RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2026.061
ScienceAsia 52 (3): 2026: ID 2026061: **1–12**

Simplified analytical pricing of moment swaps in commodity markets

Pattira Ruengsinsub[a], **Kim Pluemjai**[b], **Kittisak Chumpong**[b,c,d,*]

[a] Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok 10900 Thailand
[b] Division of Computational Science, Faculty of Science, Prince of Songkla University, Songkhla 90110 Thailand
[c] Research Center in Mathematics and Statistics with Applications, Prince of Songkla University, Songkhla 90110 Thailand
[d] Financial Mathematics, Data Science and Computational Innovations Research Unit (FDC), Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok 10900 Thailand

*Corresponding author, e-mail: kittisak.ch@psu.ac.th

Received 12 Sep 2025, Accepted 20 Jun 2026
Available online 30 Jun 2026

ABSTRACT: Moment swaps allow market participants to manage risks and take positions on higher-order realized moments of asset log-returns, such as variance, skewness, and kurtosis. Although well studied in equity markets, their application to commodity markets remains limited because commodity prices exhibit mean reversion, seasonality, and storage-related effects. This paper proposes a closed-form approach for pricing discretely sampled moment swaps under a time-dependent mean-reverting commodity price model, in which the log-price follows a trending Ornstein–Uhlenbeck (O–U) process. We derive the conditional moment-generating function of the standard O–U process and use complete Bell polynomials with recurrence relations to obtain closed-form conditional moments. These results are transformed to the trending log-price process through a centered auxiliary O–U process. We then derive conditional central moments, mixed moments, covariance, and correlation, and rederive the fair delivery prices of variance, skewness, and kurtosis swaps directly from the realized log-return payoff using the one-step innovation representation. The resulting formulas are explicit, interpretable, and computationally efficient. Monte Carlo simulations based on the same log-price formulation show strong agreement with the closed-form values, supporting the accuracy and practical usefulness of the proposed method for hedging, volatility forecasting, and higher-order risk management in commodity markets.

KEYWORDS: moment swaps, discrete sampling, mean-reverting model, trending O–U process, closed-form formula

MSC2020: 60G65 91G20 91G80

INTRODUCTION

Derivative pricing is a cornerstone of modern financial engineering, providing essential tools for valuing and managing risks associated with complex financial instruments. Traditional derivatives, such as forwards, futures, and options, rely on models that focus primarily on the dynamics of the underlying asset's price or volatility. While widely utilized, these instruments emphasize first- and second-order characteristics, such as price levels and variance, often neglecting higher-order moments like skewness and kurtosis, which offer critical insights into market risks and opportunities. Moment swaps address this limitation by targeting the realized higher moments of the log-returns of a specified underlying asset. As forward contracts on moments such as skewness and kurtosis, they enable market participants to manage risks or speculate on features of the price distribution beyond volatility, broadening the scope of traditional derivatives.

Recent studies, including those by Schoutens [1] and Rompolis and Tzavalis [2], emphasize the critical role of moment swaps in addressing market shocks within a discrete sampling method. Variance swaps, corresponding to second-order moments, are frequently used by speculators to gain exposure to future variance levels and to mitigate volatility risk in portfolios. Skewness swaps, tied to third-order moments, offer protection against changes in the symmetry of return distributions, while kurtosis swaps, reflecting fourth-order moments, safeguard against extreme jumps or shifts in tail behavior. These studies further demonstrate that integrating variance and higher-moment swaps into European option hedging strategies yields better performance compared to traditional delta hedging approaches. This underscores the significance of defining and pricing higher-moment swaps as effective tools for managing skewness and kurtosis risks.

The pricing of moment swaps when stocks are the underlying assets has been extensively studied by many researchers, including variance swaps, as well as skewness and kurtosis swaps [2–5]. However, relatively little attention has been given to pricing moment swaps of the log-returns for commodities in a discrete sampling method, despite their unique characteristics, such as mean-reversion, seasonality, and storage costs, which differentiate them from stocks and introduce additional complexities to derivative pricing models. In 2016, Weraprasertsakun and Rujivan [6] presented an analytical formula for pricing variance swaps based on Schwartz's one-factor model. Building on this, Chumpong et al [7] derived closed-form formulas for pricing skewness and kurtosis swaps under the same model, utilizing the conditional moments proposed by Chumpong et al [8]. In 2021, Rujivan [9] extended Schwartz's two-factor model by incorporating stochastic convenience yields to price moment swaps. While this extension offered a more comprehensive representation of commodity dynamics, the resulting pricing formulas were extremely complex. To address this, Duangpan et al [10] later simplified the approach, improving its analytical and practical usability. However, neither study accounted for a time-dependent mean level in the log-price dynamics, limiting their ability to capture dynamic market conditions and evolving trends in commodity prices. Although Schwartz's two-factor model has been adopted for commodity price modeling because it captures both the spot price and convenience yield dynamics, its complexity often limits analytical tractability. In contrast, Schwartz's one-factor model offers a simpler method that remains effective for many practical applications. Building on this tractability, we extend the one-factor mean-reverting process by allowing the mean level of the log-price process to vary over time and by incorporating the corresponding correction term. This formulation leads to a trending Ornstein–Uhlenbeck (O–U) process for the log-price itself, which is consistent with realized log-return payoffs used in moment swap pricing.

This work provides three key contributions. First, we derive a closed-form expression for the conditional moment-generating function of the standard O–U process using the Feynman–Kac theorem. The corresponding polynomial-exponential expectations and conditional moments are then obtained through complete Bell polynomials and recurrence relations. These results are applied to the centered auxiliary process and transformed back to the trending log-price process, ensuring that the state variable used in the moment formulas is consistent with the realized log-return payoff. Second, we derive conditional central moments, conditional mixed moments, covariance, and correlation for the trending log-price process. The mixed moment formula is written with the correct binomial expansion and time arguments, allowing the dependence between different observation dates to be handled consistently. Third, we rederive the fair delivery prices of discretely sampled variance, skewness, and kurtosis swaps directly from the realized log-return payoff. The resulting strike formulas are expressed through the one-step innovation representation of the log-price increment and are validated by Monte Carlo simulations based on the same model formulation.

TIME-DEPENDENT MEAN-REVERTING MODEL

Let $(\Omega, \mathscr{F}, \mathbb{Q})$ be a risk neutral probability space with a filtration $(\mathscr{F}_t)_{t\geqslant 0}$ and a risk-neutral probability measure $\mathbb{Q}$. Under this probability space, the dynamics of the commodity price S_t are assumed to follow a time-dependent mean-reverting model, described by

$$\mathrm{d}S_t = \kappa\left(\mu(t) + \frac{\mu'(t)}{\kappa} - \ln(S_t)\right) S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t, \quad (1)$$

$S_0 > 0$, where $\mu(t)$ is the adjusted time-dependent mean level, with the correction term $\mu'(t)$ accounting for its rate of change over time; $\kappa > 0$ is the mean-reversion speed; σ is the volatility of the commodity prices; and W_t is a standard Brownian motion. This model generalizes the classic Schwartz model, which assumes a constant mean μ and thus lacks a time-dependent adjustment [7]. While suitable for stable markets, this assumption limits the model's ability to track price trends in dynamic environments. By allowing $\mu(t)$ to vary over time and incorporating the derivative $\mu'(t)$ as a correction term, our model offers greater flexibility and responsiveness. Specifically, the term $\mu'(t)/\kappa$ ensures that the mean-reverting target adapts to real-time changes in market conditions, making it ideal for commodities whose prices are influenced by evolving trends such as seasonal effects or economic shifts. This generalization provides a more accurate method for capturing the behavior of prices in markets with variable fundamentals.

