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A complementary Hyers-Ulam stability of alternative equation of Jensen type

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ABSTRACT: We studied the stability of the alternative functional equation

$$\|f(x-y)-2f(x)+f(x+y)\|\,\|f(x-y)+f(x+y)\|=0 \tag{*}$$

for all $x,y\in G$, where $f:G\to B$ is a function from commutative group G to a real Banach space B, and found that if

$$\|f(x-y)-2f(x)+f(x+y)\|\leqslant\delta \quad \text{or} \quad \|f(x-y)+f(x+y)\|\leqslant\delta$$

for each $x,y\in G$, then there exists a solution $f^*:G\to B$ of (*) such that $\|f(x)-f^*(x)\|\leqslant 12\delta$. The general solution of (*) was also achieved.

KEYWORDS: alternative equation, stability, Jensen's functional equation

MSC2020: 39B82 39B52

INTRODUCTION

The topic of alternative functional equations has been studied quite widely [1–4]. One of the alternative equations recently studied is the equation of Jensen type

$$\begin{aligned} f(x-y)-2f(x)+f(x+y)=0 \quad \text{or} \\ \alpha f(x-y)+\beta f(x)+\gamma f(x+y)=0 \end{aligned} \tag{1}$$

for some fixed real numbers α,β,γ. This is an alternative equation of a form of Jensen's equation

$$f(x-y)-2f(x)+f(x+y)=0. \tag{2}$$

It is well known that solutions of (2) are of the form $A(x)+f(e)$, where A is an additive function.

Srisawat [5] has studied Hyers-Ulam stability of (1) in almost every cases of α,β,γ. The only cases remained to be investigated are:

1. When $\beta=0$ and $\alpha=\gamma\neq 0$.
2. When $\alpha=\gamma=\beta\neq 0$.
3. When $\beta=\alpha+\gamma$.

The patterns of solutions for these cases were given in [6]. Though those are general solutions only on cyclic groups, they give us enough light to work with stability problem.

This article will deal with the case where $\beta=0$ and $\alpha,\gamma\neq 0$. For simplicity, we assume that $\alpha=\gamma=1$. So we will study Hyers-Ulam stability of

$$\begin{aligned} f(x-y)-2f(x)+f(x+y)=0 \quad \text{or} \\ f(x-y)+f(x+y)=0, \end{aligned} \tag{3}$$

that is, we will study the inequality

$$\begin{aligned} \|f(x-y)-2f(x)+f(x+y)\|\leqslant\delta \quad \text{or} \\ \|f(x-y)+f(x+y)\|\leqslant\delta \end{aligned} \tag{4}$$

for all $x,y\in G$.

FRAMEWORK

From this point onwards, let $(G,+)$ be a commutative group, and let B be a real Banach space. We denote the set of all positive integers and the set of all integers by $\mathbb{N}$ and $\mathbb{Z}$, respectively. For conciseness, we would also like to devise some special notations.

For each $f:G\to B$ and for any $x,y\in G$, denote

$$\mathscr{F}_y^{(\alpha)}(x):=f(x-y)-\alpha f(x)+f(x+y).$$

We will use linear combinations of these forms frequently. For each $x,y\in G$, denote the statement

$$\mathscr{P}_y^{(\alpha)}(x):=\|\mathscr{F}_y^{(\alpha)}(x)\|\leqslant\delta.$$

For convenience of our approach, we define two more statements for each $x,y\in G$:

$$\begin{aligned} \mathscr{L}(x,y):=&\,\mathscr{P}_y^{(2)}(x-2y),\ \mathscr{P}_y^{(0)}(x-y),\ \mathscr{P}_y^{(0)}(x),\\ &\mathscr{P}_y^{(2)}(x+y),\ \mathscr{P}_{2y}^{(0)}(x),\ \text{and } \mathscr{P}_{2y}^{(0)}(x-y)\\ \mathscr{R}(x,y):=&\,\mathscr{P}_y^{(2)}(x+2y),\ \mathscr{P}_y^{(0)}(x+y),\ \mathscr{P}_y^{(0)}(x),\\ &\mathscr{P}_y^{(2)}(x-y),\ \mathscr{P}_{2y}^{(0)}(x),\ \text{and } \mathscr{P}_{2y}^{(0)}(x+y). \end{aligned}$$

As to the usage of these notations, we will prove that one of these will be the pattern of alternatives whenever $\|f(x)\|$ is large enough and $\mathscr{P}_y^{(0)}(x)$.

The following definition will be used to explain the solutions of (3).

Definition 1 For a commutative group G, we call a nonempty set $H \subseteq G$ a *G-convex* set when given any $x \in H$ and $y \in G$, if there exists a positive integer k such that $x+ky \in H$, then $x+y \in H$.

MAIN RESULTS

It is well known that a function $f : G \to B$ satisfies (2) if and only if $A(x) := f(x) - f(e)$ is additive. The other solutions of (3) are given in the following theorem.

Theorem 1 *Let $f : G \to B$ satisfy (3). Suppose that f does not satisfy (2) for some $x, y \in G$. Then $f(G) = \{a, -a\}$ for some $a \in B \backslash \{0\}$. Furthermore, the sets $f^{-1}\{a\}$ and $f^{-1}\{-a\}$ are both G-convex.*

Conversely, if there exist G-convex sets H_1 and H_2 such that $H_1 \cup H_2 = G$ and $f(x) = -f(y) \neq 0$ for any $x \in H_1$ and $y \in H_2$, then f satisfies (3) and not (2).

Theorem 1 can be obtained from Theorem 3 when we let $\delta = 0$, with only little work on $a \neq 0$. Note that the solution in the case when the range of f is a uniquely divisible commutative group can be proved analogously.

Due to the fact that (3) having two types of solutions, we need a criterion to distinguish between the types of approximate solutions when dealing with the stability problem. For the solution problem, the criterion "There exist $x, y \in G$ such that $\mathscr{F}_y^{(2)}(x) \neq 0$" would be suffice to imply a non-Jensen solution. Hence, it can be expected that the criteria "There exist $x, y \in G$ such that $\|\mathscr{F}_y^{(2)}(x)\|$ is large enough" would imply an approximate non-Jensen solution. This will be shown to be true.

Before proceeding to the stability of (3), we will prove several propositions and lemmas to reveal some patterns of this problem.

