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S-asymptotically ω-periodic dynamics in a fractional-order genetic regulatory network with time lags

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ABSTRACT: This paper considers a class of Caputo fractional-order genetic regulatory networks with time-varying lags in biomedicine and life sciences. Applying some features of Mittag-Leffler function and contraction mapping principle, the sufficient conditions are gained to ensure the existence and uniqueness of S-asymptotically ω-periodic solution of the model. Based on comparison principle and stability theorem of linear delayed Caputo fractional-order differential equations, global asymptotical stability of the model is also investigated. The work of this article can improve and expand some existing results.

KEYWORDS: S-asymptotical periodicity, genetic regulatory, Mittag-Leffler, asymptotical stability, comparison principle

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INTRODUCTION

A large number of DNAs, RNAs, protein molecules and small molecules in an organism and the mechanisms regulatory of genes construct the gene regulatory networks (GRNs), which are the networks of genes and their interactions within a cell. In the past few years, GRNs had caught an increasing number of scholars' attention owing to its applications in biomedicine and life sciences. GRNs are not only set in the life sciences, but also based on the control theory. At the same time, they are very practical and effective models for describing the highly dynamic and complex processes of transcriptional interactions. Therefore, GRNs are both essential to investigate a wide range of phenomena in living organisms and have lots of potential applications in biomedicine. On the other hand, the gene sequencing technology develops rapidly in recent years, various GRNs models are established to predict the dynamic behaviors of GRNs, such as Boolean network model [1], Petri networks [2], linear combination model [3], Bayesian network model [4], differential equation models [5–15] and so on.

Of which, differential equation models are widely used to describe the dynamic and static characteristics of GRNs. The differential equation models are functions composed of external environmental factors and expression levels of other genes that describe changes in gene expression, which can simulate the dynamic behaviors of GRNs beautifully. Compared with other models, differential equation models are more powerful, flexible and have more benefits to research the complex relationships in gene networks.

Over the past few decades, many literatures had learned integer order GRNs models [5, 10–14, 16, 17]. For example, in [5], by virtue of Banach fixed point theorem and novel analysis techniques, the existence, uniqueness and global exponential stability of weighted pseudo almost automorphic solution for a class of delayed fuzzy GRNs are researched. Chen and Aihara [13] studied the stability and bifurcation for a class of GRNs with time lags based on the method of local stability and bifurcation. Aouiti and Touati [16] considered the stability and global dissipativity for neutral-type fuzzy GRNs with mixed time delays utilizing Lyapunov functional method and linear matrix inequalities. Li et al [17] discussed the stability of GRNs with SUM regulatory functions according to linear matrix inequalities and Lyapunov function. Moreover, by applying Lyapunov stability theory and linear matrix inequalities techniques, Ren and Cao [12] investigated the asymptotic and robust stability of the following GRNs with time lags: for $i = 1, 2, \dots, n$,

$$\begin{cases} \dot{M}_i(t) = -a_i M_i(t) + \sum_{j=1}^{n} w_{ij} f_j(P_j(t-\sigma(t))) + B_i, \\ \dot{P}_i(t) = -c_i P_i(t) + d_i M_i(t-\tau(t)), \quad t > 0, \end{cases} \tag{1}$$

where $M_i(t)$, $P_i(t) \in \mathbb{R}$ represent the concentrations of the i-th mRNA and i-th protein at time t, respectively; $a_i > 0$ and $c_i > 0$ are the decay rates of mRNA and protein, respectively; $d_i > 0$ is the translation rate; the Hill form regulatory function

$$f_j(x) = \frac{(\frac{x}{\beta_j})^{H_j}}{1 + (\frac{x}{\beta_j})^{H_j}} \tag{2}$$

is monotonically increasing with H_j being the Hill coefficient and β_j being some real positive constant; w_{ij} is the coupling parameter; $\tau(t) \geqslant 0$ and $\sigma(t) \geqslant 0$ are time lags at time $t \geqslant 0$; $B_i = \sum_{j \in I_i} b_{ij}$ represents the basal transcriptional rate of the repressor of gene i and I_i is the set of all the j, which is a repressor of gene i; b_{ij} is the dimensionless transcriptional rate of transcription factor j to i, for $i, j = 1, 2, \dots, n$.

Compared to integer-order differential operators, fractional-order differential operators have nonlocality and memorability, which can better describe the modeling mechanisms with memorizing or genetic properties in biomedicine. Therefore, in recent years, numerous researchers studied the dynamic properties of fractional-order GRNs (FGRNs), e.g., global stability and Hopf bifurcation [18]. Taking advantage of the fractional Lyapunov method, the authors [18] gained global Mittag-Leffler stability of FGRNs below:

$$\begin{cases} {}_0^C D_t^{\alpha} M_i(t) = -a_i M_i(t) + \sum_{j=1}^n w_{ij} f_j(P_j(t)) + B_i, \\ {}_0^C D_t^{\alpha} P_i(t) = -c_i P_i(t) + d_i M_i(t),\ t > 0,\ i = 1, 2, \dots, n, \end{cases}$$

where ${}_0^C D_t^{\alpha}$ denotes the Caputo derivative of order $0 < \alpha \leqslant 1$ with

$${}_0^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^{\alpha}} \,\mathrm{d}s,$$

$f \in \mathscr{C}^1([0,\infty),\mathbb{R})$, $t > 0$ and the meanings of other parameters are same as Eq. (1).

In practical applications, periodic motion is an interesting dynamical property for the biomedicine models due to many biological and cognitive activities (e.g., heartbeat, locomotion, memorization, etc) regularly repeat. It is well known that human brain oscillates periodically. Thus, studying the periodic property for biomedicine models is necessary in order to reveal how these mechanisms of life work. Unfortunately, FGRNs can not generate nonconstant periodic oscillation, more information can be found in [19–21] and references therein. To this end, many researchers have recently devoted their efforts to asymptotically periodic solution for fractional-order models; refer to [22, 23]. For all we know, there few papers take the periodic dynamics of FGRNs into consideration. Motivated by the aforementioned argument, in this article we consider the S-asymptotically periodic oscillation and global asymptotic stability for FGRNs.

On the basis of bioinformatics theory in GRNs, we can see that it takes some time to complete the process of transcription and translation of gene information. That is to say, time lags play an vital role in the process of genetic expression. Anbalagan [24] investigated the existence and uniqueness of equilibrium points for FGRNs with feedback regulation time delays by using the Banach fixed point theorem and Cauchy-Schwarz inequality and considered the finite-time delay-free stability criteria for FGRNs by using generalized Gronwall-Bellman inequality, equivalent norm techniques and Laplace transform. In [25], the global asymptotical stability for FGRNs with time delay is learned based on Lyapunov method and comparison theorem. We refer the interested readers to articles [13, 26–28] and references therein. Meanwhile, there are lots of papers on FGRNs with time-varying lags. The article [29] focused on the stability of FGRNs with certain H_∞/passivity performance level in accordance with various results on fractional derivatives/integrals and convex property of linear matrix inequalities. Stamova and Stamov [30] studied the almost periodicity and the global Mittag-Leffler stability for a class of FGRNs and considered the effects of time-varying delays and impulsive perturbations at fixed times on the almost periodicity by the method of Lyapunov functions and appropriate conditions. Literatures [6, 12, 31] provide more details and references therein, and many studies indicate that time lags both invariable and variable may affect the stability of FGRNs.

In the translation process of GRNs, when different external stimuli effect cells or cells are at distinct stages of development, the genes contained in the expression process are different and the changes of gene expression can cause the changes of interaction, which leads to the changes of network structure. Therefore, in practical, the parameters are not invariable but change along with time. To the best of our knowledge, few researchers pay attention to the global asymptotical stability of S-asymptotically periodic solution for FGRNs with time-varying delays. Based on the above consideration, this paper deals with the dynamics behavior of FGRNs depicted by

$$\begin{cases} {}_0^C D_t^{\alpha} M_i(t) = -a_i(t) M_i(t) + \sum_{j=1}^n w_{ij}(t) f_j(P_j(t-\sigma_j(t))) + B_i(t), \\ {}_0^C D_t^{\alpha} P_i(t) = -c_i(t) P_i(t) + d_i(t) M_i(t-\tau_i(t)), \quad t > 0, \\ M_i(s) = \tilde{M}_i(s), \quad P_i(s) = \tilde{P}_i(s), \quad s \in [-\eta, 0], \end{cases} \tag{3}$$

where $M_i(t)$, $P_i(t) \in \mathbb{R}$ are defined as the same as Eq. (1); $a_i(t)$, $c_i(t)$, $w_{ij}(t)$, $d_i(t)$, $B_i(t)$ are continuous functions on $\mathbb{R}$; the parameters $a_i(t) > 0$ and $c_i(t) > 0$ are the degradation velocities of mRNA and protein molecules at time t, respectively; $d_i(t) \geqslant 0$ indicates the translation rate at time t; f_j is the nonlinear protein feedback regulation, which is usually expressed in the Hill form as (2); $B_i(t) = \sum_{j \in I_i} b_{ij}(t)$ represents the basal transcription rate of the repressor of gene i with I_i being the set of all the j which is a repressor of gene i; $b_{ij}(t) \geqslant 0$ is the dimensionless transcriptional rate of transcription factor j to i at time t; $w_{ij}(t) \in \mathbb{R}$ is the coupling parameters of the genetic network defined at time $t > 0$ as

$$w_{ij}(t) = \begin{cases} b_{ij}(t), & \text{if transcription factor } j \text{ is an activator of gene } i \\ 0, & \text{if there is no link from node } j \text{ to } i \\ -b_{ij}(t), & \text{if transcription factor } j \text{ is an repressor of gene } i \end{cases}$$

$\tau_i(t)$ and $\sigma_j(t)$ are time-varying delays; $\eta = \max_{1 \leqslant j \leqslant 2n} \sup_{t \geqslant 0}\{\tau_j(t), \sigma_j(t)\}$; $\tilde{M}_i(s)$ and $\tilde{P}_i(s)$ are differentiable functions that give the initial levels of mRNA and protein molecules in i-th gene, respectively, $i, j = 1, 2, \dots, n$. For more information, please refer to literatures [29, 32] and the references therein.