Following the storage theory assumption outlined by Schwartz [11], we assume that the convenience yield δ_t is related to the log-transformed price process through $\delta_t = \kappa Y_t$, where $Y_t = \ln S_t$ represents the log-price of the commodity. Applying Itô's lemma to (1), we obtain

$$\begin{aligned} \mathrm{d}Y_t &= \left(\kappa\left(\alpha(t) - Y_t\right) + \alpha'(t)\right)\mathrm{d}t + \sigma\,\mathrm{d}W_t, \quad (2)\\ \alpha(t) &= \mu(t) - \frac{\sigma^2}{2\kappa}. \end{aligned}$$

Thus, the log-price process itself follows the trending O–U dynamics in (2). This notation is used throughout the paper, and the realized log-return over the interval $[t_{i-1}, t_i]$ is therefore given by

$$Y_{t_i} - Y_{t_{i-1}} = \ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right).$$

For analytical convenience, we introduce the centered auxiliary process

$$Z_t = Y_t - \alpha(t).$$

Then, from (2), the process Z_t satisfies

$$\mathrm{d}Z_t = -\kappa Z_t\,\mathrm{d}t + \sigma\,\mathrm{d}W_t. \quad (3)$$

The process Z_t is used only as an auxiliary process for deriving conditional moments. The payoff definitions and the moment-swap pricing formulas are written in terms of the log-price process Y_t, not in terms of the centered process Z_t.

The adaptability of the trending O–U process in (2) makes it useful for modeling stochastic dynamics with deterministic trends across financial and engineering applications; for more details, see [12–15]. The conditional moments of this process are essential for derivative pricing, futures valuation, and risk assessment, as they describe expected values and higher-order distributional features over time. Although these moments can, in principle, be derived from the transition probability density function, the calculation becomes more involved when the mean level is time-dependent.

To derive the required conditional moments, we first consider the standard O–U process

$$\mathrm{d}X_t = \kappa(\theta - X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \tag{4}$$

where θ is the long-term mean level. The centered process (3) corresponds to the special case $\theta = 0$. Hence, the conditional moments of the trending log-price process (2) can be obtained by first deriving the conditional moments of the O–U process (4), then applying the transformation

$$Y_t = \alpha(t) + Z_t.$$

In particular, if $Y_{t_0} = y$, then the corresponding initial value of the centered process is

$$Z_{t_0} = y - \alpha(t_0).$$

This relationship will be used consistently in the conditional moment formulas and in the subsequent pricing formulas.

MAIN RESULTS

To develop a pricing method for moment swaps under the trending O–U process, it is essential to derive the conditional moments of the underlying log-price process under the risk-neutral measure. These conditional moments determine the expected payoff structure of the swaps. In addition, we also provide the unconditional moments, which offer insight into the long-term behavior of the process and are useful for asymptotic analysis or model calibration.

Conditional moments

To obtain the conditional moments of the trending log-price process (2), we begin by deriving the conditional moments of the standard O–U process (4). Moment generating function (MGF)- and Feynman–Kac-based analytical constructions have also been employed in more general stochastic settings, including nonlinear and regime-switching diffusions, to derive tractable conditional expectations and derivative pricing formulas [16, 17]. This derivation uses its MGF, which provides a basis for calculating conditional expectations involving polynomial and exponential terms. The resulting formulas are then applied to the centered auxiliary process $Z_t = Y_t - \alpha(t)$ introduced in Section "Time-Dependent Mean-Reverting Model", and finally transformed back to the log-price process Y_t. For simplicity in notation, let

$$E^{\mathbb{Q}}_{t_0}[f(X_T)] = E^{\mathbb{Q}}\left[f(X_T)\middle|\mathscr{F}_{t_0}\right] = E^{\mathbb{Q}}\left[f(X_T)\middle|X_{t_0} = x\right],$$

where $f(X_T)$ represents a function of the process X_T evaluated at time T, and x is the initial value of the process X_t at time t_0.

With this method in place, we proceed to Theorem 1, which formalizes the closed-form solution for the conditional MGF under the O–U process.

Theorem 1 *Suppose that $a \in \mathbb{R}$ and X_t follows the O–U process (4) on $[t_0, T]$ with initial value $X_{t_0} = x$. Then, the conditional MGF is defined by the following:*

$$U^{\langle a\rangle}(\tau, x, \theta) = E^{\mathbb{Q}}_{t_0}\left[\mathrm{e}^{aX_T}\right] = \mathrm{e}^{A(\tau,a,\theta)x + B(\tau,a,\theta)}, \tag{5}$$

where $\tau = T - t_0$, $x \in \mathbb{R}$ and

$$\begin{aligned} A(\tau,a,\theta) &= a\,\mathrm{e}^{-\kappa\tau}, \\ B(\tau,a,\theta) &= (1-\mathrm{e}^{-\kappa\tau})\left(a\theta + \frac{1}{4\kappa}\sigma^2 a^2\left(1+\mathrm{e}^{-\kappa\tau}\right)\right). \end{aligned} \tag{6}$$

Proof: Utilizing the Feynman-Kac theorem, the conditional MGF in (5) satisfies the following PDE:

$$\frac{\partial U^{\langle a\rangle}}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 U^{\langle a\rangle}}{\partial x^2} + \kappa(\theta - x)\frac{\partial U^{\langle a\rangle}}{\partial x} = 0 \tag{7}$$

for all $(x,t) \in \mathbb{R} \times [t_0, T]$, subject to the terminal condition

$$U^{\langle a\rangle}(0, x, \theta) = \mathrm{e}^{ax}. \tag{8}$$

Let $\tau = T - t$. We solve the PDE (7) subject to the terminal condition (8) by assuming that the solution can be written in the form (5). The required partial derivatives are

$$\begin{aligned} \frac{\partial U^{\langle a\rangle}}{\partial t} &= \left(-\frac{\mathrm{d}A}{\mathrm{d}\tau}x - \frac{\mathrm{d}B}{\mathrm{d}\tau}\right)\mathrm{e}^{Ax+B}, \\ \frac{\partial U^{\langle a\rangle}}{\partial x} &= A\,\mathrm{e}^{Ax+B}, \qquad \frac{\partial^2 U^{\langle a\rangle}}{\partial x^2} = A^2\,\mathrm{e}^{Ax+B}. \end{aligned}$$

Substituting these derivatives into (7) gives

$$\left(-\frac{\mathrm{d}A}{\mathrm{d}\tau} - \kappa A\right)x\,\mathrm{e}^{Ax+B} - \left(\frac{\mathrm{d}B}{\mathrm{d}\tau} - \kappa\theta A - \frac{1}{2}\sigma^2 A^2\right)\mathrm{e}^{Ax+B} = 0, \tag{9}$$

subject to the initial conditions $A(0,a,\theta) = a$ and $B(0,a,\theta) = 0$ by comparing the coefficients of (5) with (8). Hence, the coefficients in (6) can be directly obtained by solving the system of ODEs (9). □