Proposition 1 *Let $f : G \to B$ satisfy (4), let $a, b_1, b_2 \in G$, and let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that $\|\mathscr{F}_{b_1}^{(2)}(a)\| \leqslant \alpha_1$, $\|\mathscr{F}_{b_1+b_2}^{(2)}(a)\| \leqslant \alpha_2$, and $\|\mathscr{F}_{b_1-b_2}^{(2)}(a)\| \leqslant \alpha_3$. Suppose that $\|f(a)\| > \frac{\alpha_2+\alpha_3+2\delta}{4}$. Then $\|\mathscr{F}_{b_2}^{(2)}(a+b_1)\| \leqslant 2\alpha_1 + \alpha_2 + \alpha_3 + \delta$.*

Proof: **Case 1:** $\mathscr{P}_{b_2}^{(0)}(a-b_1)$. Then

$$\begin{aligned}
&\|\mathscr{F}_{b_2}^{(0)}(a+b_1)\| \\
&\quad = \|4f(a) + \mathscr{F}_{b_1+b_2}^{(2)}(a) + \mathscr{F}_{b_1-b_2}^{(2)}(a) - \mathscr{F}_{b_2}^{(0)}(a-b_1)\| \\
&\quad > (\alpha_2+\alpha_3+2\delta) - (\alpha_2+\alpha_3+\delta) = \delta.
\end{aligned}$$

So $\|\mathscr{F}_{b_2}^{(2)}(a+b_1)\| \leqslant \delta$.

Case 2: $\mathscr{P}_{b_2}^{(2)}(a-b_1)$. Then

$$\begin{aligned}
&\|\mathscr{F}_{b_2}^{(2)}(a+b_1)\| \\
&= \|\mathscr{F}_{b_1+b_2}^{(2)}(a) - 2\mathscr{F}_{b_1}^{(2)}(a) + \mathscr{F}_{b_1-b_2}^{(2)}(a) - \mathscr{F}_{b_2}^{(2)}(a-b_1)\| \\
&\leqslant \alpha_2 + 2\alpha_1 + \alpha_3 + \delta.
\end{aligned}$$

□

Proposition 2 *Let $\alpha \geqslant \delta \geqslant 0$ and $f : G \to B$ satisfy (4). Suppose that $x_0, y_0 \in G$ such that $\|\mathscr{F}_{y_0}^{(2)}(x_0)\| > \alpha$. Then $\|f(x_0)\| > \frac{\alpha-\delta}{2}$.*

Proof: Since $\|\mathscr{F}_{y_0}^{(2)}(x_0)\| > \alpha \geqslant \delta$, we have $\|\mathscr{F}_{y_0}^{(0)}(x_0)\| \leqslant \delta$. Then

$$\|2f(x_0)\| \geqslant \|\mathscr{F}_{y_0}^{(2)}(x_0)\| - \|\mathscr{F}_{y_0}^{(0)}(x_0)\| > \alpha - \delta.$$

□

Proposition 3 *Let $\delta \geqslant 0$ and $f : G \to B$ satisfy (4). Suppose that $x_0, y_0, z_0 \in G$, $\alpha_1, \alpha_2 \in [0, \infty)$ and a real number $\beta \neq 0$ such that*

$$\begin{aligned}
\|f(x_0)\| > \frac{\delta+\alpha_1+\alpha_2}{2|\beta|}, \quad &\|f(z_0+y_0) - \beta f(x_0)\| \leqslant \alpha_1, \\
\text{and} \quad &\|f(z_0-y_0) - \beta f(x_0)\| \leqslant \alpha_2.
\end{aligned}$$

Then $\|\mathscr{F}_{y_0}^{(0)}(z_0)\| > \delta$ (and hence $\|\mathscr{F}_{y_0}^{(2)}(z_0)\| \leqslant \delta$).

Proof: Assume all the assumptions. Then

$$\begin{aligned}
\|\mathscr{F}_{y_0}^{(0)}(z_0)\| &= \|2\beta f(x_0) + (f(z_0+y_0) - \beta f(x_0)) \\
&\qquad + (f(z_0-y_0) - \beta f(x_0))\| \\
&\geqslant \|2\beta f(x_0)\| - (\|f(z_0+y_0) - \beta f(x_0)\| \\
&\qquad + \|f(z_0-y_0) - \beta f(x_0)\|) > \delta.
\end{aligned}$$

□

Due to the nature of these alternative equations, we need to deal with any $x, y \in G$ such that both $\|\mathscr{F}_y^{(2)}(x)\|$ and $\|\mathscr{F}_y^{(0)}(x)\|$ are bounded. For such a case, we will think of $\|f(x)\|$ as "approximately zero". Since the values of f at various points in G will later be shown to be related to each other, we will be able to exclude all such cases over the entirety of G as long as $\|\mathscr{F}_{y_0}^{(2)}(x_0)\|$ is large enough (which implies that $\|f(x_0)\|$ is proportionally large) for some $x_0, y_0 \in G$.

Lemma 1 *Let $\delta \geqslant 0$ and $f : G \to B$ satisfy (4) for all $x, y \in G$. Let $a, b \in G$. Then at least one of the following holds.*

(i) $\left\|\mathscr{F}_b^{(2)}(a)\right\| \leqslant 8\delta$,

(ii) $\mathscr{L}(a, b)$,

(iii) $\mathscr{R}(a, b)$.

Proof: Suppose that (ii) and (iii) are not true. We will only consider the case where $\mathscr{P}_b^{(0)}(a)$.

Case 1: $\mathscr{P}_b^{(0)}(a-b)$ and $\mathscr{P}_b^{(2)}(a+b)$. Consider the following equations.

$$\left\|\mathscr{F}_b^{(2)}(a)\right\| = \left\|\frac{1}{3}\mathscr{F}_b^{(2)}(a-2b)+0\mathscr{F}_b^{(0)}(a-b)+0\mathscr{F}_b^{(0)}(a) - \frac{2}{3}\mathscr{F}_b^{(2)}(a+b)-\frac{1}{3}\mathscr{F}_{2b}^{(2)}(a-b)+\frac{2}{3}\mathscr{F}_{2b}^{(2)}(a)\right\|$$

$$\left\|\mathscr{F}_b^{(2)}(a)\right\| = \left\|1\mathscr{F}_b^{(2)}(a-2b)+0\mathscr{F}_b^{(0)}(a-b)-2\mathscr{F}_b^{(0)}(a) -2\mathscr{F}_b^{(2)}(a+b)-1\mathscr{F}_{2b}^{(2)}(a-b)+2\mathscr{F}_{2b}^{(0)}(a)\right\|$$

$$\left\|\mathscr{F}_b^{(2)}(a)\right\| = \left\|1\mathscr{F}_b^{(2)}(a-2b)+1\mathscr{F}_b^{(0)}(a-b)+0\mathscr{F}_b^{(0)}(a) -\mathscr{F}_b^{(2)}(a+b)-\mathscr{F}_{2b}^{(0)}(a-b)+\mathscr{F}_{2b}^{(2)}(a)\right\|$$

$$\left\|\mathscr{F}_b^{(2)}(a)\right\| = \left\|\frac{1}{4}\mathscr{F}_b^{(0)}(a-2b)-\frac{1}{2}\mathscr{F}_b^{(0)}(a-b)+\frac{1}{4}\mathscr{F}_b^{(0)}(a) -\frac{1}{2}\mathscr{F}_b^{(2)}(a+b)-\frac{1}{4}\mathscr{F}_{2b}^{(2)}(a-b)+\frac{1}{2}\mathscr{F}_{2b}^{(2)}(a)\right\|$$

$$\left\|\mathscr{F}_b^{(2)}(a)\right\| = \left\|\frac{1}{2}\mathscr{F}_b^{(0)}(a-2b)-\frac{1}{2}\mathscr{F}_b^{(0)}(a-b)+\frac{1}{2}\mathscr{F}_b^{(0)}(a) -\frac{1}{2}\mathscr{F}_b^{(2)}(a+b)-\frac{1}{2}\mathscr{F}_{2b}^{(2)}(a-b)+\frac{1}{2}\mathscr{F}_{2b}^{(2)}(a)\right\|$$

$$\left\|\mathscr{F}_b^{(2)}(a)\right\| = \left\|\frac{1}{2}\mathscr{F}_b^{(0)}(a-2b)-1\mathscr{F}_b^{(0)}(a-b)-\frac{1}{2}\mathscr{F}_b^{(0)}(a) -\mathscr{F}_b^{(2)}(a+b)-\frac{1}{2}\mathscr{F}_{2b}^{(2)}(a-b)+1\mathscr{F}_{2b}^{(0)}(a)\right\|$$

$$\left\|\mathscr{F}_b^{(2)}(a)\right\| = \left\|1\mathscr{F}_b^{(0)}(a-2b)-1\mathscr{F}_b^{(0)}(a-b)+0\mathscr{F}_b^{(0)}(a) -\mathscr{F}_b^{(2)}(a+b)-\mathscr{F}_{2b}^{(0)}(a-b)+1\mathscr{F}_{2b}^{(0)}(a)\right\|.$$

Substitute (x,y) in (4) by $(a-2b,b),(a-b,2b)$, and $(a,2b)$. Whatever the alternatives for those cases are (our assumption here exclude the case where all these alternatives satisfy (ii)), together with $\mathscr{P}_b^{(0)}(a-b)$, $\mathscr{P}_b^{(0)}(a)$, and $\mathscr{P}_b^{(2)}(a+b)$, at least one of the above equations can be used to bound $\|\mathscr{F}_b^{(2)}(a)\|$ with the triangle inequality. In any case, we have

$$\|\mathscr{F}_b^{(2)}(a)\| \leqslant 8\delta.$$

Case 2: $\mathscr{P}_b^{(2)}(a-b)$ and $\mathscr{P}_b^{(0)}(a+b)$. This case is analogous to Case 1 and gives the same result.

Case 3: Other alternatives for $(x,y) \in \{(a-b,b),(a+b,b)\}$. Then we can use one of the following.

$$\left\|\mathscr{F}_b^{(2)}(a)\right\| = \left\|-\frac{1}{2}\mathscr{F}_b^{(2)}(a-b)+0\mathscr{F}_b^{(0)}(a)-\frac{1}{2}\mathscr{F}_b^{(2)}(a+b)+\frac{1}{2}\mathscr{F}_{2b}^{(2)}(a)\right\|$$

$$\left\|\mathscr{F}_b^{(2)}(a)\right\| = \left\|-\mathscr{F}_b^{(2)}(a-b)-\mathscr{F}_b^{(0)}(a)-\mathscr{F}_b^{(2)}(a+b)+\mathscr{F}_{2b}^{(0)}(a)\right\|$$

$$\left\|\mathscr{F}_b^{(2)}(a)\right\| = \left\|-\frac{1}{2}\mathscr{F}_b^{(0)}(a-b)+\mathscr{F}_b^{(0)}(a)-\frac{1}{2}\mathscr{F}_b^{(0)}(a+b)+\frac{1}{2}\mathscr{F}_{2b}^{(2)}(a)\right\|$$

$$\left\|\mathscr{F}_b^{(2)}(a)\right\| = \left\|-\mathscr{F}_b^{(0)}(a-b)+\mathscr{F}_b^{(0)}(a)-\mathscr{F}_b^{(0)}(a+b)+\mathscr{F}_{2b}^{(0)}(a)\right\|.$$

Hence

$$\|\mathscr{F}_b^{(2)}(a)\| \leqslant 4\delta.$$

In conclusion, we have $\|\mathscr{F}_b^{(2)}(a)\| \leqslant 8\delta$. □

Now we will investigate the patterns of the alternatives.

Lemma 2 *Let* $\delta \geqslant 0$ *and* $f: G \to B$ *satisfy* (4). *If* $a,b \in G$ *satisfy* $\mathscr{L}(a,b)$*, then*

$$\|f(a-2b)+f(a)\| \leqslant \delta,$$
$$\|f(a-b)+f(a)\| \leqslant \frac{5}{2}\delta,$$
$$\|f(a+b)-f(a)\| \leqslant \frac{3}{2}\delta,$$
$$\textit{and}\quad \|f(a+2b)-f(a)\| \leqslant 2\delta.$$

Proof: The result follows from

$$\|f(a-2b)+f(a)\| = \|\mathscr{F}_b^{(0)}(a-b)\| \leqslant \delta,$$

$$\|f(a+2b)-f(a)\| = \|\mathscr{F}_{2y}^{(0)}(a)-(f(a-2b)+f(a))\| \leqslant 2\delta,$$

$$\|f(a+b)-f(a)\| = \frac{1}{2}\|(f(a)-f(a+2b))+\mathscr{F}_b^{(2)}(a+b)\| \leqslant \frac{3}{2}\delta,$$

$$\|f(a-b)+f(a)\| = \|\mathscr{F}_b^{(0)}(a)-(f(a+b)-f(a))\| \leqslant \frac{5}{2}\delta.$$

□

Lemma 3 follows from Lemma 2 and the fact that $\mathscr{R}(x,y)$ is the same as $\mathscr{L}(x,-y)$.

Lemma 3 *Let* $\delta \geqslant 0$ *and* $f: G \to B$ *satisfy* (4). *If* $a,b \in G$ *such that* $\mathscr{R}(a,b)$*, then*

$$\|f(a+2b)+f(a)\| \leqslant \delta, \tag{5}$$
$$\|f(a+b)+f(a)\| \leqslant \frac{5}{2}\delta, \tag{6}$$
$$\|f(a-b)-f(a)\| \leqslant \frac{3}{2}\delta, \tag{7}$$
$$\textit{and}\quad \|f(a-2b)-f(a)\| \leqslant 2\delta. \tag{8}$$

Now we will establish a relation between points in parts of G. We start with the point $x_0 \in G$ which exists $y_0 \in G$ such that $\|\mathscr{F}^{(2)}_{y_0}(x_0)\|$ is large enough. Such point x_0 will be considered as the central point of our analysis.

Lemma 4 *Let $\delta \geqslant 0$ and $f : G \to B$ satisfy (4). Let $x_0, y_0 \in G$ such that $\|\mathscr{F}^{(2)}_{y_0}(x_0)\| > \delta$. Then at least one of the following holds.*
(i) $\|2f(x_0)\| \leqslant 9\delta$.
(ii) $\mathscr{L}(x_0, 2^n y_0)$ *for all* $n \in \mathbb{N} \cup \{0\}$.
(iii) $\mathscr{R}(x_0, 2^n y_0)$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Proof: According to Lemma 1, we consider three cases.

Case 1: $\|\mathscr{F}^{(2)}_{y_0}(x_0)\| \leqslant 8\delta$. Then

$$\|2f(x_0)\| = \|\mathscr{F}^{(0)}_{y_0}(x_0) - \mathscr{F}^{(2)}_{y_0}(x_0)\| \leqslant 9\delta.$$

Case 2: $\mathscr{L}(x_0, y_0)$. Suppose that the result (ii) of this lemma does not hold and let k be the smallest positive integer such that $\mathscr{L}(x_0, 2^k y_0)$ does not hold. We apply Lemma 1 with (a, b) replaced by $(x_0, 2^k y_0)$.

Case 2.1: $\|\mathscr{F}^{(2)}_{2^k y_0}(x_0)\| \leqslant 8\delta$. Since $\mathscr{L}(x_0, 2^{k-1} y_0)$, we have $\mathscr{P}^{(0)}_{2^k y_0}(x_0)$. So

$$\|2f(x_0)\| = \|\mathscr{F}^{(0)}_{2^k y_0}(x_0) - \mathscr{F}^{(2)}_{2^k y_0}(x_0)\| \leqslant \delta + 8\delta = 9\delta.$$

Case 2.2: $\mathscr{R}(x_0, 2^k y_0)$. Since $\mathscr{L}(x_0, 2^{k-1} y_0)$, according to Lemma 2 and Lemma 3, we have

$$\begin{aligned}
&\|2f(x_0)\| \\
&\quad = \left\| \left(f(x_0 - 2^k y_0) + f(x_0)\right) - \left(f(x_0 - 2^k y_0) - f(x_0)\right) \right\| \\
&\quad \leqslant \delta + \frac{3}{2} = \frac{5}{2}\delta.
\end{aligned}$$

So, for Case 2, either (i) or (ii) holds.

Case 3: $\mathscr{R}(x_0, y_0)$. This case is analogous to Case 2. So either (i) or (iii) holds. □

Lemma 5 *Let $f : G \to B$ satisfy (4). Suppose that $x_0, y_0 \in G$ such that $\|2f(x_0)\| > 4\delta$ and $\mathscr{L}(x_0, 2^n y_0)$ for all $n \in \mathbb{N} \cup \{0\}$. Then*

$$\|f(x_0 + n y_0) - f(x_0)\| \leqslant \begin{cases} \delta; & n = 2^k \text{ for some } k \in \mathbb{N} \cup \{0\}, \\ 2\delta; & n \text{ is one of other positive integers} \end{cases}$$

and

$$\|f(x_0 - n y_0) + f(x_0)\| \leqslant \begin{cases} \delta; & n = 2^k \text{ for some } k \in \mathbb{N}, \\ 2\delta; & n \text{ is one of other positive integers.} \end{cases}$$

Proof: With Lemma 2, we have

$$\|f(x_0 - 2^k y_0) + f(x_0)\| \leqslant \delta \text{ for all } k > 0 \quad (9)$$

$$\text{and} \quad \|f(x_0 + 2^k y_0) - f(x_0)\| \leqslant \frac{3}{2}\delta \text{ for all } k \geqslant 0.$$

Observe that, for each nonnegative integer k,

$$\begin{aligned}
&\left\|f(x_0 + 2^k y_0) - f(x_0)\right\| \\
&= \left\| \frac{1}{2}\left(f(x_0 + 2^{k+1} y_0) - f(x_0)\right) - \frac{1}{2}\mathscr{F}^{(2)}_{2^k y_0}(x_0 + 2^k y_0) \right\| \\
&= \Bigg\| \frac{1}{4}\left(f(x_0 + 2^{k+2} y_0) - f(x_0)\right) - \frac{1}{4}\mathscr{F}^{(2)}_{2^{k+1} y_0}(x_0 + 2^{k+1} y_0) \\
&\quad - \frac{1}{2}\mathscr{F}^{(2)}_{2^k y_0}(x_0 + 2^k y_0) \Bigg\| \\
&\ \ \vdots \\
&= \left\| \frac{1}{2^n}\left(f(x_0 + 2^{k+n} y_0) - f(x_0)\right) - \sum_{i=0}^{n-1} \frac{1}{2^{i+1}} \mathscr{F}^{(2)}_{2^{k+i}}(x_0 + 2^{k+i} y_0) \right\| \\
&\leqslant \frac{3}{2^{n+1}}\delta + \sum_{i=0}^{n-1} \frac{1}{2^{i+1}}\delta
\end{aligned}$$

for any $n \in \mathbb{Z}$. So

$$\|f(x_0 + 2^k y_0) - f(x_0)\| \leqslant \lim_{n\to\infty} \left(\frac{3}{2^{n+1}}\delta + \sum_{i=0}^{n-1} \frac{1}{2^{i+1}}\delta \right) = \delta \quad (10)$$

for every $k \in \mathbb{N} \cup \{0\}$. We also have

$$\begin{aligned}
&\|f(x_0 - y_0) + f(x_0)\| \\
&\quad = \|\mathscr{F}^{(0)}_{y_0}(x_0) - (f(x_0 + y_0) - f(x_0))\| \leqslant 2\delta. \quad (11)
\end{aligned}$$

Let $u : \mathbb{N} \to \mathbb{N}$ be defined by, $u(n) :=$ the number of nonzero digits in the binary representation of n. We will show that

$$\|f(x_0 + n y_0) - f(x_0)\| \leqslant 2\delta \quad (12)$$

for any positive integer n by induction on the value of $u(n)$.

For $u(n) = 1$, the result follows from (10). Suppose that $u(m) > 1$ and (12) is true whenever $u(n) < u(m)$. Let k be the largest positive integer such that $2^k \leqslant m$. Then $u(2m - 2^{k+1}) = u(m - 2^k) = u(m) - 1$.

With the fact that $\|f\left(x_0 + (2m - 2^{k+1}) y_0\right) - f(x_0)\| \leqslant 2\delta$, $\|f(x_0 + 2^{k+1} y_0) - f(x_0)\| \leqslant \delta$, and $\|2f(x_0)\| > 4\delta$. Proposition 3 implies that $\|\mathscr{F}^{(2)}_{(2^{k+1}-m)y_0}(x_0 + m y_0)\| \leqslant \delta$. So

$$\begin{aligned}
&\|2f(x_0 + m y_0) - 2f(x_0)\| \\
&\qquad \leqslant \left\|f(x_0 + (2m - 2^{k+1}) y_0) - f(x_0)\right\| \\
&\qquad\quad + \left\|f(x_0 + 2^{k+1} y_0) - f(x_0)\right\| \\
&\qquad\quad + \left\|\mathscr{F}^{(2)}_{(2^{k+1}-m)y_0}(x_0 + m y_0)\right\| \\
&\qquad \leqslant 2\delta + \delta + \delta.
\end{aligned}$$

So $\|f(x_0 + n y_0) - f(x_0)\| \leqslant 2\delta$ for all $n \in \mathbb{N}$.

In an analogous manner, we will show that

$$\|f(x_0 - n y_0) + f(x_0)\| \leqslant 2\delta \quad (13)$$

for every positive integer n.

The case $u(n)=1$ has already been done ((9) and (11)). Let a positive integer m be such that $u(m)>1$ and (13) is true whenever $u(n)<u(m)$. Let k be the largest positive integer such that $2^k \leqslant m$.

Since $u(2m-2^{k+1})=u(m-2^k)=u(m)-1$, we have

$$\|f(x_0-(2m-2^{k+1})y_0)+f(x_0)\| \leqslant 2\delta.$$

This, together with $\|f(x_0-2^{k+1}y_0)+f(x_0)\| \leqslant \delta$, $\|2f(x_0)\|>4\delta$, and Proposition 3, we got $\|\mathscr{F}^{(2)}_{(2^{k+1}-m)y_0}(x_0-my_0)\| \leqslant \delta$. So

$$\begin{aligned}\|2f(x_0-my_0)+2f(x_0)\| &\leqslant \|f(x_0-2^{k+1}y_0)+f(x_0)\| \\ &\quad+\left\|f(x_0-(2m-2^{k+1})y_0)+f(x_0)\right\| \\ &\quad+\|\mathscr{F}^{(2)}_{(2^{k+1}-m)y_0}(x_0-my_0)\| \\ &\leqslant \delta+2\delta+\delta.\end{aligned}$$

So $\|f(x_0-ny_0)+f(x_0)\| \leqslant 2\delta$ for any $n\in\mathbb{N}$. □

In Lemma 5, we obtained the pattern for stability problem on parts of G. Next, we will expand the result to entirety of G.

Lemma 6 *Let $f:G\to B$ satisfy (4) and $x_0,z_0\in G$ such that $\|2f(x_0)\|>6\delta$. Suppose that $\mathscr{P}^{(2)}_{z_0}(x_0)$ and there exists an integer $n>1$ such that $\|\mathscr{F}^{(2)}_{nz_0}(x_0)\|>\delta$. Then $\|f(x_0+z_0)-f(x_0)\| \leqslant 3\delta$.*

Proof: We can assume that n is the smallest positive integer such that $\|\mathscr{F}^{(2)}_{nz_0}(x_0)\|>\delta$. We consider the alternatives in (4) when substituting (x,y) with $(x_0+z_0,(n-1)z_0)$ and $(x_0-z_0,(n-1)z_0)$ we have the following cases.

Case 1: Both $\mathscr{P}^{(0)}_{(n-1)z_0}(x_0+z_0)$ and $\mathscr{P}^{(0)}_{(n-1)z_0}(x_0-z_0)$, or both $\mathscr{P}^{(2)}_{(n-1)z_0}(x_0+z_0)$ and $\mathscr{P}^{(2)}_{(n-1)z_0}(x_0-z_0)$. Then consider these equations.

$$\begin{aligned}\|2f(x_0)\| = \|\mathscr{F}^{(0)}_{nz_0}(x_0)-2\mathscr{F}^{(2)}_{z_0}(x_0)+\mathscr{F}^{(2)}_{(n-2)z_0}(x_0) \\ -\mathscr{F}^{(2)}_{(n-1)z_0}(x+y_0)-\mathscr{F}^{(2)}_{(n-1)z_0}(x-z_0)\|\end{aligned}$$

$$\begin{aligned}\|2f(x_0)\| = \|-\mathscr{F}^{(0)}_{nz_0}(x_0)-\mathscr{F}^{(2)}_{(n-2)z_0}(x_0) \\ +\mathscr{F}^{(0)}_{(n-1)z_0}(x+z_0)+\mathscr{F}^{(0)}_{(n-1)z_0}(x-z_0)\|.\end{aligned}$$

Whatever the alternatives are, we have $\|2f(x_0)\| \leqslant 6\delta$, a contradiction.

Case 2: $\mathscr{P}^{(2)}_{(n-1)z_0}(x_0+z_0)$ and $\mathscr{P}^{(0)}_{(n-1)z_0}(x_0-z_0)$. Then

$$\begin{aligned}\|2f(x_0)-2f(x_0+z_0)\| = \|-\mathscr{F}^{(0)}_{nz_0}(x_0)-\mathscr{F}^{(2)}_{(n-2)z_0}(x_0) \\ +\mathscr{F}^{(2)}_{(n-1)z_0}(x+z_0)+\mathscr{F}^{(0)}_{(n-1)z_0}(x-z_0)\| \leqslant 4\delta.\end{aligned}$$

Case 3: $\mathscr{P}^{(0)}_{(n-1)z_0}(x_0+z_0)$ and $\mathscr{P}^{(2)}_{(n-1)z_0}(x_0-z_0)$. Then

$$\begin{aligned}&\|2f(x_0)-2f(x_0+z_0)\| = \|\mathscr{F}^{(0)}_{nz_0}(x_0)-2\mathscr{F}^{(2)}_{z_0}(x_0) \\ &+\mathscr{F}^{(2)}_{(n-2)z_0}(x_0)-\mathscr{F}^{(0)}_{(n-1)z_0}(x+z_0)-\mathscr{F}^{(2)}_{(n-1)z_0}(x-z_0)\| \leqslant 6\delta.\end{aligned}$$

□

Lemma 7 *Let $f:G\to B$ satisfy (4), $x_0,z_0\in G$ such that $\|2f(x_0)\|>9\delta$ and $\|\mathscr{F}^{(2)}_{z_0}(x)\|>\delta$. Then at least one of the following holds.*
(i) $\|f(x_0+z_0)-f(x_0)\| \leqslant \delta$,
(ii) $\|f(x_0+z_0)+f(x_0)\| \leqslant 2\delta$.

Proof: With Lemma 4, we consider 2 cases.

Case 1: $\mathscr{L}(x_0,2^kz_0)$ for all nonnegative integers k. Then Lemma 5 yields $\|f(x_0+z_0)-f(x_0)\| \leqslant \delta$.

Case 2: $\mathscr{R}(x_0,2^kz_0)$ (which means $\mathscr{L}(x_0,2^k(-z_0)))$ for all nonnegative integers k. Then, with Lemma 5 again, we have $\|f(x_0+z_0)+f(x_0)\| \leqslant 2\delta$. □

Lemma 8 *Let $f:G\to B$ satisfy (4), $x_0,y_0,z_0\in G$ such that $\|\mathscr{F}^{(2)}_{y_0}(x_0)\|>\delta$, $\|2f(x_0)\|>9\delta$, $\mathscr{P}^{(2)}_{z_0}(x)$, and $\mathscr{P}^{(2)}_{2z_0}(x)$. Then at least one of the following holds.*
(i) $\|f(x_0+2z_0)\| \leqslant \frac{5}{2}\delta$,
(ii) $\|f(x_0+2z_0)+f(x_0)\| \leqslant 2\delta$,
(iii) $\|f(x_0+2z_0)-f(x_0)\| \leqslant 2\delta$.

Proof: With $\mathscr{P}^{(0)}_{y_0}(x_0)$, we have $\|\mathscr{F}^{(2)}_{y_0}(x_0)\| \geqslant \|-2f(x_0)+\mathscr{F}^{(0)}_{y_0}(x_0)\|>8\delta$. With Lemma 1 and, without loss of generality, we can assume that $\mathscr{L}(x_0,y_0)$. Lemma 4 and Lemma 5 yield $\|f(x_0+2y_0)-f(x_0)\| \leqslant \delta$ and $\|f(x_0-2y_0)+f(x_0)\| \leqslant \delta$.

Case 1: $\mathscr{P}^{(0)}_{z_0-y_0}(x+y_0+z_0)$. Then

$$\begin{aligned}&\|f(x_0+2z_0)+f(x_0)\| \\ &=\|\mathscr{F}^{(0)}_{z_0-y_0}(x+y_0+z_0)-(f(x_0+2y_0)-f(x_0))\| \leqslant 2\delta.\end{aligned}$$

Case 2: $\mathscr{P}^{(0)}_{z_0+y_0}(x-y_0+z_0)$. Then

$$\begin{aligned}&\|f(x_0+2z_0)-f(x_0)\| \\ &=\|\mathscr{F}^{(0)}_{z_0+y_0}(x-y_0+z_0)-(f(x_0-2y_0)+f(x_0))\| \leqslant 2\delta.\end{aligned}$$

Case 3: $\mathscr{P}^{(2)}_{z_0-y_0}(x+y_0+z_0)$ and $\mathscr{P}^{(2)}_{z_0+y_0}(x-y_0+z_0)$. Then

$$\begin{aligned}&\|2f(x_0+2z_0)-2\lambda f(x_0+z_0)\| \\ &=\|\mathscr{F}^{(2)}_{z_0-y_0}(x+y_0+z_0)+\mathscr{F}^{(2)}_{z_0+y_0}(x-y_0+z_0) \\ &\quad-\mathscr{F}^{(0)}_{2y_0}(x_0)+2\mathscr{F}^{(\lambda)}_{y_0}(x_0+z_0)\| \leqslant 5\delta\end{aligned}$$

for some $\lambda \in \{0,2\}$. If $\lambda = 0$ is applicable then $\|f(x_0 + 2z_0)\| \leqslant \frac{5}{2}\delta$. On the other hand, if $\mathscr{P}^{(2)}_{y_0}(x_0+z_0)$ then

$$\begin{aligned}\|2\mathscr{F}^{(2)}_{z_0}(x_0+z_0)\| &= \|2f(x_0)+(2f(x_0+2z_0)-4f(x_0+z_0))\| \\ &> 9\delta - 5\delta > 2\delta.\end{aligned}$$

so $\mathscr{P}^{(0)}_{z_0}(x_0+z_0)$, that is, $\|f(x_0+2z_0)+f(x_0)\| \leqslant \delta$. □

Theorem 2 *Let $f : G \to B$ satisfy (4). Also let $x_0, y_0, z_0 \in G$ such that $\|\mathscr{F}^{(2)}_{y_0}(x_0)\| > \delta$, $\|2f(x_0)\| > 9\delta$, and $\mathscr{P}^{(2)}_{kz_0}(x_0)$ for all positive integers k. Then $\|f(x_0 + kz_0) - f(x_0)\| \leqslant 5\delta$ for all integers k.*

Proof: Proposition 1 implies that $\|\mathscr{F}^{(2)}_{k_2z_0}(x_0 + k_1z_0)\| \leqslant 5\delta$ for all integers k_1, k_2. Define $G : \mathbb{Z} \to B$ by $g(k) = f(x_0 + ky_0)$ for all integers k. Then we have

$$\|g(k_1-k_2)-2g(k_1)+g(k_1+k_2)\| \leqslant 5\delta$$

for all integers k_1, k_2. By [7, Theorem 3.1], there exists $b \in B$ such that $\|f(x_0 + kz_0) - f(x_0) - kb\| \leqslant 5\delta$ for all integers k. But with our assumptions, Lemma 8 implies, for each $k \in \mathbb{N}$,

$$\begin{aligned}\|2kb\| &\leqslant 5\delta + \|f(x_0+2kz_0)-f(x_0)\| \\ &\leqslant 5\delta + \frac{5}{2}\delta + 2\|f(x_0)\|.\end{aligned}$$

Hence b is zero in B and $\|f(x_0 + kz_0) - f(x_0)\| \leqslant 5\delta$ for all k. □

Lemma 6, Lemma 7 and Theorem 2 combine into the following lemma.

Lemma 9 *Let $\delta \geqslant 0$ and $f : G \to B$ satisfy (4). Suppose that there exist $x_0, y_0 \in G$ such that $\|\mathscr{F}^{(2)}_{y_0}(x_0)\| > 10\delta$. Then, for each $z \in G$, only one of the following holds.*
(i) $\|f(x_0+z)-f(x_0)\| \leqslant 5\delta$,
(ii) $\|f(x_0+z)+f(x_0)\| \leqslant 2\delta$.

Proof: Proposition 2 implies that $\|2f(x_0)\| > 9\delta$. Let $z \in G$. Lemma 6, Lemma 7 and Theorem 2 directly imply that at least one of these results holds. Suppose that (i) is true. Then

$$\begin{aligned}\|f(x_0+z)+f(x_0)\| &= \|2f(x_0)+(f(x_0+z)-f(x_0))\| \\ &> 9\delta - 5\delta \geqslant 4\delta.\end{aligned}$$

So (i) and (ii) cannot be both true for one z. □

With Lemma 9 and the fact that $\mathscr{L}(x,-y)$ is the same statement as $\mathscr{R}(x,y)$, we are almost ready for the conclusions. The next proposition will fill the gap.

Proposition 4 *Let $\delta \geqslant 0$ and $f : G \to B$ satisfy (4). Suppose that there exist $\alpha_1, \alpha_2 \in [0,\infty)$ and $a \in B$ such that $\|a\| > \frac{\delta}{2} + \max\{\alpha_1, \alpha_2\}$ and $H_1 \cup H_2 = G$, where*

$$\begin{aligned}H_1 &= \{x \in G : \|f(x)-a\| \leqslant \alpha_1\} \\ H_2 &= \{x \in G : \|f(x)+a\| \leqslant \alpha_2\}.\end{aligned}$$

Then

- *$H_1 \cap H_2 = \varnothing$ and*
- *each of H_1 and H_2 is either empty or G-convex.*

Proof: Let $\alpha = \max\{\alpha_1, \alpha_2\}$. Firstly, if $x \in H_1 \cap H_2 \neq \varnothing$, the triangle inequality implies

$$\|a\| = \frac{1}{2}\|(f(x)+a)-(f(x)-a)\| \leqslant \alpha.$$

Next, we will prove that H_1 and H_2 are G-convex. Let $\{i, j\} = \{1,2\}$. Let $x \in H_i$, $y \in G$ and $n \in \mathbb{N}$ such that $x + ny \in H_i$. Suppose that $x + y \in H_j$. Let k be the smallest positive integer such that $x + ky \in H_i$. We will show that $\|a\| \leqslant \frac{\delta}{2} + \alpha$, contradicting our assumption.

Case 1: k is even. Then $x + \frac{k}{2}y \in H_j$ and either

$$\begin{aligned}\|4a\| = \Big\|&\mathscr{F}^{(2)}_{\frac{k}{2}y}\Big(x+\frac{k}{2}y\Big) - \big(f(x)+(-1)^i a\big) \\ &+ 2\Big(f(x+\frac{k}{2}y)+(-1)^j a\Big) \\ &- \big(f(x+ky)+(-1)^i a\big)\Big\| \leqslant \delta + 4\alpha \\ \text{or} \quad \|2a\| = \Big\|&\mathscr{F}^{(0)}_{\frac{k}{2}y}\Big(x+\frac{k}{2}y\Big) - \big(f(x)+(-1)^i a\big) \\ &- \big(f(x+ky)+(-1)^i a\big)\Big\| \leqslant \delta + 2\alpha.\end{aligned}$$

So $\|a\| \leqslant \frac{\delta}{2} + \alpha$.

Case 2: k is odd and $x + (k+1)y \in H_i$.

The fact $k > 1$ implies that $\frac{k+1}{2} < k$. So $x + \frac{k+1}{2}y \in H_j$ and we can use the same argument as Case 1 by replacing k with $k+1$.

Case 3: k is odd and $x + (k+1)y \in H_j$. Then either

$$\begin{aligned}\|4a\| &= \big\| -\mathscr{F}^{(2)}_y(x+ky) + \big(f(x+(k-1)y)+(-1)^j a\big) \\ &\quad - 2(f(x+ky)+(-1)^i a) \\ &\quad + \big(f(x+(k+1)y)+(-1)^j a\big)\big\| \leqslant \delta + 4\alpha \\ \text{or } \|2a\| &= \big\| -\mathscr{F}^{(0)}_y(x+ky) + \big(f(x+(k-1)y)+(-1)^j a\big) \\ &\quad + \big(f(x+(k+1)y)+(-1)^j a\big)\| \leqslant \delta + 2\alpha.\end{aligned}$$

In any case, we have $\|a\| \leqslant \frac{\delta}{2} + \alpha$. □

Theorem 3 *Let $\delta \geqslant 0$ and $f : G \to B$ satisfy (4). Then one of the following results holds:*
(i) There exists $A : G \to B$ which satisfies (2) for all $x, y \in G$ and $\|f(x) - f(e) - A(x)\| \leqslant 12\delta$ for all $x \in G$.
(ii) There exists $g : G \to B$ which satisfies (3) for all $x, y \in G$ and not satisfy (2) for some $x_0, y_0 \in G$. Also, $\|f(x) - g(x)\| \leqslant 5\delta$. Furthermore, there exists $a \in B\backslash\{0\}$ and a partition H_1, H_2 of G such that $g(H_1) = \{a\}$ and $g(H_2) = \{-a\}$. The sets H_1 and H_2 are G-convex.

Proof: If $\|\mathscr{F}^{(2)}_y(x)\| \leqslant 12\delta$ for all $x, y \in G$, then the result (i) can be obtained from direct method (can be found in [8, Theorem 1] and [7, Theorem 3.1], for instance). So from now on, we will assume that there exists $x_0, y_0 \in G$ such that $\|\mathscr{F}_{y_0}^{(2)}(x_0)\| > 12\delta$ (and hence $\|2f(x_0)\| > 11\delta$).

According to Lemma 4, either $\mathscr{L}(x_0, 2^n y_0)$ for all $n \in \mathbb{N} \cup \{0\}$ or $\mathscr{R}(x_0, 2^n y_0)$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\mathscr{R}(x_0, 2^n y_0)$ and $\mathscr{L}(x_0, 2^n(-y_0))$ are the same statements, we can assume without loss of generality that $\mathscr{L}(x_0, 2^n y_0)$ for all $n \in \mathbb{N} \cup \{0\}$. Let

$$H_1 = \{x \in G : \|f(x) - f(x_0)\| \leqslant 5\delta\}$$
$$H_2 = \{x \in G : \|f(x) + f(x_0)\| \leqslant 2\delta\}.$$

Lemma 4, Lemma 5, Lemma 9 yield $x \in H_1 \cup H_2$ for all $x \in G$.

Lemma 9 implies that H_1 and H_2 are disjoint. Proposition 4 implies that they are G-convex (none of them is empty according to Lemma 5).

Define $g : G \to B$ by

$$g(x) = \begin{cases} f(x_0); & x \in H_1 \\ -f(x_0); & x \in H_2. \end{cases}$$

Then g is the function we desire. It is straightforward to show that g is a solution of (3) (and not of (2), since $\|f(x_0)\| \neq 0$). □

Note that the function g in Theorem 3 (ii) is not necessarily unique.

The next example shows a function f which satisfies (4), is unbounded and $\|\mathscr{F}_y(x)\| \leqslant 2\delta$ for all $x, y \in G$. Such functions never satisfy Hyers-Ulam stability of (2) with bound δ.

Example 1 Let $f : \mathbb{Z} \to (-\infty, \infty)$ defined by

$$f(x) = \begin{cases} -\frac{5}{4}; & x = 0, \\ \frac{5}{4}; & x = 1, \\ \frac{x}{2} - \frac{1}{4}; & \text{otherwise.} \end{cases}$$

Then f satisfies (4) for $\delta = 1$, and $\|\mathscr{F}_y^{(2)}(x)\| \leqslant 2$ for all $x, y \in \mathbb{Z}$.

The next theorem explains the absent of examples where f is unbounded and $\|f(x) - f(e) - A(x)\|$ reached large values. For each $x \in G$, we denote $\langle x \rangle$ as the subgroup of G generated by x.

Theorem 4 *Let $\delta \geqslant 0$ and $f : G \to B$ satisfy (4) and $\|\mathscr{F}_y^{(2)}(x)\| \leqslant 12\delta$ for all $x, y \in G$. Then at least one of the following holds.*

(i) There exists $A : G \to B$ which satisfies (2) for all $x, y \in G$ and $\|f(x) - f(e) - A(x)\| \leqslant 4\delta$ for all $x \in G$, and $\|\mathscr{F}_y^{(2)}(x)\| \leqslant 5\delta$ for all $x, y \in G$.

(ii) $\|f(x) - f(e)\| \leqslant 12\delta$ for all $x \in G$.

Proof: We already have an additive $A : G \to B$ such that $\|f(x) - f(e) - A(x)\| \leqslant 12\delta$ for all $x \in G$. Suppose that A is not a zero function and let $x \in G$. We consider two cases.

Case 1: $A(x) \neq 0$. Let integer $m > \frac{25\delta + 2\|f(e)\|}{\|2A(x)\|}$. Then $\|A(2mx) + 2f(e)\| > 25\delta$. So

$$\begin{aligned} &\|\mathscr{F}_y^{(0)}(mx)\| \\ &\quad = \|A(2mx) + 2f(e) + (f(mx - y) - f(e) - A(mx - y)) \\ &\qquad + (f(mx + y) - f(e) - A(mx + y))\| \\ &\quad > 25\delta - (12\delta + 12\delta) = \delta \end{aligned}$$

for all $y \in G$ and integers k. Since m only needs to be large enough, this is also true for any $M \geqslant m$. Hence $\|\mathscr{F}_{x_2}^{(2)}(x_1)\| \leqslant \delta$ for all $x_1, x_2 \in \langle Mx \rangle$. [7, Theorem 3.1] implies that $\|f(Mx) - f(e) - A(Mx)\| \leqslant \delta$ (since $A(Mx) := \lim_{k\to\infty} \dfrac{f(2^k Mx) - f(e)}{2^{k+1}}$, it is still the same A.) So

$$\begin{aligned} &\|f(x) - f(e) - A(x)\| \\ &= \|\mathscr{F}_{mx}^{(2)}((m+1)x) + 2(f((m+1)x) - f(e) - A((m+1)x)) \\ &\quad - (f((2m+1)x) - f(e) - A((2m+1)x))\| \\ &\leqslant \delta + 2\delta + \delta = 4\delta. \end{aligned}$$

Case 2: $A(x) = 0$. Let $w_0 \in G$ such that $A(w_0) \neq 0$ and $k > \frac{25\delta + 2\|f(e)\|}{\|2A(w_0)\|}$ be an integer.

For any $K \geqslant k$, we get $K > \frac{25\delta + 2\|f(e)\|}{\|2A(w_0)\|}$. Then $1 > \frac{25\delta + 2\|f(e)\|}{\|A(2Kw_0)\|} = \frac{25\delta + 2\|f(e)\|}{\|A(x + 2Kw_0)\|}$.

Let $x^* = x + 2Kw_0$ and use the same arguments in Case 1 (with $m = 1$), we have $\|f(x + 2Kw_0) - f(e) - A(x + 2Kw_0)\| \leqslant \delta$ for all $K > k$ and $\|\mathscr{F}_{2kw_0}^{(2)}(x + 2kw_0)\| \leqslant \delta$. Hence

$$\begin{aligned} &\|f(x) - f(e) - A(x)\| \\ &= \|\mathscr{F}_{2kw_0}^{(2)}(x + 2kw_0) + 2(f(x + 2kw_0) - f(e) - A(x + 2kw_0)) \\ &\quad - (f(x + 4kw_0) - f(e) - A(x + 4kw_0))\| \\ &\leqslant 4\delta. \end{aligned}$$

Lastly, let $x, y \in G$. Since A is nonzero, there exists $x^* \in G$ such that $A(x^*) \neq 0$. Let m be a positive integer such that $\|A(mx^*)\| > 25\delta + 2\|f(e)\|$. Then $\mathscr{P}_w^{(2)}(mx^*)$ for all $w \in G$. We then use Proposition 1 with $(a, b_1, b_2) = (mx^*, x - mx^*, y)$ to imply that $\|\mathscr{F}_y^{(2)}(x)\| \leqslant 5\delta$. This finishes the proof. □

We gave a criterion for a function which satisfies (4) to determine the type of solution of (3) that is close to it. Let $S = \sup\{\|\mathscr{F}_y^{(2)}(x)\| : x, y \in G\}$.

(i) If $S \leqslant 5\delta$, f is near a solution of Jensen's equation (2).

(ii) If $S \in (5\delta, 12\delta]$, f is nearly constant.

(iii) If $S > 12\delta$, f is near a solution of (3) which is not a solution of (2).

Our result also implies that S is always finite.

Also note that for $S \leqslant 12\delta$, some functions might be near a nonlinear solution. In such cases, $\|f(x)\|$ are relatively small for all $x \in G$, so they can also be treated as nearly zero. Further criterions regarding these functions can be a future topic.

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