The main contributions of this paper can be highlighted below.

1) By means of Mittag-Leffler function kernels and its some important features, the Volterra integral expression has been achieved for FGRNs (3).

2) The existence and uniqueness of S-asymptotically periodic oscillation for FGRNs (3) are studied based on contraction mapping principle. Additionally, novel and concise conditions are derived for global asymptotical stability of the periodic oscillation by applying comparison principle and stability theorem of fractional-order differential equations.

3) The influence of time-varying delays on dynamic behaviors (e.g., asymptotical periodicity, global asymptotical stability) for FGRNs (3) is discussed.

Notations: $\mathbb{N}$ is the set of positive integers; $\mathbb{R}^n$ is the n-dimensional real vector space; $\mathbb{R}_0^n$ describes the n-dimensional nonnegative real vector space; $\mathbb{C}$ is the set of complex numbers and $\mathscr{C}^n(J,\mathbb{R}^n)$ denotes the space consisting of n-order continuous differentiable functions from J to $\mathbb{R}^n$.

CAPUTO FRACTIONAL DERIVATIVE AND SOME LEMMAS

Definition 1 ([33]) The two-parameter Mittag-Leffler function is defined as

$$\mathbb{E}_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad z \in \mathbb{C}, \quad \alpha,\beta > 0. \tag{4}$$

Especially, $\mathbb{E}_1(z) = e^z$; $\mathbb{E}_{\alpha,1}(z) = \mathbb{E}_\alpha(z)$; $\mathbb{E}_{1,2}(z) = \frac{e^z-1}{z}$.

Lemma 1 ([33])

$$\frac{\mathrm{d}}{\mathrm{d}z}\big[z^\alpha \mathbb{E}_{\alpha,\alpha+1}(\lambda z^\alpha)\big] = z^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda z^\alpha),$$

where $\alpha,\lambda,z \in \mathbb{C}$.

Lemma 2 ([19]) *If* $\lambda > 0$ *and* $\alpha \in (0,1]$, *then* $\lim_{t\to\infty} t^\alpha \mathbb{E}_{\alpha,\alpha+1}(-\lambda t^\alpha) = \frac{1}{\lambda}$ *and* $t^\alpha \mathbb{E}_{\alpha,\alpha+1}(-\lambda t^\alpha) \leqslant \frac{1}{\lambda}$ *for* $t \geqslant 0$.

Lemma 3 ([21]) *If* $a,\lambda > 0$ *and* $\alpha \in (0,1]$, *then*

$$\lim_{t\to\infty} \mathbb{E}_\alpha(-\lambda t^\alpha) = 0,$$
$$\lim_{t\to\infty} \int_0^a (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-\lambda(t-s)^\alpha]\,\mathrm{d}s = 0.$$

Lemma 4 ([33]) *If the Laplace transform for* $f(t) \in \mathscr{C}^n([0,\infty),\mathbb{R})$ *is defined as*

$$F(s) = \mathscr{L}\{f(t);s\} = \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t, \quad s \in \mathbb{C}.$$

Then

(*i*) $\mathscr{L}\{{}_0^C D_t^\alpha f(t);s\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0)$, $0 < n-1 < \alpha < n$, $n \in \mathbb{N}$, $t \geqslant 0$, $s \in \mathbb{C}$;

(*ii*) $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$.

Let

$$\eta_j(t) = \begin{cases} \tau_j(t), & t > 0, j = 1,2,\ldots,n, \\ \sigma_j(t), & t > 0, j = n+1,n+2,\ldots,2n. \end{cases} \tag{5}$$

Lemma 5 ([34]) *Suppose that* $u_i \geqslant 0$, $v_i \geqslant 0$ *and* $u_i, v_i \in \mathscr{C}([0,\infty),\mathbb{R})$, *considering the following fractional-order differential system*

$$\begin{cases} {}_0^C D_t^\alpha u_i(t) \leqslant -a_i u_i(t) + b_i \sum_{j=1}^{2n} u_j(t-\eta_j(t)), \quad t > 0, \\ u_i(t) = \varphi_i(t) \geqslant 0, \quad t \in [-\eta,0], \end{cases}$$

and the following linear fractional-order differential system

$$\begin{cases} {}_0^C D_t^\alpha v_i(t) = -a_i v_i(t) + b_i \sum_{j=1}^{2n} v_j(t-\eta_j(t)), \quad t > 0, \\ v_i(t) = \varphi_i(t) \geqslant 0, \quad t \in [-\eta,0], \end{cases}$$

where $i = 1,2,\ldots,2n$. *If* $a_i > 0$ *and* $b_i > 0$, *then* $u_i(t) \leqslant v_i(t)$, $\forall t \geqslant 0$, $i = 1,2,\ldots,2n$.

Lemma 6 ([35]) *Assume that* $x \in \mathscr{C}^1([t_0,\infty),\mathbb{R})$, *then* ${}_0^C D_{t_0}^\alpha x^2(t) \leqslant 2x(t){}_0^C D_{t_0}^\alpha x(t)$, $\forall t \in [t_0,\infty)$, $0 < \alpha < 1$.

Let $\mathrm{SAP}_\omega(\mathbb{R}_0^{2n}) = \Big\{(M,P)^\top \in \mathscr{C}([0,\infty),\mathbb{R}^{2n}) : M = (M_1,M_2,\ldots,M_n)^\top,\ P = (P_1,P_2,\ldots,P_n)^\top,\ M_i$ and P_i are S-asymptotically ω-periodic functions with nonnegative initial conditions $\tilde{M}_i(s)$ and $\tilde{P}_i(s)$, $s \in [-\eta,0]$, $i = 1,2,\ldots,n\Big\}$. $\mathrm{SAP}_\omega(\mathbb{R}_0^{2n})$ is a Banach space endowed with the norm $\|x\|_\infty := \sup_{t\geqslant 0}\max_{1\leqslant i\leqslant n}\{|x_i(t)|,|y_i(t)|\}$.

We switch FGRNs (3) into the following system

$$\begin{cases} {}_0^C D_t^\alpha M_i(t) = -AM_i(t) + (A-a_i(t))M_i(t) \\ \qquad + \sum_{j=1}^n w_{ij}(t) f_j(P_j(t-\sigma_j(t))) + B_i(t), \\ {}_0^C D_t^\alpha P_i(t) = -CP_i(t) + (C-c_i(t))P_i(t) \\ \qquad + d_i(t)M_i(t-\tau_i(t)), \quad t > 0, \\ M_i(s) = \tilde{M}_i(s),\ P_i(s) = \tilde{P}_i(s),\ s \in [-\eta,0], \end{cases} \tag{6}$$

$i = 1,2,\ldots,n$, where A and C are undetermined constants.

In view of Eq. (6), we investigate the following system

$$\begin{cases} {}_0^C D_t^\alpha M_i(t) = -AM_i(t) + (A-a_i(t))\varphi_i^M(t) \\ \qquad + \sum_{j=1}^n w_{ij}(t) f_j(\varphi_j^P(t-\sigma_j(t))) + B_i(t), \\ {}_0^C D_t^\alpha P_i(t) = -CP_i(t) + (C-c_i(t))\varphi_i^P(t) \\ \qquad + d_i(t)\varphi_i^M(t-\tau_i(t)), \quad t > 0, \\ M_i(s) = \tilde{M}_i(s),\ P_i(s) = \tilde{P}_i(s),\ s \in [-\eta,0],\ i = 1,2,\ldots,n, \end{cases}$$

for any $\varphi = (\varphi_1^M,\ldots,\varphi_n^M,\varphi_1^P,\ldots,\varphi_n^P)^\top \in \mathrm{SAP}_\omega(\mathbb{R}_0^{2n})$.

Define operator $T:\varphi\to x^{\varphi}$, $\forall\varphi\in \mathrm{SAP}_{\omega}(\mathbb{R}_0^{2n})$ as follows.

$$\begin{aligned}T\varphi&=((T\varphi)_1^M,\dots,(T\varphi)_n^M,(T\varphi)_1^P,\dots,(T\varphi)_n^P)^\top\\&=(M_1^{\varphi},\dots,M_n^{\varphi},P_1^{\varphi},\dots,P_n^{\varphi})^\top=x^{\varphi},\end{aligned}\tag{7}$$

where

$$\begin{aligned}(T\varphi)_i^M(t)&=M_i^{\varphi(t)}=\tilde{M}_i(0)\mathbb{E}_{\alpha}(-At^{\alpha})\\&+\int_0^t(t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-A(t-s)^{\alpha}]\Big[(A-a_i(s))\varphi_i^M(s)\\&+\sum_{j=1}^n w_{ij}(s)f_j(\varphi_j^P(s-\sigma_j(s)))+B_i(s)\Big]\mathrm{d}s,\\(T\varphi)_i^P(t)&=P_i^{\varphi(t)}=\tilde{P}_i(0)\mathbb{E}_{\beta}(-Ct^{\alpha})\\&+\int_0^t(t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-C(t-s)^{\alpha}]\Big[(C-c_i(s))\varphi_i^P(s)\\&+d_i(s)\varphi_i^M(s-\tau_i(s))\Big]\mathrm{d}s,\quad t>0,\\(T\varphi)_i^M(s)&=M_i^{\varphi(s)}=\tilde{M}_i(s),\\(T\varphi)_i^P(s)&=P_i^{\varphi(s)}=\tilde{P}_i(s),\quad s\in[-\eta,0],\ i=1,2,\dots,n.\end{aligned}\tag{8}$$

If T has a unique fixed point $\varphi^*\in SAP_{\omega}(\mathbb{R}_0^{2n})$, then $\varphi^*=T\varphi^*=x^{\varphi^*}$ is a unique S-asymptotically ω-periodic oscillation of FGRNs (3).

Remark 1 If $\alpha=1$ in (8), then (8) is turned to the following Volterra integral expression of the solution for first order genetic regulatory networks with time lags,

$$\begin{aligned}M_i^{\varphi(t)}&=\tilde{M}_i(0)\,\mathrm{e}^{-At}+\int_0^t\mathrm{e}^{-A(t-s)}\Big[(A-a_i(s))\varphi_i^M(s)\\&+\sum_{j=1}^n w_{ij}(s)f_j(\varphi_j^P(s-\sigma_j(s)))+B_i(s)\Big]\mathrm{d}s,\\P_i^{\varphi(t)}&=\tilde{P}_i(0)\,\mathrm{e}^{-Ct}+\int_0^t\mathrm{e}^{-C(t-s)}\Big[(C-c_i(s))\varphi_i^P(s)\\&+d_i(s)\varphi_i^M(s-\tau_i(s))\Big]\mathrm{d}s,\quad t>0,\\M_i^{\varphi(s)}&=\tilde{M}_i(s),\ P_i^{\varphi(s)}=\tilde{P}_i(s),\quad s\in[-\eta,0],\ i=1,2,\dots,n.\end{aligned}\tag{9}$$

Referring to (9), the almost periodic dynamics of first order GRNs had been considered in [15]. However, few papers concentrate on the periodic dynamics of FGRNs. So the work in this paper complements some existing results.

Remark 2 Eq. (3) is an extension of integer-order GRNs [15] and some other models associated with different fractional-order derivatives and time-varying delays [16, 31].

S-ASYMPTOTICAL *ω*-PERIODICITY

In this section, we will research S-asymptotically ω-periodic oscillation of FGRNs (3) on the basis of contraction mapping principle and some appropriate assumptions.

Define $\|x\|_1=\max_{1\leqslant i\leqslant n}|x_i|$ for all $x=(x_1,x_2,\dots,x_n)^\top\in\mathbb{R}^n$. For any bounded function $f(t)\in\mathscr{C}([0,\infty),\mathbb{R})$, let $\bar{f}=\sup_{t\geqslant0}|f(t)|$ and $\underline{f}=\inf_{t\geqslant0}|f(t)|$.

Definition 2 ([23]) Suppose that there exists $\omega>0$ such that $\lim_{t\to\infty}\|x(t+\omega)-x(t)\|_1=\lim_{t\to\infty}\max_{1\leqslant i\leqslant n}|x_i(t+\omega)-x_i(t)|=0$ for $x=(x_1,x_2,\dots,x_n)^\top\in\mathscr{C}([t_0,+\infty),\mathbb{R}^n)$, then x is S-asymptotically ω-periodic.

Suppose the following conditions hold.

(H_1) $a_i(t)>0$, $c_i(t)>0$, $d_i(t)\geqslant0$, $b_{ij}(t)\geqslant0$ and $B_i(t)$ are S-asymptotically ω-periodic functions; $\tau_i(t)$ and $\sigma_j(t)$ are nonnegative ω-periodic functions, $\forall t\geqslant0$, $i,j=1,2,\dots,n$.

(H_2) It has a positive number L_j^f ensuring that $|f_j(x)-f_j(y)|\leqslant L_j^f|x-y|$, $\forall x,y\in\mathbb{R}$, $j=1,2,\dots,n$.

(H_3) $\sum_{j=1}^n L_j^f\bar{w}_{ij}<\underline{a}_i$ and $\bar{d}_i<\underline{c}_i$, $i=1,2,\dots,n$.

From (H_3), there exist two positive constants $A>\overline{a}_i$ and $C>\bar{c}_i$ for $i=1,2,\dots,n$ ensuring that

$$0<\xi=\max_{1\leqslant i\leqslant n}\left\{\frac{A-\underline{a}_i+\sum_{j=1}^n\bar{w}_{ij}L_j^f}{A},\ \frac{C-\underline{c}_i+\bar{d}_i}{C}\right\}<1.\tag{10}$$

Theorem 1 *Suppose that (H_1)–(H_3) are fulfilled, then FGRNs (3) owns a unique S-asymptotically periodic oscillation.*

Proof: Let $T:SAP_{\omega}(\mathbb{R}_0^{2n})\to\mathscr{C}([0,\infty),\mathbb{R}^{2n})$ be defined as that in Eqs. (7). Firstly, it proves that $T:SAP_{\omega}(\mathbb{R}_0^{2n})\to SAP_{\omega}(\mathbb{R}_0^{2n})$. For $\varphi=(\varphi_1^M,\dots,\varphi_n^M,\varphi_1^P,\dots,\varphi_n^P)^\top\in SAP_{\omega}(\mathbb{R}_0^{2n})$, $\forall\epsilon>0$, it has $t_1>0$ such that

$$|\varphi_i^M(t+\omega)-\varphi_i^M(t)|<\epsilon,\quad|\varphi_i^P(t+\omega)-\varphi_i^P(t)|<\epsilon,$$

$$\begin{aligned}&|\varphi_i^M(t+\omega-\tau_i(t+\omega))-\varphi_i^M(t-\tau_i(t))|\\&\quad=|\varphi_i^M(t+\omega-\tau_i(t))-\varphi_i^M(t-\tau_i(t))|<\epsilon,\end{aligned}$$

$$\begin{aligned}&|\varphi_j^P(t+\omega-\sigma_j(t+\omega))-\varphi_j^P(t-\sigma_j(t))|\\&\quad=|\varphi_j^P(t+\omega-\sigma_j(t))-\varphi_j^P(t-\sigma_j(t))|<\epsilon,\end{aligned}$$

$$\begin{aligned}&|a_i(t+\omega)-a_i(t)|<\epsilon,\quad|c_i(t+\omega)-c_i(t)|<\epsilon,\\&|d_i(t+\omega)-d_i(t)|<\epsilon,\quad|w_{ij}(t+\omega)-w_{ij}(t)|<\epsilon,\end{aligned}$$

$$|B_i(t+\omega)-B_i(t)|<\epsilon,\ \ t>t_1,\ \ i,j=1,2,\dots,n.\tag{11}$$

By means of the asymptotical periodicity of φ, $\|\varphi\|_\infty < \infty$.

For any $\varphi \in SAP_\omega(\mathbb{R}_0^{2n})$, according to the first equation of Eqs. (8), $\tilde{M}_i(0) \geqslant 0$, $A > \bar{a}_i$, $0 \leqslant f_j \leqslant 1$, and

$$\begin{aligned}&\sum_{j=1}^n w_{ij}(s) f_j(\varphi_j^P(s-\sigma_j(s))) + B_i(s)\\ &= \sum_{j\in\tilde{I}_i} w_{ij}(s) f_j(\varphi_j^P(s-\sigma_j(s)))\\ &\quad + \sum_{j\in I_i} b_{ij}(s)[1-f_j(\varphi_j^P(s-\sigma_j(s)))] \geqslant 0, \quad s>0,\end{aligned}$$

where I_i is the set of all the j, which is a repressor of gene i, $\tilde{I}_i$ is the complement set of I_i, $i,j = 1,2,\dots,n$. Combining Lemma 1 and Lemma 2, it yields $(T\varphi)_i^M(t) \geqslant 0$, $i=1,2,\dots,n$. Similarly, $(T\varphi)_i^P(t) \geqslant 0$, $i=1,2,\dots,n$. Thus, $T\varphi \geqslant 0$.

By the first equation of Eqs. (8), for $t>0$, it gains

$$\begin{aligned}(T\varphi)_i^M(t+\omega) &= \tilde{M}_i(0)\mathbb{E}_\alpha[-A(t+\omega)^\alpha]\\ &\quad + \int_0^{t+\omega} (t+\omega-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-A(t+\omega-s)^\alpha]\Big[(A-a_i(s))\varphi_i^M(s)\\ &\quad + \sum_{j=1}^n w_{ij}(s) f_j(\varphi_j^P(s-\sigma_j(s))) + B_i(s)\Big]\mathrm{d}s\\ &= \tilde{M}_i(0)\mathbb{E}_\alpha[-A(t+\omega)^\alpha] + \int_{-\omega}^t (t-s)^{\alpha-1}\\ &\quad \times \mathbb{E}_{\alpha,\alpha}[-A(t-s)^\alpha]\Big[(A-a_i(s+\omega))\varphi_i^M(s+\omega)\\ &\quad + \sum_{j=1}^n w_{ij}(s+\omega) f_j(\varphi_j^P(s+\omega-\sigma_j(s+\omega))) + B_i(s+\omega)\Big]\mathrm{d}s,\end{aligned}$$

which obtains

$$\begin{aligned}&(T\varphi)_i^M(t+\omega) - (T\varphi)_i^M(t) = \tilde{M}_i(0)\mathbb{E}_\alpha[-A(t+\omega)^\alpha]\\ &\quad - \tilde{M}_i(0)\mathbb{E}_\alpha(-At^\alpha) + \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-A(t-s)^\alpha]\\ &\quad \times \Big[(A-a_i(s+\omega))\varphi_i^M(s+\omega) - (A-a_i(s))\varphi_i^M(s)\\ &\quad + \sum_{j=1}^n w_{ij}(s+\omega) f_j(\varphi_j^P(s+\omega-\sigma_j(s)))\\ &\quad - \sum_{j=1}^n w_{ij}(s) f_j(\varphi_j^P(s-\sigma_j(s))) + B_i(s+\omega) - B_i(s)\Big]\mathrm{d}s\\ &\quad + \int_{-\omega}^0 (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-A(t-s)^\alpha]\Big[(A-a_i(s+\omega))\varphi_i^M(s+\omega)\\ &\quad + \sum_{j=1}^n w_{ij}(s+\omega) f_j(\varphi_j^P(s+\omega-\sigma_j(s))) + B_i(s+\omega)\Big]\mathrm{d}s\\ &= I_{i1}(t) + I_{i2}(t) + I_{i3}(t) + I_{i4}(t) + I_{i5}(t) + I_{i6}(t)\\ &\quad + I_{i7}(t) + I_{i8}(t) + I_{i9}(t),\end{aligned}$$

where, for $i=1,2,\dots,n$,

$$\begin{aligned}I_{i1}(t) &= \tilde{M}_i(0)\{\mathbb{E}_\alpha[-A(t+\omega)^\alpha] - \mathbb{E}_\alpha(-At^\alpha)\},\\ I_{i2}(t) &= \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-A(t-s)^\alpha]\\ &\quad \times [\varphi_i^M(s+\omega) - \varphi_i^M(s)](A-a_i(s+\omega))\,\mathrm{d}s,\\ I_{i3}(t) &= \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-A(t-s)^\alpha]\varphi_i^M(s)\\ &\quad \times [a_i(s) - a_i(s+\omega)]\,\mathrm{d}s,\\ I_{i4}(t) &= \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-A(t-s)^\alpha]\sum_{j=1}^n \big[w_{ij}(s+\omega)\\ &\quad - w_{ij}(s)\big] f_j(\varphi_j^P(s+\omega-\sigma_j(s)))\,\mathrm{d}s,\\ I_{i5}(t) &= \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-A(t-s)^\alpha]\sum_{j=1}^n w_{ij}(s)\\ &\quad \times \Big[f_j(\varphi_j^P(s+\omega-\sigma_j(s))) - f_j(\varphi_j^P(s-\sigma_j(s)))\Big]\mathrm{d}s,\\ I_{i6}(t) &= \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-A(t-s)^\alpha][B_i(s+\omega) - B_i(s)]\,\mathrm{d}s,\\ I_{i7}(t) &= \int_{-\omega}^0 (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-A(t-s)^\alpha](A-a_i(s+\omega))\varphi_i^M(s+\omega)\,\mathrm{d}s,\\ I_{i8}(t) &= \int_{-\omega}^0 (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-A(t-s)^\alpha]\\ &\quad \times \sum_{j=1}^n w_{ij}(s+\omega) f_j(\varphi_j^P(s+\omega-\sigma_j(s)))\,\mathrm{d}s,\\ I_{i9}(t) &= \int_{-\omega}^0 (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-A(t-s)^\alpha]B_i(s+\omega)\,\mathrm{d}s.\end{aligned}$$

By Lemma 3, for $\epsilon > 0$, there exists $t_2 > t_1$ ensuring

$$|I_{i1}(t)| < \epsilon, \quad \forall t > t_2, \quad i=1,2,\dots,n. \tag{12}$$

Obviously, when $t \geqslant 0$, $\mathbb{E}_{\alpha,\alpha}[-At^\alpha] \geqslant 0$. Applying Lemma 1, it concludes

$$\begin{aligned}|I_{i2}(t)| &\leqslant \Big|\int_0^{t_1} (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-A(t-s)^\alpha]\\ &\qquad \times [\varphi_i^M(s+\omega) - \varphi_i^M(s)](A-a_i(s+\omega))\,\mathrm{d}s\Big|\\ &\qquad + \Big|\int_{t_1}^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-A(t-s)^\alpha]\\ &\qquad \times [\varphi_i^M(s+\omega) - \varphi_i^M(s)](A-a_i(s+\omega))\,\mathrm{d}s\Big|\\ &\leqslant 2A\|\varphi\|_\infty \int_0^{t_1} (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-A(t-s)^\alpha]\,\mathrm{d}s\\ &\qquad + A\epsilon \int_{t_1}^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-A(t-s)^\alpha]\,\mathrm{d}s\end{aligned}$$

$$= 2A\|\varphi\|_\infty \int_0^{t_1} (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-A(t-s)^\alpha]\,\mathrm{d}s - A\epsilon(t-s)^\alpha \mathbb{E}_{\alpha,\alpha+1}[-A(t-s)^\alpha]\Big|_{t_1}^{t}$$

$$= 2A\|\varphi\|_\infty \int_0^{t_1} (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-A(t-s)^\alpha]\,\mathrm{d}s + A\epsilon(t-t_1)^\alpha \mathbb{E}_{\alpha,\alpha+1}[-A(t-t_1)^\alpha],$$

where $t > t_1$, $i = 1,2,\dots,n$. Based on Lemma 2–Lemma 3, it has $t_3 > t_2$ such that

$$|I_{i2}(t)| < 2\epsilon, \quad t > t_3, \quad i = 1,2,\dots,n. \tag{13}$$

In the same way, it has $t_4 > t_3$ such that

$$|I_{i3}(t)| < \frac{2\|\varphi\|_\infty}{A}\epsilon, \tag{14}$$

$$|I_{i4}(t)| < \frac{2}{A}\sum_{j=1}^{n}(L_j^f\|\varphi\|_\infty + |f_j(0)|)\epsilon, \tag{15}$$

$$|I_{i5}(t)| < \frac{2}{A}\sum_{j=1}^{n}\bar{w}_{ij}L_j^f\epsilon, \tag{16}$$

$$|I_{i6}(t)| < \frac{2}{A}\epsilon, \tag{17}$$

$$|I_{i7}(t)| < A\|\varphi\|_\infty\epsilon, \tag{18}$$

$$|I_{i8}(t)| < \sum_{j=1}^{n}\bar{w}_{ij}(L_j^f\|\varphi\|_\infty + |f_j(0)|)\epsilon, \tag{19}$$

$$|I_{i9}(t)| < \bar{B}_i\epsilon, \quad t > t_4,\ i = 1,2,\dots,n. \tag{20}$$

Combining (12)–(20), there exists $M_1 > 0$ large enough such that, for $i = 1,2,\dots,n$,

$$|(T\varphi)_i^M(t+\omega) - (T\varphi)_i^M(t)| < M_1\epsilon, \quad t > t_4. \tag{21}$$

On the other hand, from the second equation of Eq. (8), for $t > 0$, it has

$$(T\varphi)_i^P(t+\omega) = \tilde{P}_i(0)\mathbb{E}_\alpha[-C(t+\omega)^\alpha] + \int_0^{t+\omega}(t+\omega-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-C(t+\omega-s)^\alpha] \times \Big[(C-c_i(s))\varphi_i^P(s) + d_i(s)\varphi_i^M(s-\tau_i(s))\Big]\mathrm{d}s$$

$$= \tilde{P}_i(0)\mathbb{E}_\alpha[-C(t+\omega)^\alpha] + \int_{-\omega}^{t}(t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-C(t-s)^\alpha] \times\Big[(C-c_i(s+\omega))\varphi_i^P(s+\omega) + d_i(s+\omega)\varphi_i^M((s+\omega)-\tau_i(s))\Big]\mathrm{d}s,$$

which gets

$$\begin{aligned}
&(T\varphi)_i^P(t+\omega) - (T\varphi)_i^P(t)\\
&= \tilde{P}_i(0)\mathbb{E}_\alpha[-C(t+\omega)^\alpha] - \tilde{P}_i(0)\mathbb{E}_\alpha[-C(t)^\alpha]\\
&+ \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-C(t-s)^\alpha]\times\\
&\Big[(C-c_i(s+\omega))\varphi_i^P(s+\omega) - (C-c_i(s))\varphi_i^P(s)\\
&+ d_i(s+\omega)\varphi_i^M((s+\omega)-\tau_i(s)) - d_i(s)\varphi_i^M((s)-\tau_i(s))\Big]\mathrm{d}s\\
&+ \int_{-\omega}^0 (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-C(t-s)^\alpha]\times\\
&\Big[(C-c_i(s+\omega))\varphi_i^P(s+\omega) + d_i(s+\omega)\varphi_i^M((s+\omega)-\tau_i(s))\Big]\mathrm{d}s\\
&= J_{i1}+J_{i2}+J_{i3}+J_{i4}+J_{i5}+J_{i6}+J_{i7},
\end{aligned}$$

where, for $i = 1,2,\dots,n$,

$$J_{i1} = \tilde{P}_i(0)\{\mathbb{E}_\alpha[-C(t+\omega)^\alpha] - \mathbb{E}_\alpha(-Ct^\alpha)\},$$

$$J_{i2} = \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-C(t-s)^\alpha](C-c_i(s+\omega)) \times[\varphi_i^P(s+\omega) - \varphi_i^P(s)]\,\mathrm{d}s,$$

$$J_{i3} = \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-C(t-s)^\alpha]\varphi_i^P(s)[c_i(s) - c_i(s+\omega)]\,\mathrm{d}s,$$

$$J_{i4} = \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-C(t-s)^\alpha][d_i(s+\omega) - d_i(s)] \times\varphi_i^M(s+\omega-\tau_i(s))\,\mathrm{d}s,$$

$$J_{i5} = \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-C(t-s)^\alpha]d_i(s) \times[\varphi_i^M(s+\omega-\tau_i(s)) - \varphi_i^M(s-\tau_i(s))]\,\mathrm{d}s,$$

$$J_{i6} = \int_{-\omega}^0 (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-C(t-s)^\alpha](C-c_i(s+\omega))\varphi_i^P(s+\omega)\,\mathrm{d}s,$$

$$J_{i7} = \int_{-\omega}^0 (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-C(t-s)^\alpha]d_i(s+\omega)\varphi_i^M(s+\omega-\tau_i(s))\,\mathrm{d}s.$$

By Lemma 3, for $\epsilon > 0$, there exists $t_5 > t_4$ such that

$$|J_{i1}(t)| < \epsilon, \quad \forall t > t_5,\ i = 1,2,\dots,n. \tag{22}$$

Making use of $\mathbb{E}_{\alpha,\alpha}[-Ct^\alpha] \geqslant 0$ for $t \geqslant 0$ and Lemma 1, it yields

$$\begin{aligned}
|J_{i2}| &\leqslant \Big|\int_0^{t_1}(t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-C(t-s)^\alpha](C-c_i(s+\omega))\\
&\quad\times[\varphi_i^P(s+\omega) - \varphi_i^P(s)]\,\mathrm{d}s\Big| + \Big|\int_{t_1}^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-C(t-s)^\alpha]\\
&\quad\times(C-c_i(s+\omega))[\varphi_i^P(s+\omega) - \varphi_i^P(s)]\,\mathrm{d}s\Big|\\
&\leqslant 2C\|\varphi\|_\infty\int_0^{t_1}(t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-C(t-s)^\alpha]\,\mathrm{d}s\\
&\quad + C\epsilon\int_{t_1}^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-C(t-s)^\alpha]\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= 2C\|\varphi\|_\infty \int_0^{t_1} (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-C(t-s)^\alpha]\,\mathrm{d}s \\
&\quad - C\epsilon(t-s)^\alpha \mathbb{E}_{\alpha,\alpha+1}[-C(t-s)^\alpha]\Big|_{t_1}^{t} \\
&= 2C\|\varphi\|_\infty \int_0^{t_1} (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-C(t-s)^\alpha]\,\mathrm{d}s \\
&\quad + C\epsilon(t-t_1)^\alpha \mathbb{E}_{\alpha,\alpha+1}[-C(t-t_1)^\alpha],
\end{aligned}
$$

where $t > t_1$, $i = 1,2,\dots,n$. By Lemma 2–Lemma 3, there exists $t_6 > t_5$ such that

$$|J_{i2}(t)| < 2\epsilon, \quad t > t_6,\ i = 1,2,\dots,n. \tag{23}$$

Analogously, there exists $t_7 > t_6$ such that

$$
\begin{aligned}
&|J_{i3}(t)| < \frac{2\|\varphi\|_\infty}{C}\epsilon, \\
&|J_{i4}(t)| < \frac{2\|\varphi\|_\infty}{C}\epsilon, \\
&|J_{i5}(t)| < \frac{2}{C}\bar{d}_i\epsilon, \\
&|J_{i6}(t)| < C\|\varphi\|_\infty\epsilon, \\
&|J_{i7}(t)| < \bar{d}_i\|\varphi\|_\infty\epsilon, \quad t > t_7,\ \ i = 1,2,\dots,n.
\end{aligned} \tag{24}
$$

By (22)–(24), there exists a positive number M_2 large enough such that, for $i = 1,2,\dots,n$,

$$|(T\varphi)_i^P(t+\omega) - (T\varphi)_i^P(t)| < M_2\epsilon, \quad t > t_7. \tag{25}$$

From (21) and (25), it deduces that

$$\|(T\varphi)(t+\omega) - (T\varphi)(t)\|_\infty < \max_{1\leqslant i\leqslant n}\{M_1, M_2\}\epsilon,$$

thus $T\varphi \in SAP_\omega(\mathbb{R}_0^{2n})$.

Next, we will demonstrate that mapping T is contractive. For $\varphi,\psi \in SAP_\omega(\mathbb{R}_0^{2n})$, by Lemma 2, from the first equation of Eqs. (8), it yields

$$
\begin{aligned}
&|(T\varphi)_i^M(t) - (T\psi)_i^M(t)| \\
&= \bigg|\int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-A(t-s)^\alpha]\Big\{(A-a_i(s))[\varphi_i^M(s) - \psi_i^M(s)] \\
&\quad + \sum_{j=1}^n w_{ij}(s)\Big[f_j(\varphi_j^P(s-\sigma_j(s))) - f_j(\psi_j^P(s-\sigma_j(s)))\Big]\Big\}\,\mathrm{d}s\bigg| \\
&\leqslant \Big[A - \underline{a}_i + \sum_{j=1}^n \bar{w}_{ij}L_j^f\Big]\|\varphi-\psi\|_\infty \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-A(t-s)^\alpha]\,\mathrm{d}s \\
&= t^\alpha \mathbb{E}_{\alpha,\alpha+1}(-At^\alpha)\Big[A - \underline{a}_i + \sum_{j=1}^n \bar{w}_{ij}L_j^f\Big]\|\varphi-\psi\|_\infty \\
&\leqslant \frac{1}{A}\Big[A - \underline{a}_i + \sum_{j=1}^n \bar{w}_{ij}L_j^f\Big]\|\varphi-\psi\|_\infty, \quad t \geqslant 0,\ i = 1,2,\dots,n,
\end{aligned} \tag{26}
$$

which implies that

$$
\begin{aligned}
\|T\varphi(t) - T\psi(t)\|_\infty &\leqslant \max_{1\leqslant i\leqslant n}\frac{1}{A}\Big[A - \underline{a}_i + \sum_{j=1}^n \bar{w}_{ij}L_j^f\Big]\|\varphi-\psi\|_\infty \\
&\leqslant \xi\|\varphi-\psi\|_\infty.
\end{aligned} \tag{27}
$$

Similarly,

$$
\begin{aligned}
&|(T\varphi)_i^P(t) - (T\psi)_i^P(t)| \\
&= \bigg|\int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-C(t-s)^\alpha]\Big[(C-c_i(s))[\varphi_i^P(s) - \psi_i^P(s)] \\
&\quad + d_i(s)[\varphi_i^M(s-\tau_i(s)) - \psi_i^M(s-\tau_i(s))]\Big]\,\mathrm{d}s\bigg| \\
&\leqslant (C - \underline{c}_i + \bar{d}_i)\|\varphi-\psi\|_\infty \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-C(t-s)^\alpha]\,\mathrm{d}s \\
&= t^\alpha \mathbb{E}_{\alpha,\alpha+1}(-Ct^\alpha)(C - \underline{c}_i + \bar{d}_i)\|\varphi-\psi\|_\infty \\
&\leqslant \frac{1}{C}(C - \underline{c}_i + \bar{d}_i)\|\varphi-\psi\|_\infty, \quad t \geqslant 0,\ i = 1,2,\dots,n,
\end{aligned} \tag{28}
$$

which gets

$$
\begin{aligned}
\|T\varphi(t) - T\psi(t)\|_\infty &\leqslant \max_{1\leqslant i\leqslant n}\frac{1}{C}(C - \underline{c}_i + \bar{d}_i)\|\varphi-\psi\|_\infty \\
&\leqslant \xi\|\varphi-\psi\|_\infty.
\end{aligned} \tag{29}
$$

Referring to (10), (27) and (29), it yields

$$\|T\varphi(t) - T\psi(t)\|_\infty \leqslant \xi\|\varphi-\psi\|_\infty \leqslant \|\varphi-\psi\|_\infty,$$

which indicates that T is a contractive mapping. Therefore, T has a unique fixed point $\varphi^* = T\varphi^*$ and $\varphi^* \in SAP_\omega(\mathbb{R}_0^{2n})$ is a unique S-asymptotically periodic oscillation of FGRNs (3). The proof is end. □

Remark 3 In organisms, periodic phenomenon exists in all kinds of physiology. The interaction in organisms is periodic and there exists a feedback loop. That is to say, in GRNs, the interactions of DNAs, RNAs, protein molecules and small molecules are regularly repeat. Hence, it is of great significance to study periodicity of GRNs. In the past few years, some literatures had discussed the periodic dynamics of integer-order GRNs [8, 15, 20, 36]. Remarkably, few scholars discuss the periodic oscillation for FGRNs. To some extent, the results in this paper fill the gap.

Remark 4 Recently, there are some papers with regard to exponential periodicity [21], asymptotical periodicity [22, 23], mean almost periodicity [37] and so on. However, there is just very little research on S-asymptotical ω-periodicity. In this regard, Theorem 1 is an extension of the existing results and the method in this article can provide ideas for researchers who study the qualitative analysis of fractional-order case.

GLOBAL ASYMPTOTICAL STABILITY

In this section, by virtue of comparison principle and stability theorem of delayed fractional-order differential equations, the global asymptotical stability of FGRNs (3) is learned based on some novel conditions. Let

$$
\begin{aligned}
&L^f = \max_{1\leqslant j\leqslant n} L_j^f,\ m = \min_{1\leqslant i\leqslant n}\Big\{2\underline{a}_i - \sum_{j=1}^n \bar{w}_{ij}L_j^f,\ 2\underline{c}_i - \bar{d}_i\Big\}, \\
&M = \max_{1\leqslant i\leqslant n}\big\{\bar{w}_{i*}L^f,\ \bar{d}_i\big\},\ \bar{w}_{i*} = \max_{1\leqslant j\leqslant n}\bar{w}_{ij},\ i,j = 1,2,\dots,n.
\end{aligned}
$$

Theorem 2 *If condition (H_2) and the following condition hold*

(H_4) $m > \max_{1\leqslant i\leqslant 2n} \frac{2nM}{1-\dot{\eta}_i^+}$, *where* $\dot{\eta}_i^+ = \sup_{t\geqslant 0} \dot{\eta}_i(t) < 1$, $i = 1, 2, \ldots, 2n$.

Then FGRNs (3) is globally asymptotically stable.

Proof: Assume that $(M, P)^\top = (M_1, \ldots, M_n, P_1, \ldots, P_n)^\top$ and $(\tilde{M}, \tilde{P})^\top = (\tilde{M}_1, \ldots, \tilde{M}_n, \tilde{P}_1, \ldots, \tilde{P}_n)^\top$ are two solutions of FGRNs (3). Let $x_i = M_i - \tilde{M}_i$, $y_i = P_i(t) - \tilde{P}_i$, $U = (u_1, \ldots, u_{2n})^\top$, where $u_i = x_i$ and $u_{n+i} = y_i$, $i = 1, 2, \ldots, n$.

It acquires from the first equation of FGRNs (3) that

$$ {}_0^C D_t^\alpha x_i(t) = -a_i(t)[M_i(t) - \tilde{M}_i(t)] + \sum_{j=1}^n w_{ij}(t)\big[f_j(P_j(t-\sigma_j(t))) - f_j(\tilde{P}_j(t-\sigma_j(t)))\big], $$

which deduces from Lemma 6 that, for $t > 0$,

$$ \begin{aligned} {}_0^C D_t^\alpha x_i^2(t) &\leqslant 2x_i(t)\, {}_0^C D_t^\alpha x_i(t) \\ &= -2a_i(t)x_i^2(t) + 2x_i(t)\sum_{j=1}^n w_{ij}(t) \times \big[f_j(P_j(t-\sigma_j(t))) - f_j(\tilde{P}_j(t-\sigma_j(t)))\big] \\ &\leqslant -2a_i(t)x_i^2(t) + 2\sum_{j=1}^n \bar{w}_{ij}L_j^f |x_i(t)||P_j(t-\sigma_j(t)) - \tilde{P}_j(t-\sigma_j(t))| \\ &\leqslant -2\underline{a}_i x_i^2(t) + \sum_{j=1}^n 2\Big[\sqrt{\bar{w}_{ij}L_j^f}|x_i(t)|\Big]\Big[\sqrt{\bar{w}_{ij}L_j^f}|y_j(t-\sigma_j(t))|\Big] \\ &\leqslant -2\underline{a}_i x_i^2(t) + \sum_{j=1}^n \bar{w}_{ij}L_j^f\left[x_i^2(t) + y_j^2(t-\sigma_j(t))\right] \\ &\leqslant -\Big(2\underline{a}_i - \sum_{j=1}^n \bar{w}_{ij}L_j^f\Big)x_i^2(t) + \bar{w}_{i*}L^f\sum_{j=1}^n y_j^2(t-\sigma_j(t)), \end{aligned} \tag{30} $$

where $i = 1, 2, \ldots, n$. Thus, by (5) and (30), it gets

$$ \begin{aligned} {}_0^C D_t^\alpha u_i^2(t) &\leqslant -mu_i^2(t) + \bar{w}_{i*}L^f\sum_{j=1}^n u_{n+j}^2(t-\sigma_j(t)), \\ &= -mu_i^2(t) + 0\times\sum_{j=1}^n u_j^2(t-\tau_j(t)) + \bar{w}_{i*}L^f\sum_{j=1}^n u_{n+j}^2(t-\sigma_j(t)), \\ &\leqslant -mu_i^2(t) + \bar{w}_{i*}L^f\sum_{j=1}^{2n} u_j^2(t-\eta_j(t)) \\ &\leqslant -mu_i^2(t) + M\sum_{j=1}^{2n} u_j^2(t-\eta_j(t)),\quad t>0,\ i=1,2,\ldots,n. \end{aligned} \tag{31} $$

In the same way, by the second equation of FGRNs (3), it has

$$ {}_0^C D_t^\alpha y_i(t) = -c_i(t)[P_i(t) - \tilde{P}_i(t)] + d_i(t)[M_i(t-\tau_i(t)) - \tilde{M}_i(t-\tau_i(t))], $$

which obtains from Lemma 6 that

$$ \begin{aligned} {}_0^C D_t^\alpha y_i^2(t) &\leqslant 2y_i(t)\, {}_0^C D_t^\alpha y_i(t) \\ &= -2c_i(t)y_i^2(t) + 2d_i(t)y_i(t)x_i(t-\tau_i(t)) \\ &\leqslant -2\underline{c}_i y_i^2(t) + 2\bar{d}_i y_i(t)|x_i(t-\tau_i(t))| \\ &\leqslant -\big(2\underline{c}_i - \bar{d}_i\big)y_i^2(t) + \bar{d}_i x_i^2(t-\tau_i(t)),\quad t>0. \end{aligned} \tag{32} $$

Combining (5) and (32), it yields for $i = 1, 2, \ldots, n$,

$$ \begin{aligned} {}_0^C D_t^\alpha u_{n+i}^2(t) &\leqslant -mu_{n+i}^2(t) + \bar{d}_i u_i^2(t-\tau_i(t)) \\ &= -mu_{n+i}^2(t) + \bar{d}_i\sum_{j=1}^{2n} u_j^2(t-\eta_j(t)) \\ &\leqslant -mu_{n+i}^2(t) + M\sum_{j=1}^{2n} u_j^2(t-\eta_j(t)),\quad t>0. \end{aligned} \tag{33} $$

From (31) and (33), it shows

$$ {}_0^C D_t^\alpha u_i^2(t) \leqslant -mu_i^2(t) + M\sum_{j=1}^{2n} u_j^2(t-\eta_j(t)),\quad t>0,\ i=1,2,\ldots,2n. \tag{34} $$

Next, considering the linear FGRNs as follows:

$$ \begin{cases} {}_0^C D_t^\alpha v_i(t) = -mv_i(t) + M\sum_{j=1}^{2n} v_j(t-\eta_j(t)), & t>0, \\ v_i(s) = u_i^2(s) \geqslant 0, & s\in[-\eta, 0],\ i=1,2,\ldots,2n. \end{cases} $$

Suppose that $\mu_j(t)$ is the inverse function for $t - \eta_j(t)$, i.e., $\mu_j(t-\eta_j(t)) = t$, $j = 1, 2, \ldots, 2n$. Let $V_i(s) \geqslant 0$ be the Laplace transform of $v_i(t) \geqslant 0$, $i = 1, 2, \ldots, 2n$. According to Eqs. (6) and Lemma 4, it gets

$$ \begin{aligned} s^\alpha V_i(s) - s^{\alpha-1}u_i^2(0) &= -mV_i(s) + M\sum_{j=1}^{2n}\int_0^\infty e^{-st}v_j(t-\eta_j(t))\,dt \\ &= -mV_i(s) + M\sum_{j=1}^{2n}\int_{-\eta_j(0)}^\infty \frac{e^{-s[p+\eta_j(\mu_j(p))]}}{1-\dot{\eta}_j(\mu_j(p))}v_j(p)\,dp \\ &\leqslant -mV_i(s) + \sum_{j=1}^{2n}\frac{M}{1-\dot{\eta}_j^+}\int_{-\eta_j(0)}^\infty e^{-sp}v_j(p)\,dp \quad (s>0) \\ &= -mV_i(s) + \sum_{j=1}^{2n}\frac{M}{1-\dot{\eta}_j^+}\Big[V_j(s) + \int_{-\eta_j(0)}^0 e^{-st}u_j^2(t)\,dt\Big] \quad (s>0). \end{aligned} $$

Set $V = \sum_{i=1}^{2n} V_i$. So

$$ \Big[s^\alpha + m - \max_{1\leqslant i\leqslant 2n}\frac{2nM}{1-\dot{\eta}_i^+}\Big]V(s) \leqslant s^{\alpha-1}\sum_{i=1}^{2n}u_i^2(0) + \sum_{i=1}^{2n}\sum_{j=1}^{2n}\frac{M}{1-\dot{\eta}_j^+}\int_{-\eta_j(0)}^0 e^{-st}u_j^2(t)\,dt,\quad s>0, $$

which concludes

$$sV(s) \leqslant \left[s^\alpha + m - \max_{1\leqslant i\leqslant 2n} \frac{2nM}{1-\dot{\eta}_i^+}\right]^{-1}\left[s^\alpha \sum_{i=1}^{2n} u_i^2(0) + \sum_{i=1}^{2n}\sum_{j=1}^{2n} \frac{Ms}{1-\dot{\eta}_j^+}\int_{-\eta_j(0)}^{0} \mathrm{e}^{-st} u_j^2(t)\,\mathrm{d}t\right],$$

where $s>0$. By employing Lemma 4, $\lim_{t\to\infty} v_i(t) \leqslant \lim_{s\to 0^+} sV(s) = 0$, $i=1,2,\dots,2n$. In light of Lemma 5, $\lim_{t\to+\infty} u_i^2(t) \leqslant \lim_{t\to+\infty} v_i(t) = 0$ and FGRNs (3) is globally asymptotically stable. The proof is end. □

Combining Theorem 1 and Theorem 2, it indicates

Theorem 3 *If (H_1)–(H_4) hold, then FGRNs (3) possesses a unique globally asymptotically stable S-asymptotically ω-periodic oscillation.*

Remark 5 If $\eta_j(t) \equiv 0$ $(j=1,2,\dots,2n)$, then (34) turns into ${}_0^C D_t^\alpha U(t) \leqslant -kU(t)$, where $U(t) = \sum_{i=1}^{2n} u_i(t)$, $k = m - 2nM$, $\forall t>0$. Hence, $U(t) \leqslant U(0)\mathbb{E}_\alpha(-kt^\alpha)$, $\forall t>0$. Suppose that $k>0$, it is easy to derive FGRNs (3) is Mittag-Leffler stability, which is same as the results in literatures [18].

Remark 6 Owing to the existence of time lags in FGRNs (3), it is difficult to yield Mittag-Leffler stability of FGRNs with time lags by using the methods in literatures [18]. Besides, the authors in literature [25] discussed global stability of FGRNs with constant lags, but the method in article [25] is unable to deal with FGRNs with time variable lags. Nevertheless, Theorem 2–Theorem 3 in this article get the global asymptotical stability of a unique S-asymptotically ω-periodic oscillation for FGRNs with time variable lags. Therefore, the work of this article extends and complements the results in literatures [18, 25].

NUMERICAL EXAMPLES

This section presents two numerical examples to illustrate the effectiveness of the results in this article.

Example 1 Considering the following FGRNs with periodic coefficients:

$$\begin{cases} {}_0^C D_t^{0.6} M_i(t) = -a_i(t)M_i(t) \\ \qquad + \sum_{j=1}^{n} w_{ij}(t) f_j(P_j(t-\sigma_j(t))) + B_i(t), \\ {}_0^C D_t^{0.6} P_i(t) = -c_i(t)P_i(t) + d_i M_i(t-\tau_i(t)),\ t>0, \end{cases} \tag{35}$$

where $a_i(t) = 4.5 + 1.5\sin t$, $c_i(t) = 1.3$, $d_i(t) = \frac{2}{3} + \frac{1}{3}\sin t$, $\tau_i(t) = 1 + 0.2\sin t$, $\sigma_j(t) = 1 + 0.2\cos t$, $L_j^f = 0.1$, $B_i(t) = 1+\sin t$, $f_j(P_j) = \frac{P_j^2}{1+P_j^2}$, $i,j=1,2$,

$$w_{ij}(t) = \begin{pmatrix} w_{11}(t) & w_{12}(t) \\ w_{21}(t) & w_{22}(t) \end{pmatrix} = \begin{pmatrix} 0 & -1-\sin t \\ -1-\sin t & 0 \end{pmatrix}$$

It is easy to deduce that

$$\sum_{j=1}^{2} L_j^f \bar{w}_{ij} = \sum_{j=1}^{2}(0.1*2 + 0.1*2) = 0.4 < 3 = \underline{a}_i,$$
$$\bar{d}_i = 1 < 1.3 = \underline{c}_i, \quad i=1,2,$$

i.e., (H_1)–(H_3) are satisfied, thus, by Theorem 1, system (35) has a unique S-asymptotically 2π-periodic oscillation, see Fig. 1–Fig. 2.

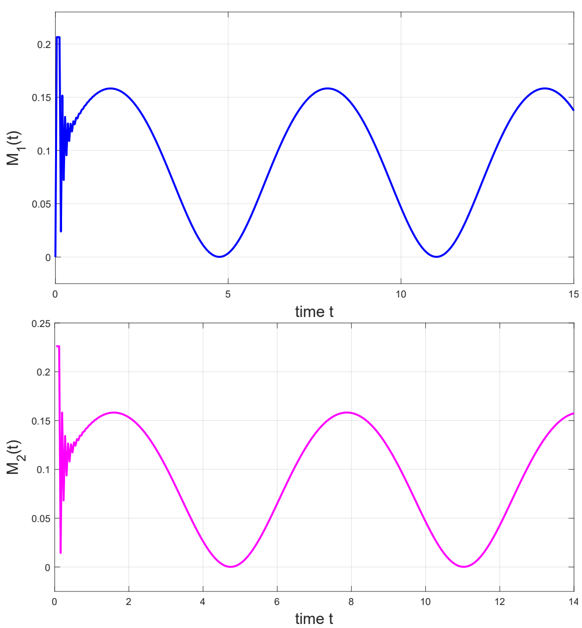

Fig. 1 State variables $(M_1, M_2)^\top$ of system (35).

Furthermore, it has $m=1.3$, $M=1$, $\dot{\eta}_j^+ = 0.2$, $\frac{M}{1-\dot{\eta}_j^+} = 1.25 < m = 1.3$, which implies that (H_4) is fulfilled in Theorem 2. Therefore, on the basis of Theorem 2, system (35) is globally asymptotically stable, see Fig. 3–Fig. 4.

Fig. 1 and Fig. 2 paint the S-asymptotically 2π-periodic trajectories of state variables $(M_1, M_2)^\top$ and $(P_1, P_2)^\top$ in system (35), respectively. In contrast to the periodic solutions, the asymptotically periodic solutions oscillate irregularly at the beginning and show periodicity after a period of time. Fig. 3 and Fig. 4 portray the trajectories of state variables $(M_1, M_2)^\top$ and $(P_1, P_2)^\top$ in system (35) with different initial values, respectively, and demonstrate the trajectories of state variables in system (35) with different initial values will tend to the same trajectory for some time, i.e., system (35) is globally asymptotically stable.

Example 2 Considering the asymptotically periodic

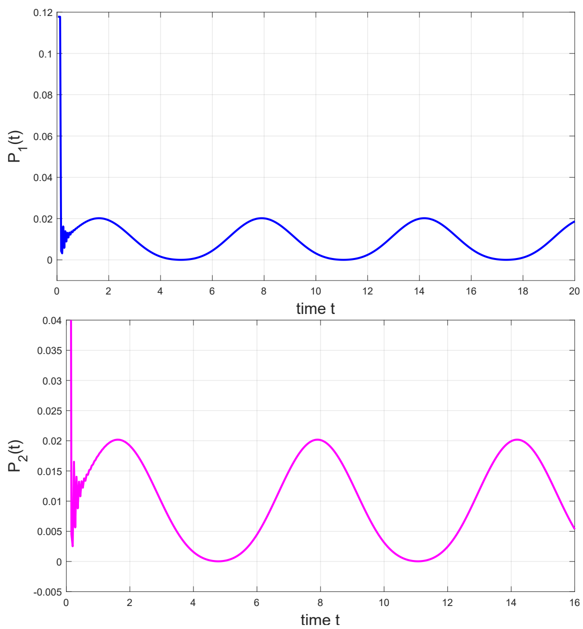

Fig. 2 State variables $(P_1, P_2)^\top$ of system (35).

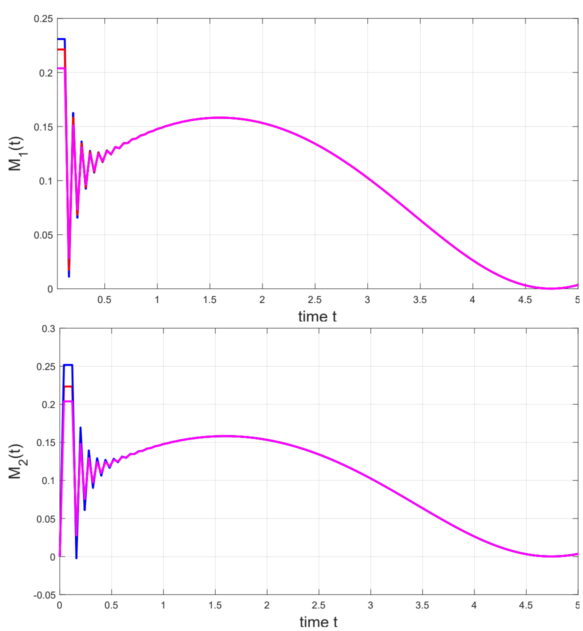

Fig. 3 Stability of state variables $(M_1, M_2)^\top$ of system (35).

FGRNs below:

$$\begin{cases} {}_0^C D_t^{\frac{1}{5}} M_i(t) = -a_i(t) M_i(t) \\ \qquad + \sum_{j=1}^{n} w_{ij}(t) f_j(P_j(t - \sigma_j(t))) + B_i(t), \\ {}_0^C D_t^{\frac{1}{5}} P_i(t) = -c_i(t) P_i(t) + d_i M_i(t - \tau_i(t)),\ t > 0, \end{cases} \tag{36}$$

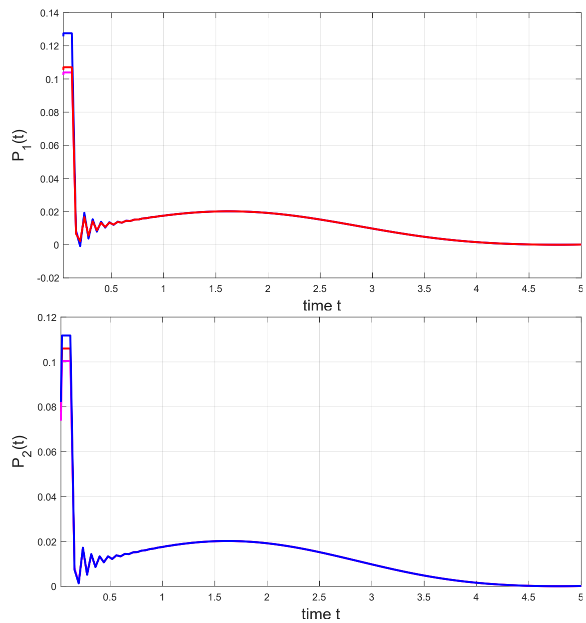

Fig. 4 Stability of state variables $(P_1, P_2)^\top$ of system (35).

where

$$a_i(t) = \frac{t}{1+t}(4.5 + 1.5\sin t), \quad c_i(t) = \frac{1.3t}{1+t},$$
$$d_i(t) = \frac{0.1t}{1+t}\left(\frac{2}{3} + \frac{1}{3}\sin t\right), \quad f_j(P_j) = \frac{P_j^2}{1+P_j^2},$$
$$L_j^f = 0.1, \quad B_i = 1 + \cos t, \quad \tau_i(t) = 1 + 0.2\sin t,$$
$$\sigma_j(t) = 1 + 0.2\cos t, \quad i, j = 1, 2,$$

$$w_{ij}(t) = \begin{pmatrix} w_{11}(t) & w_{12}(t) \\ w_{21}(t) & w_{22}(t) \end{pmatrix} = \begin{pmatrix} 0 & -1-\cos t \\ -1-\cos t & 0 \end{pmatrix}.$$

Clearly,

$$\sum_{j=1}^{2} L_j^f \bar{w}_{ij} = \sum_{j=1}^{2} (0.1 * 2 + 0.1 * 2) = 0.4 < 3 = \underline{a}_i,$$
$$\bar{d}_i = 0.1 < 1.3 = \underline{c}_i, \quad i = 1, 2.$$

Hence, (H_1)–(H_3) in Theorem 1 hold. From Theorem 1, system (36) has a unique S-asymptotically periodic oscillation, see Fig. 5–Fig. 6.

Besides, $m = 1.3$, $M = 0.1$, $\dot{\eta}^+ = 0.2$, $\frac{M}{1-\dot{\eta}^+} = 0.125 < m = 1.3$. So (H_4) in Theorem 2 holds. By Theorem 2, system (36) is globally asymptotically stable, see Fig. 7–Fig. 8.

System (36) learns a class of FGRNs with asymptotically periodic coefficients. Fig. 5–Fig. 6 respectively plot the S-asymptotically periodic trajectories of state variables $(M_1, M_2)^\top$ and $(P_1, P_2)^\top$ in system (36) and Fig. 7–Fig. 8 respectively describe the trajectories of state variables $(M_1, M_2)^\top$ and $(P_1, P_2)^\top$ in system

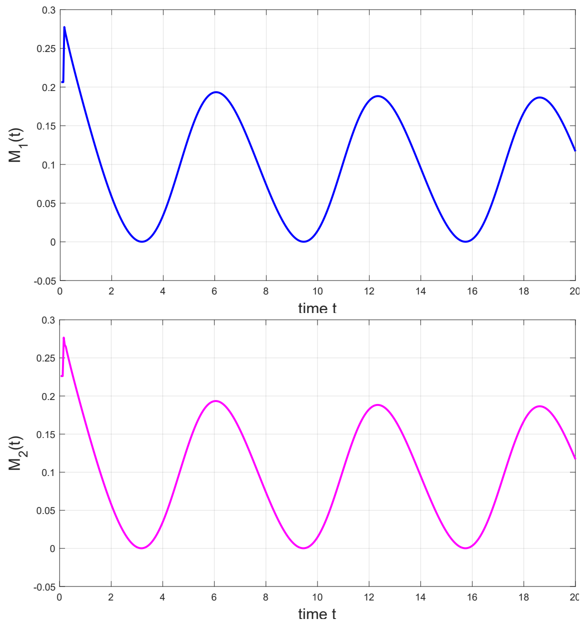

Fig. 5 State variables $(M_1, M_2)^\top$ of system (36).

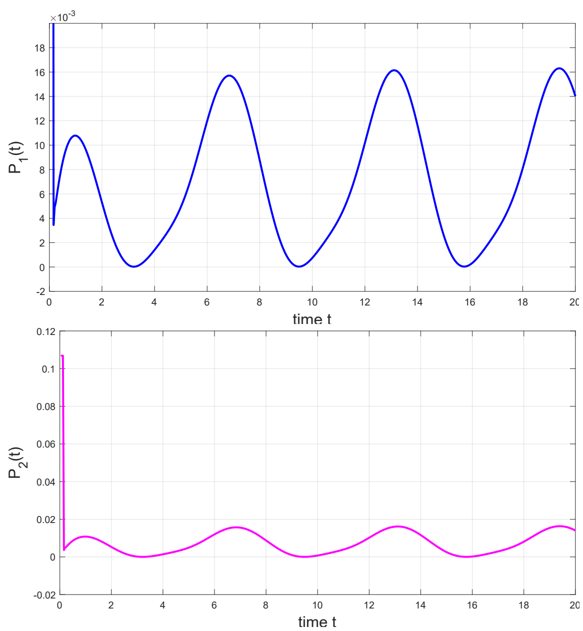

Fig. 6 State variables $(P_1, P_2)^\top$ of system (36).

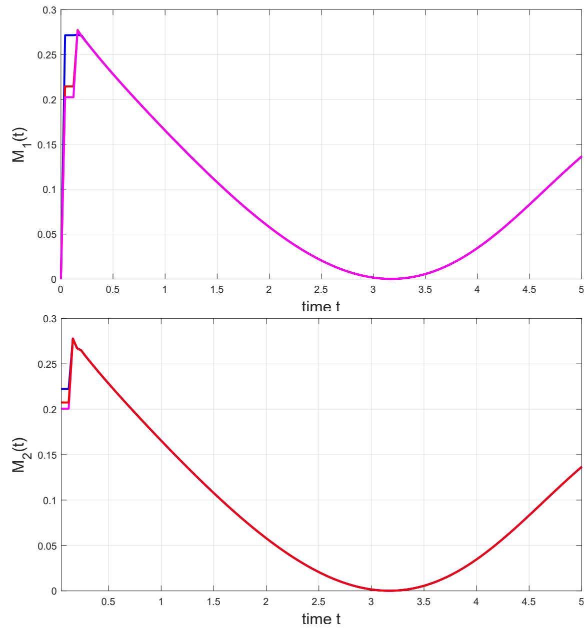

Fig. 7 Stability of state variables $(M_1, M_2)^\top$ of system (36).

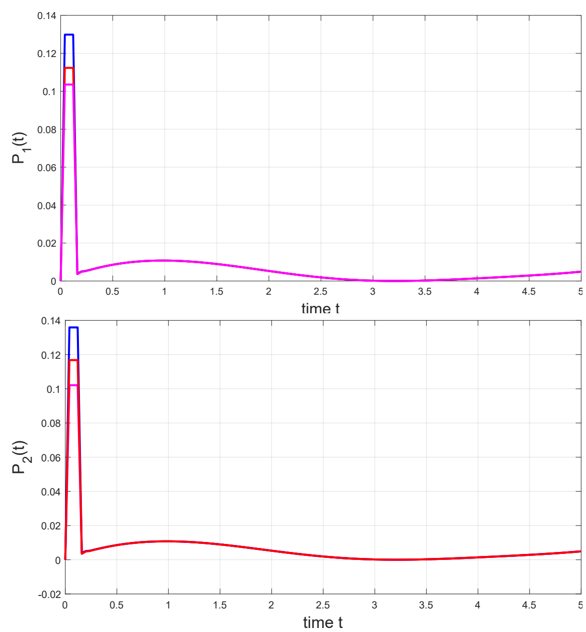

Fig. 8 Stability of state variables $(P_1, P_2)^\top$ of system (36).

(36) with different initial values. Compared to system (35), the influence of asymptotically periodic coefficients on state variables is significant, especially in the initial time period.

CONCLUSION

In this paper, a class of FGRNs with time-varying lags has been addressed. And several results are derived for FGRNs as follows. (i) By using the contraction mapping theorem, the existence and uniqueness of S-asymptotically ω-periodic oscillation is achieved for FGRNs with time variable lags. (ii) Based on comparison theorem and stability criteria of fractional-order differential equations with delays, we obtain the global asymptotical stability of S-asymptotically ω-periodic oscillations for FGRNs. In light of (H_4), we can see that the global asymptotical stability of FGRNs (3) depends on time variable lags $\tau_i(t)$ and $\sigma_j(t)(i,j=1,2,\dots,n)$, but has no relation with fractional order α.

There are a few issues need to be considered in the further, which are listed as below.

(1) Whether the work of this paper can be extended to FGRNs with order $\alpha > 1$?

(2) It is worthy to focus on other GRNs, such as Boolean network model [1], Petri networks [2], linear combination model [3] and Bayesian network model [4], etc.

(3) The techniques in this paper can be applied to study other classical GRNs, for example, stochastic GRNs [6], impulsive GRNs [30] and cyclic GRNs [36], etc.

(4) The research ideas in this article can be used to investigate other types of dynamics, e.g., almost periodic dynamics [37, 38], control [39].

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