We next recall the complete Bell polynomial representation used to differentiate the exponential form of the MGF. If $f = f(a)$ is sufficiently differentiable, then

$$\frac{\mathrm{d}^n}{\mathrm{d}a^n}\,\mathrm{e}^{f(a)} = \mathrm{e}^{f(a)} B_n\left(f_a, f_{aa}, f_a^{(3)}, \dots, f_a^{(n)}\right),$$

where B_n is the complete Bell polynomial and $f_a^{(r)} = \partial^r f / \partial a^r$. In the present setting, this identity gives the conditional expectation of the product of a polynomial and an exponential function as follows:

$$\begin{aligned} E_{t_0}^{\mathbb{Q}}\left[X_T^n\,\mathrm{e}^{aX_T}\right] &= U^{\langle a\rangle}(\tau,x,\theta) K_n(f_a, f_{aa}, f_a^{(3)}, \dots, f_a^{(n)}) \\ &= \mathrm{e}^f K_n(f_a, f_{aa}, f_a^{(3)}, \dots, f_a^{(n)}), \end{aligned} \tag{10}$$

where $f = A(\tau,a,\theta)x + B(\tau,a,\theta)$, as defined in (6). The quantity K_n denotes the determinant representation of the complete Bell polynomial:

$$K_n(f_a, f_{aa}, f_a^{(3)}, \dots, f_a^{(n)}) = \begin{vmatrix} \frac{f_a}{0!} & \frac{f_{aa}}{1!} & \frac{f_a^{(3)}}{2!} & \cdots & \frac{f_a^{(n)}}{(n-1)!} \\ -1 & \frac{f_a}{0!} & \frac{f_{aa}}{1!} & \cdots & \frac{f_a^{(n-1)}}{(n-2)!} \\ 0 & -2 & \frac{f_a}{0!} & \cdots & \frac{f_a^{(n-2)}}{(n-3)!} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-n & \frac{f_a}{0!} \end{vmatrix}, \tag{11}$$

with initial value $K_0 = 1$. Equivalently, if $M^{(n)} = (m_{ij}^{(n)})_{1\leqslant i,j\leqslant n}$ denotes the matrix in (11), then its entries are given by

$$m_{ij}^{(n)} = \begin{cases} \dfrac{f_a^{(j-i+1)}}{(j-i)!}, & j \geqslant i, \\ 1-i, & j = i-1, \\ 0, & j < i-1. \end{cases}$$

Since $A(\tau,a,\theta)$ and $B(\tau,a,\theta)$ are polynomials in a with degrees less than 3, the third and higher-order partial derivatives of the function f with respect to a vanish. The complete Bell polynomial in (11) becomes

$$\begin{aligned} K_n &= K_n(f_a, f_{aa}, 0, \dots, 0) \\ &= \begin{vmatrix} f_a & f_{aa} & 0 & \cdots & 0 \\ -1 & f_a & f_{aa} & \cdots & 0 \\ 0 & -2 & f_a & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & f_{aa} \\ 0 & 0 & \cdots & 1-n & f_a \end{vmatrix} \\ &= f_a K_{n-1} + (n-1) f_{aa} K_{n-2} \end{aligned} \tag{12}$$

where the last equality is obtained by using the cofactor expansion along the last column with initial values $K_0 = 1$ and $K_1 = f_a$.

To obtain an explicit closed-form formula from the second order linear recurrence relation (12), we first derive the useful lemmas from counting sets, as proposed by Laohakosol et al [18] and Sutthimat et al [19]. Let $D_0 = D_1 = \{\varnothing\}$ and for $n \geqslant 2$,

$$\begin{aligned} D_n = \{X \in \mathscr{P}([0:n-2]) \,:\, &|X| = 0,1 \\ &\text{or } |u-v| \geqslant 2 \text{ for all } u,v \in X \text{ with } u \neq v\} \end{aligned}$$

where $[i:j] := \{i, i+1, \dots, j\}$ for $i \leqslant j$, $\mathscr{P}([i:j])$ denotes the power set of $[i:j]$, and $|X|$ denotes the cardinality of X. We now present Lemma 1, which establishes the identity necessary for proving Lemma 2 and plays a pivotal role in simplifying the recurrence relation in Theorem 2.

Lemma 1 *For each $n \geqslant 2$ and $k \geqslant 0$, we have*

$$\sum_{X\in D_n,\ |X|=k} \left(\prod_{i\in X}(i+1)\right) = \begin{cases} 1, & \text{for } k = 0, \\ \displaystyle\sum_{\substack{X\in D_{n-1}\\|X|=k}}\left(\prod_{i\in X}(i+1)\right) + (n-1)\sum_{\substack{X\in D_{n-2}\\|X|=k-1}}\left(\prod_{i\in X}(i+1)\right), & \text{for } 1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor, \\ 0, & \text{for } k > \lfloor \frac{n}{2} \rfloor. \end{cases}$$

Proof: The demonstration is split into three cases contingent on the condition k. For $k = 0$ and $k > \lfloor \frac{n}{2} \rfloor$, these cases are straightforward to verify, as they follow directly from the definition of D_n. For interesting case when $1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor$, we consider the structure of D_{n+1}. Specifically, we decompose the sum over D_{n+1} based on whether each set $X \in D_{n+1}$ also belongs to D_n or to $D_{n+1} \backslash D_n$. This gives us:

$$\begin{aligned} \sum_{X\in D_{n+1},\ |X|=k}\left(\prod_{i\in X}(i+1)\right) &= \sum_{X\in D_n,\ |X|=k}\left(\prod_{i\in X}(i+1)\right) \\ &\quad + \sum_{X\in D_{n+1}\backslash D_n,\ |X|=k}\left(\prod_{i\in X}(i+1)\right) \end{aligned} \tag{13}$$

In the second term on the right-hand side of (13), each set $X \in D_{n+1} \backslash D_n$ necessarily contains the element $n-1$. Therefore, each product is a multiple of n:

$$\begin{aligned} \sum_{\substack{X\in D_{n+1}\backslash D_n\\|X|=k}}\left(\prod_{i\in X}(i+1)\right) &= n \sum_{\substack{X\backslash\{n-1\}\in D_{n+1}\backslash D_n\\|X\backslash\{n-1\}|=k-1}}\left(\prod_{i\in X\backslash\{n-1\}}(i+1)\right) \\ &= n \sum_{\substack{X\in D_{n-1}\\|X|=k-1}}\left(\prod_{i\in X}(i+1)\right). \end{aligned} \tag{14}$$

Combining (13) and (14), we obtain the desired recursive relation, which completes the proof. □

Lemma 2 *For each $n \in \mathbb{N}$ and $0 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor$, we have*

$$\sum_{\substack{X\in D_n\\|X|=k}}\left(\prod_{i\in X}(i+1)\right) = \frac{n!}{(n-2k)!\,k!\,2^k}. \tag{15}$$

Proof: Let

$$S_{n,k} = \sum_{X\in D_n,\ |X|=k} \Big(\prod_{i\in X}(i+1)\Big).$$

For $k=0$, we have $S_{n,0}=1$, which agrees with (15). For $k>\lfloor n/2\rfloor$, both sides are zero by the definition of D_n. We prove the remaining cases by induction on n. The formula is immediate for $n=0$ and $n=1$. Suppose that it holds for all smaller indices. By Lemma 1, for $1\leqslant k\leqslant \lfloor n/2\rfloor$,

$$S_{n,k} = S_{n-1,k} + (n-1)S_{n-2,k-1}.$$

Using the induction hypothesis, we obtain

$$S_{n,k} = \frac{(n-1)!}{(n-1-2k)!k!2^k} + \frac{(n-1)(n-2)!}{(n-2k)!(k-1)!2^{k-1}},$$

where the first term is interpreted as zero when $k>\lfloor (n-1)/2\rfloor$. For the nonzero case, the right-hand side becomes

$$\frac{(n-1)!(n-2k)}{(n-2k)!k!2^k} + \frac{(n-1)!(2k)}{(n-2k)!k!2^k} = \frac{n!}{(n-2k)!k!2^k}.$$

The boundary case $k=n/2$ when n is even is obtained from the same identity, with the first term equal to zero. Therefore, (15) holds for all admissible n and k. □

Theorem 2 *Suppose that $a\in\mathbb{R}$, $n=0,1,2,\dots$, and X_t follows the O–U process (4) on $[t_0,T]$ with initial value $X_{t_0}=x$. The closed-form formula for the conditional expectation of the product of a polynomial and an exponential function can be expressed by*

$$\begin{aligned} V^{\langle n\rangle}(\tau,x,a,\theta) &= E^{\mathbb{Q}}_{t_0}\big[X_T^n\, \mathrm{e}^{aX_T}\big] \\ &= U^{\langle a\rangle}(\tau,x,\theta)\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\frac{n!}{(n-2k)!k!2^k} \\ &\quad\times\Big(\mathrm{e}^{-\kappa\tau}x + C(\tau,a,\theta)\Big)^{n-2k} D^k(\tau) \end{aligned} \tag{16}$$

where $\tau=T-t_0$, $x\in\mathbb{R}$ and

$$\begin{aligned} C(\tau,a,\theta) &= \theta\,\mathrm{e}^{-\kappa\tau}(\mathrm{e}^{\kappa\tau}-1) + a(1-\mathrm{e}^{-2\kappa\tau})\frac{\sigma^2}{2\kappa}, \\ D(\tau) &= (1-\mathrm{e}^{-2\kappa\tau})\frac{\sigma^2}{2\kappa}. \end{aligned}$$

Proof: The case $n=0$ follows directly from the definition of $U^{\langle a\rangle}$. For $n\geqslant 1$, the identity (10) gives

$$V^{\langle n\rangle}(\tau,x,a,\theta) = U^{\langle a\rangle}(\tau,x,\theta)K_n.$$

For $n\geqslant 2$, define $X+\{1\}=\{x+1; x\in X\}$. According to Laohakosol et al [18], the second order linear recurrence (12) has an explicit closed form solution given by

$$\begin{aligned} K_n &= \sum_{X\in D_n,\ 0\in X}\Big(\prod_{i\in X}(i+1)f_{aa}\Big)\Big(\prod_{i\in[1:n-1]\setminus(X\cup(X+\{1\}))} f_a\Big) \\ &\quad + f_a\sum_{X\in D_n,\ 0\notin X}\Big(\prod_{i\in X}(i+1)f_{aa}\Big)\Big(\prod_{i\in[1:n-1]\setminus(X\cup(X+\{1\}))} f_a\Big) \\ &= \sum_{X\in D_n,\ 0\in X}\Big(\prod_{i\in X}(i+1)\Big)(f_a)^{n-1-(2|X|-1)}(f_{aa})^{|X|} \\ &\quad + f_a\sum_{X\in D_n,\ 0\notin X}\Big(\prod_{i\in X}(i+1)\Big)(f_a)^{n-1-2|X|}(f_{aa})^{|X|} \\ &= \sum_{X\in D_n}\Big(\prod_{i\in X}(i+1)\Big)(f_a)^{n-2|X|}(f_{aa})^{|X|} \\ &= \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\Big[\sum_{X\in D_n,\ |X|=k}\Big(\prod_{i\in X}(i+1)\Big)(f_a)^{n-2k}(f_{aa})^k\Big] \\ &= \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\frac{n!}{(n-2k)!k!2^k}(f_a)^{n-2k}(f_{aa})^k \end{aligned} \tag{17}$$

where we use Lemma 2 in the final equality. Since

$$\begin{aligned} f_a &= x\,\mathrm{e}^{-\kappa\tau} + \theta\,\mathrm{e}^{-\kappa\tau}(\mathrm{e}^{\kappa\tau}-1) + a(1-\mathrm{e}^{-2\kappa\tau})\frac{\sigma^2}{2\kappa} \\ &= x\,\mathrm{e}^{-\kappa\tau} + C(\tau,a,\theta) \end{aligned}$$

and

$$f_{aa} = (1-\mathrm{e}^{-2\kappa\tau})\frac{\sigma^2}{2\kappa} = D(\tau),$$

we conclude, by (10), (12) and (17), that the proof is complete. □

Theorem 2 provides the closed-form formula (16) for the conditional expectation of the product of a polynomial and an exponential function under the dynamics of the O–U process. This result serves as a foundation for deriving specific conditional moments by assigning appropriate parameter values. Corollary 1 builds on this method by setting $a=0$ and gives the conditional n-moment of the O–U process.

Corollary 1 *According to Theorem 2 with $a=0$, the conditional n-moment can be expressed in the following form:*

$$\begin{aligned} V^{\langle n\rangle}(\tau,x,0,\theta) &= E^{\mathbb{Q}}_{t_0}\big[X_T^n\big] \\ &= \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\frac{n!}{(n-2k)!k!2^k}\Big(\mathrm{e}^{-\kappa\tau}x + C(\tau,0,\theta)\Big)^{n-2k} D^k(\tau). \end{aligned}$$

Proof: Corollary 1 follows directly from Theorem 2 by setting $a=0$ in (16). □

We now specialize Corollary 1 to the centered O–U process (3), which corresponds to the case $\theta=0$. Define

$$W^{\langle n\rangle}(\tau,x) = V^{\langle n\rangle}(\tau,x,0,0).$$

Then

$$W^{\langle n\rangle}(\tau,x)=\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\frac{n!}{(n-2k)!k!2^k}\left(\mathrm{e}^{-\kappa\tau}x\right)^{n-2k}D^k(\tau). \quad (18)$$

This formula gives the conditional n-moment of the auxiliary centered process Z_t over a time interval of length τ.

In Thamrongrat's formulation [20], calculating each moment $W^{\langle k\rangle}$ involves handling complex expressions and requires significant computation due to the intricate structure of the O–U process with a deterministic trend. Thamrongrat's approach applies the Binomial theorem to express the k-th moment, effectively transforming the problem but resulting in a series of nested terms. This complexity can make direct computation challenging and computationally intensive, especially for higher-order moments.

In contrast, our approach introduces a simplified closed-form solution for $W^{\langle k\rangle}$, as presented in (18). This solution provides a direct and efficient expression, reducing the computational complexity inherent in Thamrongrat's method. The primary advantage of our formula lies in its streamlined structure, allowing each term in the sum of $W^{\langle k\rangle}$ to be expressed in a straightforward, closed form. This avoids the need for repeated expansions or nested terms, which are often required in Thamrongrat's method. Consequently, our approach facilitates easier evaluation of $W^{\langle k\rangle}$ for various values of k without the need for complex iterative calculations. This makes the formula practical and adaptable to a range of applications, even when calculating moments for larger values.

Furthermore, the computational efficiency of our closed-form expression extends the applicability of $W^{\langle k\rangle}$ to more advanced processes, such as those with deterministic trends, with reduced complexity. By removing the need for iterative or nested computations, our approach enables researchers to work with the trending O–U process in applied settings where computational resources may be limited, and time efficiency is essential. This simplified formula not only enhances the accessibility of $W^{\langle k\rangle}$ for theoretical analysis but also significantly improves its usability in practical applications, distinguishing it from Thamrongrat's approach.

To further expand the application of our results, we present an easy-to-use formula for the conditional moments of the trending log-price process, which will be particularly useful in the pricing of moment swaps.

Theorem 3 *Suppose that $n=0,1,2,\dots$ and Y_t follows the trending O–U process (2). Let $0\leqslant s<t$ and suppose that $Y_s=y$. The closed-form formula for the conditional moment of the log-price process is*

$$\begin{aligned}\mathscr{A}^{\langle n\rangle}_{s,t}(y)&=E^{\mathbb{Q}}\left[Y_t^n \mid Y_s=y\right]\\&=\sum_{j=0}^{n}\binom{n}{j}(\alpha(t))^{n-j}W^{\langle j\rangle}\left(t-s,y-\alpha(s)\right). \quad (19)\end{aligned}$$

Here, $y\in\mathbb{R}$, and $W^{\langle j\rangle}$ is defined as in (18) for $j=0,1,\dots,n$.

Proof: Since $Z_t=Y_t-\alpha(t)$, we have $Y_t=\alpha(t)+Z_t$. Moreover, if $Y_s=y$, then $Z_s=y-\alpha(s)$. Applying the binomial theorem gives

$$Y_t^n=\sum_{j=0}^{n}\binom{n}{j}(\alpha(t))^{n-j}Z_t^j.$$

Taking the conditional expectation given $Y_s=y$, and using the conditional moment formula for the centered process Z_t, yields

$$\begin{aligned}&E^{\mathbb{Q}}\left[Y_t^n \mid Y_s=y\right]\\&\quad=\sum_{j=0}^{n}\binom{n}{j}(\alpha(t))^{n-j}E^{\mathbb{Q}}\left[Z_t^j \mid Z_s=y-\alpha(s)\right].\end{aligned}$$

Since Z_t follows the centered O–U process (3), the last conditional expectation is

$$E^{\mathbb{Q}}\left[Z_t^j \mid Z_s=y-\alpha(s)\right]=W^{\langle j\rangle}(t-s,y-\alpha(s)).$$

This proves (19). □

Unconditional moment

For completeness, we provide the equilibrium moments of the standard O–U process (4). These moments are useful as reference quantities for the centered auxiliary process and for comparison with the time-dependent case.

Theorem 4 *Suppose that $a\in\mathbb{R},n\in\mathbb{N}$ and X_t follows the O–U process (4) on $[t_0,T]$. Then, the unconditional expectation of polynomial-exponential expectation at equilibrium is provided by*

$$\begin{aligned}&\lim_{\tau\to\infty}V^{\langle n\rangle}(\tau,x,a,\theta)\\&\quad=\mathrm{e}^{a\theta+\frac{1}{4\kappa}\sigma^2a^2}\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\frac{n!}{(n-2k)!k!2^k}\left(\theta+a\frac{\sigma^2}{2\kappa}\right)^{n-2k}\left(\frac{\sigma^2}{2\kappa}\right)^k,\end{aligned}$$

where $\tau=T-t_0$.

Proof: We start by analyzing the behavior of the terms of $A(\tau,a,\theta),B(\tau,a,\theta),C(\tau,a,\theta)$ and $D(\tau)$ as $\tau\to\infty$. Observe that

$$A(\tau,a,\theta)\to 0,\quad B(\tau,a,\theta)\to a\theta+\frac{1}{4\kappa}\sigma^2a^2,$$
$$C(\tau,a,\theta)\to\theta+a\frac{\sigma^2}{2\kappa},\quad \text{and}\quad D(\tau)\to\frac{\sigma^2}{2\kappa}.$$

Since the power function is continuous, taking the limit as $\tau\to\infty$ in $V^{\langle n\rangle}(\tau,x,a,\theta)$, yields

$$\begin{aligned}&\lim_{\tau\to\infty}V^{\langle n\rangle}(\tau,x,a,\theta)\\&=\mathrm{e}^{a\theta+\frac{1}{4\kappa}\sigma^2a^2}\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\frac{n!}{(n-2k)!k!2^k}\left(\theta+a\frac{\sigma^2}{2\kappa}\right)^{n-2k}\left(\frac{\sigma^2}{2\kappa}\right)^k,\end{aligned}$$

as required. □

We now present Corollary 2, which extends the established moments to address specialized cases.

Corollary 2 *According to Theorem 4 with $a = 0$, the unconditional moment can be expressed in the following form:*

$$\lim_{\tau\to\infty} V^{\langle n\rangle}(\tau,x,0,\theta) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(n-2k)!k!2^k}\theta^{n-2k}\left(\frac{\sigma^2}{2\kappa}\right)^k.$$

Proof: This result follows immediately from Theorem 4 by setting $a = 0$. □

STATISTICAL PROPERTIES

In this section, we investigate key statistical characteristics of the trending O–U process, which are essential for understanding the dynamics of its distribution under the risk-neutral measure. While the previous section focused on deriving conditional moments, here we extend those results to obtain higher-order statistical measures, such as conditional central moments, covariance, and correlation. These quantities play an important role in quantifying risk and shape features—such as dispersion, asymmetry, and tail behavior—of the underlying process. Moreover, they serve as the foundation for computing skewness and kurtosis, which are widely used in volatility modeling and the pricing of higher-moment derivatives.

Corollary 3 *Let $0 \leqslant s < t$, and suppose that the log-price process Y_t follows the trending O–U process in (2), with initial value $Y_s = y$. Then, the conditional n-th central moment of Y_t, given $Y_s = y$, is given by*

$$\begin{aligned}\mu_{s,t}^{\langle n\rangle}(y) &= E^{\mathbb{Q}}\left[\left(Y_t - E^{\mathbb{Q}}[Y_t \mid Y_s = y]\right)^n \mid Y_s = y\right] \\ &= \sum_{\ell=0}^{n}\binom{n}{\ell}(-1)^{n-\ell}\mathscr{A}_{s,t}^{\langle\ell\rangle}(y)\left(\mathscr{A}_{s,t}^{\langle 1\rangle}(y)\right)^{n-\ell}, \end{aligned} \quad (20)$$

where $\mathscr{A}_{s,t}^{\langle\ell\rangle}(y)$ denotes the conditional moment defined in (19), for $\ell = 0, 1, \dots, n$.

Proof: Applying the binomial theorem to expand the power, we obtain

$$\begin{aligned}&\left(Y_t - E^{\mathbb{Q}}[Y_t \mid Y_s = y]\right)^n \\ &\quad= \sum_{\ell=0}^{n}\binom{n}{\ell}(-1)^{n-\ell}Y_t^{\ell}\left(E^{\mathbb{Q}}[Y_t \mid Y_s = y]\right)^{n-\ell}.\end{aligned}$$

Taking the conditional expectation given $Y_s = y$, we have

$$\begin{aligned}\mu_{s,t}^{\langle n\rangle}(y) = \sum_{\ell=0}^{n}\binom{n}{\ell}(-1)^{n-\ell}E^{\mathbb{Q}}[Y_t^{\ell} \mid Y_s = y] \\ \times\left(E^{\mathbb{Q}}[Y_t \mid Y_s = y]\right)^{n-\ell}.\end{aligned}$$

Finally, by Theorem 3, we have $E^{\mathbb{Q}}[Y_t^{\ell} \mid Y_s = y] = \mathscr{A}_{s,t}^{\langle\ell\rangle}(y)$ and $E^{\mathbb{Q}}[Y_t \mid Y_s = y] = \mathscr{A}_{s,t}^{\langle 1\rangle}(y)$. Substituting these expressions yields (20). □

The conditional variance, skewness, and kurtosis can be directly derived from Corollary 3. Specifically, setting $n = 2$, 3, and 4 yields the second, third, and fourth conditional central moments, respectively. The conditional skewness and kurtosis are then obtained by normalizing these moments as follows:

$$\mathrm{Skew}_{s,t}^{\mathbb{Q}}(Y) = \frac{\mu_{s,t}^{\langle 3\rangle}(y)}{\left(\mu_{s,t}^{\langle 2\rangle}(y)\right)^{3/2}}, \quad \mathrm{Kurt}_{s,t}^{\mathbb{Q}}(Y) = \frac{\mu_{s,t}^{\langle 4\rangle}(y)}{\left(\mu_{s,t}^{\langle 2\rangle}(y)\right)^{2}}.$$

Corollary 4 *Let $n_1, n_2 \in \mathbb{N}$. Consider the trending O–U process defined in (2), with initial condition $Y_{t_0} = y$, and let $0 \leqslant t_0 < T_1 < T_2$. Then, the conditional mixed moment of orders n_1 and n_2 is given by*

$$\begin{aligned}&\mathscr{M}_{t_0,T_1,T_2}^{\langle n_1,n_2\rangle}(y) = E^{\mathbb{Q}}\left[Y_{T_1}^{n_1}Y_{T_2}^{n_2} \mid Y_{t_0} = y\right] \quad (21)\\ &= \sum_{j=0}^{n_2}\binom{n_2}{j}(\alpha(T_2))^{n_2-j}\sum_{k=0}^{\lfloor j/2\rfloor}\frac{j!}{(j-2k)!k!2^k} \\ &\quad\times \mathrm{e}^{-\kappa(T_2-T_1)(j-2k)}D^k(T_2-T_1) \\ &\quad\times\left(\sum_{\ell=0}^{j-2k}\binom{j-2k}{\ell}(-1)^{j-2k-\ell}(\alpha(T_1))^{j-2k-\ell}\mathscr{A}_{t_0,T_1}^{\langle n_1+\ell\rangle}(y)\right),\end{aligned}$$

where $\mathscr{A}_{t_0,T_1}^{\langle n_1+\ell\rangle}(y)$ is defined in (19).

Proof: By the tower property of conditional expectation, we have

$$E^{\mathbb{Q}}\left[Y_{T_1}^{n_1}Y_{T_2}^{n_2} | Y_{t_0}{=}y\right] = E^{\mathbb{Q}}\left[Y_{T_1}^{n_1}E^{\mathbb{Q}}\left[Y_{T_2}^{n_2} | Y_{T_1}\right] | Y_{t_0}{=}y\right].$$

Using Theorem 3, the conditional moment $E^{\mathbb{Q}}\left[Y_{T_2}^{n_2} \mid Y_{T_1}\right]$ can be written as

$$\begin{aligned}&E^{\mathbb{Q}}\left[Y_{T_2}^{n_2} \mid Y_{T_1}\right] \\ &\quad= \sum_{j=0}^{n_2}\binom{n_2}{j}(\alpha(T_2))^{n_2-j}W^{\langle j\rangle}\left(T_2-T_1, Y_{T_1}-\alpha(T_1)\right).\end{aligned}$$

By (18), the function $W^{\langle j\rangle}$ can be expanded as

$$\begin{aligned}W^{\langle j\rangle}\left(T_2-T_1, Y_{T_1}-\alpha(T_1)\right) = \sum_{k=0}^{\lfloor j/2\rfloor}\frac{j!}{(j-2k)!k!2^k} \\ \times\left(\mathrm{e}^{-\kappa(T_2-T_1)}\left(Y_{T_1}-\alpha(T_1)\right)\right)^{j-2k}D^k(T_2-T_1).\end{aligned}$$

Expanding the power of $Y_{T_1} - \alpha(T_1)$ using the binomial theorem yields:

$$\begin{aligned}&(Y_{T_1}-\alpha(T_1))^{j-2k} \\ &\quad= \sum_{\ell=0}^{j-2k}\binom{j-2k}{\ell}(-1)^{j-2k-\ell}Y_{T_1}^{\ell}(\alpha(T_1))^{j-2k-\ell}.\end{aligned}$$

Substituting these expressions into the tower-property identity and using

$$E^{\mathbb{Q}}\left[Y_{T_1}^{n_1+\ell} \mid Y_{t_0}=y\right]=\mathscr{A}_{t_0,T_1}^{\langle n_1+\ell\rangle}(y),$$

we obtain (21). This completes the proof. □

In addition, the general formula for conditional mixed moments of the form

$$E^{\mathbb{Q}}\left[Y_{T_1}^{n_1}Y_{T_2}^{n_2}\cdots Y_{T_k}^{n_k} \mid Y_{t_0}=y\right],$$

where $n_1, n_2, \ldots, n_k \in \mathbb{N}$ and $0 \leqslant t_0 < T_1 < T_2 < \cdots < T_k$, can be derived recursively by applying the tower property together with Theorem 3.

Corollary 5 *Let $0 \leqslant t_0 < T_1 < T_2$, and suppose that $Y_{t_0} = y$. Then, the conditional covariance between Y_{T_1} and Y_{T_2} under the trending O–U process in (2) is given by*

$$\begin{aligned}\operatorname{Cov}^{\mathbb{Q}}&\left[Y_{T_1}, Y_{T_2} \mid Y_{t_0}=y\right]\\ &=\left(\alpha(T_2)-\alpha(T_1)\,\mathrm{e}^{-\kappa(T_2-T_1)}\right)\mathscr{A}_{t_0,T_1}^{\langle 1\rangle}(y)\\ &\quad+\mathrm{e}^{-\kappa(T_2-T_1)}\mathscr{A}_{t_0,T_1}^{\langle 2\rangle}(y)-\mathscr{A}_{t_0,T_1}^{\langle 1\rangle}(y)\mathscr{A}_{t_0,T_2}^{\langle 1\rangle}(y)\\ &=\mathrm{e}^{-\kappa(T_2-T_1)}\left(\mathscr{A}_{t_0,T_1}^{\langle 2\rangle}(y)-\left(\mathscr{A}_{t_0,T_1}^{\langle 1\rangle}(y)\right)^2\right).\end{aligned} \tag{22}$$

Furthermore, the conditional correlation is given by

$$\operatorname{Corr}^{\mathbb{Q}}\left[Y_{T_1}, Y_{T_2}|Y_{t_0}=y\right]=\frac{\operatorname{Cov}^{\mathbb{Q}}\left[Y_{T_1}, Y_{T_2}|Y_{t_0}=y\right]}{\sqrt{\mu_{t_0,T_1}^{\langle 2\rangle}(y)\mu_{t_0,T_2}^{\langle 2\rangle}(y)}}. \tag{23}$$

Here,

$$\mu_{t_0,T_i}^{\langle 2\rangle}(y)=\mathscr{A}_{t_0,T_i}^{\langle 2\rangle}(y)-\left(\mathscr{A}_{t_0,T_i}^{\langle 1\rangle}(y)\right)^2, \qquad i=1,2.$$

Proof: By the definition of the conditional covariance, we have

$$\begin{aligned}\operatorname{Cov}^{\mathbb{Q}}\left[Y_{T_1}, Y_{T_2} \mid Y_{t_0}=y\right]&=E^{\mathbb{Q}}\left[Y_{T_1}Y_{T_2} \mid Y_{t_0}=y\right]\\ &\quad-E^{\mathbb{Q}}\left[Y_{T_1} \mid Y_{t_0}=y\right]E^{\mathbb{Q}}\left[Y_{T_2} \mid Y_{t_0}=y\right].\end{aligned}$$

The first term is obtained from Corollary 4 by setting $n_1 = 1$ and $n_2 = 1$, which gives

$$\begin{aligned}&E^{\mathbb{Q}}\left[Y_{T_1}Y_{T_2} \mid Y_{t_0}=y\right]=\\ &\left(\alpha(T_2)-\alpha(T_1)\,\mathrm{e}^{-\kappa(T_2-T_1)}\right)\mathscr{A}_{t_0,T_1}^{\langle 1\rangle}(y)+\mathrm{e}^{-\kappa(T_2-T_1)}\mathscr{A}_{t_0,T_1}^{\langle 2\rangle}(y).\end{aligned}$$

Moreover,

$$\begin{aligned}E^{\mathbb{Q}}\left[Y_{T_1} \mid Y_{t_0}=y\right]&=\mathscr{A}_{t_0,T_1}^{\langle 1\rangle}(y),\\ E^{\mathbb{Q}}\left[Y_{T_2} \mid Y_{t_0}=y\right]&=\mathscr{A}_{t_0,T_2}^{\langle 1\rangle}(y).\end{aligned}$$

Substituting these expressions proves the first equality in (22). The second equality follows from the identity

$$\mathscr{A}_{t_0,T_2}^{\langle 1\rangle}(y)=\alpha(T_2)+\mathrm{e}^{-\kappa(T_2-T_1)}\left(\mathscr{A}_{t_0,T_1}^{\langle 1\rangle}(y)-\alpha(T_1)\right),$$

which is obtained from the conditional mean formula in Theorem 3. The formula for the conditional correlation in (23) follows immediately from its definition. □

PRICING MOMENT SWAPS

The dynamics of the underlying asset price S_t is assumed to follow the time-dependent mean-reverting model, described by (1). Schoutens [1] defined the annualized realized m-moment, $m \geqslant 2$, based on discrete sampling over the contract duration $[0, T]$ with maturity time $T > 0$, given by

$$\mathrm{MOMS}^{\langle m\rangle}=\frac{AF}{N}\sum_{i=1}^{N}\ln^m\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right),$$

where S_{t_i} are the closing prices of the underlying asset observed at times t_i, for $i = 0, 1, 2, \ldots, N$. Here, N denotes the total number of observations, and AF is the annualization factor that adjusts the calculation to reflect an annualized higher moment. For instance, when the sampling frequency is daily, $AF = 252$, assuming that there are 252 trading days per year; if weekly, then $AF = 52$; and if monthly, then $AF = 12$, with adjustments for other frequencies accordingly.

Typically, we set $T = \frac{N}{AF}$ and assume equally-spaced discrete observations, where $t_i = i\Delta t$ and $\Delta t = t_i - t_{i-1} = T/N > 0$ for all $i = 1, 2, \ldots, N$. Hence, the typical formula for the realized m-moment can be expressed as

$$\mathrm{MOMS}^{\langle m\rangle}=\frac{AF}{N}\sum_{i=1}^{N}\ln^m\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)=\frac{1}{T}\sum_{i=1}^{N}\left(Y_{t_i}-Y_{t_{i-1}}\right)^m,$$

where $Y_t = \ln S_t$ represents the log-price process following the trending O–U dynamics in (2). In Chumpong et al [7], it is shown that the fair strike price for discretely sampled moment swaps can be expressed as

$$\begin{aligned}K^m&=E^{\mathbb{Q}}\left[\mathrm{MOMS}^{\langle m\rangle} \mid Y_0=y_0\right]\\ &=\frac{1}{T}\sum_{i=1}^{N}E^{\mathbb{Q}}\left[\left(Y_{t_i}-Y_{t_{i-1}}\right)^m \mid Y_0=y_0\right]\\ &=\frac{1}{T}\sum_{i=1}^{N}\sum_{k=0}^{m}\binom{m}{k}(-1)^k E^{\mathbb{Q}}\left[Y_{t_{i-1}}^{k}Y_{t_i}^{m-k} \mid Y_0=y_0\right],\end{aligned} \tag{24}$$

where

$$y_0=Y_0=\ln S_0=\frac{\delta_0}{\kappa}.$$

To obtain implementable formulas, we use the one-step transition of the trending O–U process. Let

$$\begin{aligned}\rho&=\mathrm{e}^{-\kappa\Delta t}, & \eta&=\rho-1,\\ \beta_i&=\alpha(t_i)-\rho\,\alpha(t_{i-1}), & \nu&=D(\Delta t).\end{aligned}$$

Then, from $Y_t = \alpha(t) + Z_t$ and the centered O–U dynamics of Z_t, we have

$$Y_{t_i}-Y_{t_{i-1}}=\eta Y_{t_{i-1}}+\beta_i+\varepsilon_i,$$

where, conditionally on $Y_{t_{i-1}}$, the innovation ε_i is normally distributed with mean zero and variance v. Moreover, define

$$\mathscr{A}_i^{\langle r\rangle} = \mathscr{A}_{0,t_i}^{\langle r\rangle}(y_0), \qquad r = 0,1,2,\ldots,$$

with the convention $\mathscr{A}_i^{\langle 0\rangle} = 1$ and $\mathscr{A}_0^{\langle r\rangle} = y_0^r$ for $r \geqslant 1$.

Theorem 5 *Suppose that S_t follows the time-dependent mean-reverting model (1) and $m \geqslant 2$ is an integer. Then, the fair delivery price of the m-moment swap is given by*

$$K^m(T,\Delta t,\delta_0) = \frac{1}{T}\sum_{i=1}^{N}\sum_{r=0}^{\lfloor m/2\rfloor}\frac{m!}{(m-2r)!r!2^r}v^r \times \sum_{\ell=0}^{m-2r}\binom{m-2r}{\ell}\eta^\ell\beta_i^{m-2r-\ell}\mathscr{A}_{i-1}^{\langle\ell\rangle}. \quad (25)$$

Here, $\Delta t = T/N$, $t_i = i\Delta t$, $i = 0,1,\ldots,N$, $\delta_0 = \kappa\ln(S_0)$, and $\mathscr{A}_{i-1}^{\langle\ell\rangle} = \mathscr{A}_{0,t_{i-1}}^{\langle\ell\rangle}(y_0)$.

Proof: From the one-step representation,

$$Y_{t_i} - Y_{t_{i-1}} = \eta Y_{t_{i-1}} + \beta_i + \varepsilon_i,$$

where ε_i is conditionally independent of $Y_{t_{i-1}}$, with

$$E[\varepsilon_i] = 0, \quad E[\varepsilon_i^{2r}] = \frac{(2r)!}{r!2^r}v^r, \quad E[\varepsilon_i^{2r+1}] = 0.$$

Therefore,

$$E^{\mathbb{Q}}\Big[\big(Y_{t_i} - Y_{t_{i-1}}\big)^m \mid Y_0 = y_0\Big] = \sum_{r=0}^{\lfloor m/2\rfloor}\frac{m!}{(m-2r)!r!2^r} \times v^r E^{\mathbb{Q}}\Big[\big(\eta Y_{t_{i-1}} + \beta_i\big)^{m-2r} \mid Y_0 = y_0\Big].$$

Expanding the last power by the binomial theorem gives

$$E^{\mathbb{Q}}\Big[\big(\eta Y_{t_{i-1}}+\beta_i\big)^{m-2r} \mid Y_0 = y_0\Big] = \sum_{\ell=0}^{m-2r}\binom{m-2r}{\ell}\eta^\ell\beta_i^{m-2r-\ell}\mathscr{A}_{i-1}^{\langle\ell\rangle}.$$

Substituting this expression into (24) yields (25). □

The general result in Theorem 5 provides the fair delivery price for a moment swap of arbitrary order $m \geqslant 2$. In particular, when $m = 2,3$, and 4, the resulting contracts are referred to as variance, skewness, and kurtosis swaps, respectively. These products are designed to capture exposure to specific distributional characteristics of asset returns, namely volatility, asymmetry, and tail risk. The following corollaries present the corresponding formulas for $m = 2,3$, and 4. Each formula is written in terms of the conditional moments of $Y_{t_{i-1}}$, which is the information available before the one-step innovation over $[t_{i-1}, t_i]$.

Corollary 6 *According to Theorem 5, when $m = 2$, the fair delivery price of the variance swap is given by*

$$K^2(T,\Delta t,\delta_0) = \frac{1}{T}\sum_{i=1}^{N}\Big[\eta^2\mathscr{A}_{i-1}^{\langle 2\rangle} + 2\eta\beta_i\mathscr{A}_{i-1}^{\langle 1\rangle} + \beta_i^2 + v\Big].$$

Proof: The result follows by setting $m = 2$ in (25). □

Corollary 7 *According to Theorem 5, when $m = 3$, the fair delivery price of the skewness swap is given by*

$$K^3(T,\Delta t,\delta_0) = \frac{1}{T}\sum_{i=1}^{N}\Big[\eta^3\mathscr{A}_{i-1}^{\langle 3\rangle} + 3\eta^2\beta_i\mathscr{A}_{i-1}^{\langle 2\rangle} + 3\eta(\beta_i^2 + v)\mathscr{A}_{i-1}^{\langle 1\rangle} + \beta_i^3 + 3\beta_i v\Big].$$

Proof: The result follows by setting $m = 3$ in (25). □

Corollary 8 *According to Theorem 5, when $m = 4$, the fair delivery price of the kurtosis swap is given by*

$$K^4(T,\Delta t,\delta_0) = \frac{1}{T}\sum_{i=1}^{N}\Big[\eta^4\mathscr{A}_{i-1}^{\langle 4\rangle} + 4\eta^3\beta_i\mathscr{A}_{i-1}^{\langle 3\rangle} + 6\eta^2(\beta_i^2+v)\mathscr{A}_{i-1}^{\langle 2\rangle} + 4\eta(\beta_i^3+3\beta_i v)\mathscr{A}_{i-1}^{\langle 1\rangle} + \beta_i^4 + 6\beta_i^2 v + 3v^2\Big].$$

Proof: The result follows by setting $m = 4$ in (25). □

EXPERIMENTAL VALIDATIONS

In this section, we empirically validate the closed-form formulas derived in Theorem 3 for the conditional moments of the trending log-price process and Corollaries 6–8 for the fair delivery prices of variance, skewness, and kurtosis swaps. The validation is performed by Monte Carlo (MC) simulations based on the exact one-step transition of the centered O–U process. This simulation procedure is consistent with the formulation

$$Y_t = \alpha(t) + Z_t, \qquad \mathrm{d}Z_t = -\kappa Z_t\,\mathrm{d}t + \sigma\,\mathrm{d}W_t,$$

where $Y_t = \ln S_t$ is the log-price process.

Let $\{\widehat{Y}_{t_i}\}_{i=0}^N$ denote a time-discretized sample path of the log-price process Y_t on the interval $[0,T]$, where $0 = t_0 < t_1 < \cdots < t_N = T$ is a uniform partition. Since the centered process $Z_t = Y_t - \alpha(t)$ follows the centered O–U dynamics, the exact one-step update for Y_t is

$$\widehat{Y}_{t_{i+1}} = \alpha(t_{i+1}) + \mathrm{e}^{-\kappa\Delta t}\big(\widehat{Y}_{t_i} - \alpha(t_i)\big) + \sqrt{D(\Delta t)}\,\xi_i,$$

where

$$D(\Delta t) = \big(1 - \mathrm{e}^{-2\kappa\Delta t}\big)\frac{\sigma^2}{2\kappa},$$

$\Delta t = t_{i+1} - t_i$, and $\{\xi_i\}_{i=0}^{N-1}$ are independent standard normal random variables. The initial condition is set as

$$\widehat{Y}_{t_0} = y_0 = \ln S_0 = \frac{\delta_0}{\kappa}.$$

This initial value is consistent with the relation $\delta_0 = \kappa \ln S_0$ and with the payoff definition based on realized log-returns.

We fix the baseline parameters at $t_0 = 0$, $\kappa = 0.099$, and $\sigma = 0.129$. Unless otherwise specified, maturities up to $T = 5$, and each experiment is based on 50,000 MC paths (N_p). By simulating a large number of independent trajectories, we obtain empirical estimates of conditional moments and realized moment swap payoffs, which are then compared with their closed-form counterparts to assess accuracy and robustness. All numerical experiments are conducted using Python on a machine equipped with an 11th Gen Intel® Core™ i7-1165G7 CPU at 2.80 GHz and 8 GB of RAM, running a 64-bit Windows 10 operating system.

Conditional moments

To examine the accuracy of Theorem 3, which provides closed-form formulas for the conditional moments of the trending log-price process, we consider four representative specifications of the time-dependent mean function $\mu(t)$. These functions are selected to illustrate constant (c), linear (ℓ), smooth periodic (s), and periodic piecewise continuous (p) patterns that are relevant in commodity price modeling:

$$\mu_c(t) = \mu + \frac{\sigma^2}{2\kappa}, \qquad \mu_\ell(t) = \mu t,$$
$$\mu_s(t) = \mu + 1.5\sin\left(\frac{\pi}{2} + \pi t\right),$$
$$\mu_p(t) = \begin{cases} \mu + 6\tau, & \tau \in \left[0, \frac{1}{3}\right], \\ \mu + 6\left(\tau - \frac{1}{3}\right), & \tau \in \left(\frac{1}{3}, \frac{2}{3}\right], \\ \mu + 6\left(\tau - \frac{2}{3}\right), & \tau \in \left(\frac{2}{3}, 1\right], \end{cases} \quad \tau = \frac{t}{T} - \left\lfloor \frac{t}{T} \right\rfloor.$$

For each specification of $\mu(t)$, the closed-form value of the n-th conditional moment is computed from

$$\mathscr{A}_{0,T}^{\langle n \rangle}(y_0) = E^{\mathbb{Q}}\left[Y_T^n \mid Y_0 = y_0\right],$$

where the MC estimator is computed from the simulated terminal values by

$$\widehat{\mathscr{A}}_{0,T}^{\langle n \rangle}(y_0) = \frac{1}{N_p}\sum_{p=1}^{N_p}\left(\widehat{Y}_T(\omega_p)\right)^n.$$

Fig. 1 compares the closed-form conditional moments (solid lines) with MC estimates (markers) obtained from the exact one-step simulation scheme. The results are shown for multiple moment orders and for four specifications of $\mu(t)$. The MC estimates (markers) almost perfectly match the closed-form formulas (solid lines) across different configurations and initial values y_0, confirming the accuracy and robustness of the proposed results.

Table 1 Percent relative error summary for variance, skewness, and kurtosis swaps.

Moment swaps	Mean (%)	Median (%)	Std. (%)	P95 (%)	Max (%)
K^2	0.0153	0.0137	0.0110	0.0341	0.0751
K^3	0.0219	0.0190	0.0149	0.0492	0.0899
K^4	0.1104	0.0364	0.4625	0.1755	4.9637

Pricing moment swaps

We next validate Corollaries 6–8, which provide closed-form expressions for the fair delivery prices of variance (K^2), skewness (K^3), and kurtosis (K^4) swaps. The MC estimator of the fair strike is then given by

$$\widehat{K}^m = \frac{1}{N_p}\sum_{p=1}^{N_p}\frac{1}{T}\left(\sum_{i=1}^{N}\left(\widehat{Y}_{t_i}(\omega_p) - \widehat{Y}_{t_{i-1}}(\omega_p)\right)^m\right),$$

for $\omega_p \in \Omega$. This estimator is computed from realized log-returns of the simulated log-price process $Y_t = \ln S_t$, in agreement with the payoff definition in Section "Pricing Moment Swaps".

In the simulations, we fix $N = 1,260$ to discretize the interval $[0, T]$, corresponding to a step size T/N. The trend function is chosen as linear, $\mu(t) = \mu t$, and the initial condition is set to $y_0 = \delta_0/\kappa$, where both δ_0 and T are varied across a grid of values.

Fig. 2 illustrates the fair delivery prices of K^2, K^3, and K^4 swaps across (T, δ_0). The surfaces represent the closed-form results, while the markers denote the MC estimates. The two are visually indistinguishable, confirming the high level of agreement between the closed-form formulas and MC simulation.

To quantify the accuracy of the simulation results, we compute the percentage relative error (PRE) between the MC estimates $\widehat{K}^m$ and the closed-form values K^m, defined as

$$\mathrm{PRE} = \frac{|\widehat{K}^m - K^m|}{|K^m|} \times 100\%.$$

Table 1 summarizes the PRE statistics for the three types of moment swaps. The mean and median PRE values are very small, below 0.03% for K^2 and K^3, confirming the excellent accuracy of the proposed method. For K^4, although the maximum PRE reaches approximately 5% in extreme cases, the median error remains below 0.05%, which is still remarkably precise given the inherent difficulty of estimating fourth-order moments. