}$, so all the above comparison results can be directly extended to proper single weak regular splittings over proper cones of $A \in \mathbb{R}^{m\times n}$. Hence, according to Remark 1 and Theorem 4, we have the following results directly.

Theorem 5 *Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $A = U_1 - V_1 = U_2 - V_2$ be proper single weak regular splittings over proper cones of $A \in \mathbb{R}^{m\times n}$. If any one of the following conditions*

(i) $A^{\dagger}V_2 \geqslant_{K_1} A^{\dagger}V_1 >_{K_1} 0$;

(ii) $A^{\dagger}U_2 \geqslant_{K_1} A^{\dagger}U_1 >_{K_1} 0$,

holds, then $\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2) < 1$.

Theorem 6 *Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $A = U_1 - V_1$ be a proper single regular splitting over proper cones and $A = U_2 - V_2$ be a proper single weak regular splitting over proper cones. If any one of the following conditions*

(i) $A^{\dagger}V_2 \geqslant_{K_1} A^{\dagger}V_1 >_{K_1} 0$;

(ii) $A^{\dagger}U_2 \geqslant_{K_1} A^{\dagger}U_1 >_{K_1} 0$,

holds, then $\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2) < 1$.

When we consider proper single weak regular splittings over proper cones of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$ over proper cones, by applying Remark 1 and Corollary 4, the following corollaries can be obtained directly.

Corollary 5 *Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $A = U_1 - V_1 = U_2 - V_2$ be proper single weak regular splittings over proper cones of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$ over proper cones. If any one of the following conditions*

(i) $(V_2 - V_1) \in \pi(K_1, K_2)$ and $A^{\dagger}V_1 >_{K_1} 0$;

(ii) $(U_2 - U_1) \in \pi(K_1, K_2)$ and $A^{\dagger}U_1 >_{K_1} 0$,

holds, then $\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2) < 1$.

Corollary 6 *Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $A = U_1 - V_1$ be a proper single weak regular splitting over proper cones and $A = U_2 - V_2$ be a proper single regular splitting over proper cones. If $A^{\dagger} \in \pi(K_2, K_1)$ and any one of the following conditions*

(i) $(V_2 - V_1) \in \pi(K_1, K_2)$ and $A^{\dagger}V_1 >_{K_1} 0$;

(ii) $(U_2 - U_1) \in \pi(K_1, K_2)$ and $A^{\dagger}U_1 >_{K_1} 0$,

holds, then $\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2) < 1$.

Comparison results for proper single splittings over proper cones of different rectangular matrices

In the rest of this section, we consider comparison results between the spectral radii of matrices arising from proper single splittings over proper cones of different rectangular matrices. Let $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ be proper single regular splittings over proper cones of $A_1 \in \mathbb{R}^{m\times n}$ and $A_2 \in \mathbb{R}^{m\times n}$, respectively. Now, let us list the first comparing result in the following theorem.

Theorem 7 *Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Assume that $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ are proper single regular splittings over proper cones of $A_1 \in \mathbb{R}^{m\times n}$ and $A_2 \in \mathbb{R}^{m\times n}$, respectively. If any one of the following conditions*

(i) $A_2^{\dagger}V_2 \geqslant_{K_1} A_1^{\dagger}V_1 >_{K_1} 0$ and $A_1^{\dagger}V_1 \neq A_2^{\dagger}V_2$;

(ii) $A_2^{\dagger}U_2 \geqslant_{K_1} A_1^{\dagger}U_1 >_{K_1} 0$ and $A_1^{\dagger}U_1 \neq A_2^{\dagger}U_2$,

holds, then $\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2) < 1$.

Proof:

(i) Under the assumption $A_2^{\dagger}V_2 \geqslant_{K_1} A_1^{\dagger}V_1 >_{K_1} 0$, according to Theorem 1, we can imply

$$\rho(U_i^{\dagger}V_i) = \frac{\rho(A_i^{\dagger}V_i)}{1+\rho(A_i^{\dagger}V_i)} < 1$$

for $i = 1, 2$. So what we need to show now is that $\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2)$. To do this, we first need to demonstrate that $\rho(A_1^{\dagger}V_1) < \rho(A_2^{\dagger}V_2)$.

As $A_2^{\dagger}V_2 \geqslant_{K_1} A_1^{\dagger}V_1 >_{K_1} 0$, so [16, Corollary 3.29] and [13, Corollary 2.6] yield $\rho(A_1^{\dagger}V_1) < \rho(A_2^{\dagger}V_2)$. Let $f(\lambda) = \frac{\lambda}{1+\lambda}$, it is easy to see that $f(\lambda)$ is a strictly increasing function for $\lambda \geqslant 0$. Hence, we have $\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2)$.

(ii) By Theorem 3, the assumption $A_2^{\dagger}U_2 \geqslant_{K_1} A_1^{\dagger}U_1 >_{K_1} 0$ implies $\rho(U_i^{\dagger}V_i) < 1$, where $i = 1, 2$. In the following, in order to prove that $\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2)$, we first show that $\rho(A_1^{\dagger}U_1) < \rho(A_2^{\dagger}U_2)$.

Applying the condition $A_2^{\dagger}U_2 \geqslant_{K_1} A_1^{\dagger}U_1 >_{K_1} 0$ to Theorem 3, we can get that

$$\rho(U_i^{\dagger}V_i) = \frac{\rho(A_i^{\dagger}U_i) - 1}{\rho(A_i^{\dagger}U_i)}.$$

In the meantime, [16, Corollary 3.29] and [13, Corollary 2.6] imply $\rho(A_1^{\dagger}U_1) < \rho(A_2^{\dagger}U_2)$. Since $f(\lambda) = \frac{\lambda-1}{\lambda}$ is a strictly increasing function for $\lambda > 0$, so the inequality $\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2)$ is true. □

The example given below shows that $A_2^{\dagger}V_2 \geqslant_{K_1} A_1^{\dagger}V_1 >_{K_1} 0$ or $A_2^{\dagger}U_2 \geqslant_{K_1} A_1^{\dagger}U_1 >_{K_1} 0$ cannot be dropped in Theorem 7.

Example 4 Consider proper cones $K_1=\{\mathbf{x}\in\mathbb{R}^3|(x_2^2+x_3^2)^{\frac{1}{2}}\leqslant x_1\}$ and $K_2=\{\mathbf{x}\in\mathbb{R}^2|(x_2^2)^{\frac{1}{2}}\leqslant x_1\}$. Let

$$A_1=\begin{pmatrix}\frac{1}{6} & 0 & 0\\ \frac{1}{8} & 3 & 1\end{pmatrix} \text{ and } A_2=\begin{pmatrix}\frac{1}{8} & 0 & 0\\ \frac{1}{8} & \frac{5}{2} & 1\end{pmatrix}.$$

Assume that A_1 and A_2 are splitted as

$$A_1=U_1-V_1 \text{ and } A_2=U_2-V_2$$

with

$$U_1=\begin{pmatrix}\frac{1}{2} & 0 & 0\\ \frac{3}{8} & 3 & 1\end{pmatrix},\quad V_1=\begin{pmatrix}\frac{1}{3} & 0 & 0\\ \frac{1}{4} & 0 & 0\end{pmatrix}$$

and

$$U_2=\begin{pmatrix}\frac{1}{4} & 0 & 0\\ \frac{3}{16} & \frac{5}{2} & 1\end{pmatrix},\quad V_2=\begin{pmatrix}\frac{1}{8} & 0 & 0\\ \frac{1}{16} & 0 & 0\end{pmatrix},$$

respectively. It is easy to see that

$$U_1^{\dagger}=\begin{pmatrix}2 & 0\\ -0.225 & 0.3\\ -0.075 & 0.1\end{pmatrix} \text{ and } U_2^{\dagger}=\begin{pmatrix}4 & 0\\ -0.2586 & 0.3448\\ -0.1034 & 0.1379\end{pmatrix}.$$

It is easy to verify that $A_1=U_1-V_1$ and $A_2=U_2-V_2$ are proper single regular splittings over proper cones of $A_1\in\mathbb{R}^{m\times n}$ and $A_2\in\mathbb{R}^{m\times n}$, respectively.

Following the operations, we have

$$A_2^{\dagger}V_2-A_1^{\dagger}V_1=\begin{pmatrix}-1 & 0 & 0\\ -0.0216 & 0 & 0\\ -0.0086 & 0 & 0\end{pmatrix}\ngeq_{K_1} 0$$

and

$$A_2^{\dagger}U_2-A_1^{\dagger}U_1=\begin{pmatrix}-1 & 0 & 0\\ -0.0216 & -0.0379 & 0.0448\\ -0.0086 & 0.0448 & 0.0379\end{pmatrix}\ngeq_{K_1} 0,$$

i.e., $A_2^{\dagger}V_2\ngeq_{K_1}A_1^{\dagger}V_1$ and $A_2^{\dagger}U_2\ngeq_{K_1}A_1^{\dagger}U_1$.

In fact, $\rho(U_2^{\dagger}V_2)=0.5<0.6667=\rho(U_1^{\dagger}V_1)$.

The conclusion of Theorem 7 can also be achieved by replacing proper single regular splittings $A_i=U_i-V_i$ over proper cones with proper single weak regular splittings over proper cones, for $i=1,2$. The following is the exact statement of this result.

Theorem 8 *Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Assume that $A_1=U_1-V_1$ and $A_2=U_2-V_2$ are proper single weak regular splittings over proper cones of $A_1\in\mathbb{R}^{m\times n}$ and $A_2\in\mathbb{R}^{m\times n}$, respectively. If any one of the following conditions*

(i) $A_2^{\dagger}V_2\geqslant_{K_1}A_1^{\dagger}V_1>_{K_1}0$ *and* $A_1^{\dagger}V_1\neq A_2^{\dagger}V_2$;

(ii) $A_2^{\dagger}U_2\geqslant_{K_1}A_1^{\dagger}U_1>_{K_1}0$ *and* $A_1^{\dagger}U_1\neq A_2^{\dagger}U_2$,

holds, then $\rho(U_1^{\dagger}V_1)<\rho(U_2^{\dagger}V_2)<1$.

Proof: By Remark 1, we know that any proper single regular splitting over proper cones of one rectangular matrix is a proper single weak regular splitting over proper cones of the rectangular matrix. Therefore, it follows from Theorem 7 that $\rho(U_1^{\dagger}V_1)<\rho(U_2^{\dagger}V_2)<1$. □

Clearly, for the different types of proper single splittings over proper cones of different rectangular matrices, we can get the following result.

Theorem 9 *Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $A_1=U_1-V_1$ be a proper single weak regular splitting over proper cones and $A_2=U_2-V_2$ be a proper single regular splitting over proper cones. If any one of the following conditions*

(i) $A_1^{\dagger}V_2\geqslant_{K_1}A_1^{\dagger}V_1>_{K_1}0$ *and* $A_1^{\dagger}V_1\neq A_2^{\dagger}V_2$;

(ii) $A_2^{\dagger}U_2\geqslant_{K_1}A_1^{\dagger}U_1>_{K_1}0$ *and* $A_1^{\dagger}U_1\neq A_2^{\dagger}U_2$,

holds, then $\rho(U_1^{\dagger}V_1)<\rho(U_2^{\dagger}V_2)<1$.

The proof of Theorem 9 is very similar to the proof of Theorem 8, so we omitted it here. For Theorem 9, the numerical example similar to Example 4 can be constructed as follows.

Example 5 Consider proper cones $K_1=\{\mathbf{x}\in\mathbb{R}^3|(x_2^2+x_3^2)^{\frac{1}{2}}\leqslant x_1\}$ and $K_2=\{\mathbf{x}\in\mathbb{R}^2|(x_2^2)^{\frac{1}{2}}\leqslant x_1\}$. Let

$$A_1=\begin{pmatrix}\frac{1}{6} & 0 & 0\\ \frac{1}{10} & \frac{5}{2} & 2\end{pmatrix} \text{ and } A_2=\begin{pmatrix}\frac{1}{2} & 0 & 0\\ \frac{1}{4} & \frac{5}{2} & 0\end{pmatrix}$$

be splitted as

$$A_1=U_1-V_1 \text{ and } A_2=U_2-V_2,$$

respectively, here

$$U_1=\begin{pmatrix}1 & 0 & 0\\ \frac{11}{10} & \frac{5}{2} & 2\end{pmatrix},\quad V_1=\begin{pmatrix}\frac{5}{6} & 0 & 0\\ 1 & 0 & 0\end{pmatrix}$$

and

$$U_2=\begin{pmatrix}2 & 0 & 0\\ -\frac{1}{2} & 3 & 0\end{pmatrix},\quad V_2=\begin{pmatrix}\frac{3}{2} & 0 & 0\\ -\frac{3}{4} & \frac{1}{2} & 0\end{pmatrix}.$$

It is easy to see that

$$U_1^{\dagger}=\begin{pmatrix}1 & 0\\ -0.2683 & 0.2439\\ -0.2146 & 0.1951\end{pmatrix} \text{ and } U_2^{\dagger}=\begin{pmatrix}0.5 & 0\\ 0.0833 & 0.3333\\ 0 & 0\end{pmatrix}.$$

Clearly, $A_1=U_1-V_1$ is a proper single weak regular splitting over proper cones and $A_2=U_2-V_2$ is a proper single regular splitting over proper cones.

Following the operations, we have

$$A_2^{\dagger}V_2-A_1^{\dagger}V_1=\begin{pmatrix}-2 & 0 & 0\\ -0.722 & 0.2 & 0\\ -0.0976 & 0 & 0\end{pmatrix}\ngeq_{K_1} 0$$

and

$$A_2^\dagger U_2 - A_1^\dagger U_1 = \begin{pmatrix} -2 & 0 & 0 \\ -0.722 & 0.5902 & -0.4878 \\ -0.0976 & -0.4878 & -0.3902 \end{pmatrix} \ngeq_{K_1} 0,$$

i.e., $A_2^\dagger V_2 \ngeq_{K_1} A_1^\dagger V_1$ and $A_2^\dagger U_2 \ngeq_{K_1} A_1^\dagger U_1$.

In fact, we have $\rho(U_1^\dagger V_1) = 0.8333 > 0.75 = \rho(U_2^\dagger V_2)$.

The above example shows that $A_1^\dagger V_2 \geqslant_{K_1} A_1^\dagger V_1 >_{K_1} 0$ or $A_2^\dagger U_2 \geqslant_{K_1} A_1^\dagger U_1 >_{K_1} 0$ cannot be dropped in Theorem 9. As the same time, we found $U_1^\dagger \ngeq 0$, but $U_1^\dagger \geqslant_{K_1} 0$, which shows that the iterative method based on splittings over proper cones has stronger applicability.

CONCLUSION

In order to solve the rectangular system by the iterative method based on splittings over proper cones, we establish the concepts of proper single regular and weak regular splittings over proper cones in this paper. Furthermore, we propose convergence results of the proper single regular splitting over proper cones of general rectangular matrices and semimonotone matrices over proper cones respectively. To analyze the convergence rate of iterative systems, we present comparison theorems of different splittings over proper cones of one rectangular matrix, and comparison theorems for proper single regular and/or weak regular splittings over proper cones of different rectangular matrices are given. It should be noted that the theoretical results obtained in this paper are more general than the splitting results over field. Moreover, the numerical examples given in the paper show that the iterative method based on splittings over proper cones may has stronger applicability.

It is well known that comparison theorems between the spectral radii of iteration matrices are useful tools to analyze the convergent rate of iteration methods or to judge the effectiveness of preconditioners. Therefore, the research results obtained in this paper can be applied to the following two aspects in the future:

(i) The research results in the paper can be applied to analyze the convergent rate of the regularized iterative method [14] based on proper single regular and weak regular splittings over proper cones.

(ii) Comparison results in the paper can be used to judge the efficiency of the preconditioners for rectangular linear systems.

Acknowledgements: The research has been supported by the Postgraduate Research Funding Project of Northwest Normal University (No. 2020KYZZ001118), the "innovation star" project for excellent postgraduates in Gansu Province (No. 2021CXZX-264) and the Foundation for Distinguished Young Scholars of Gansu Province (No. 20JR5RA540).

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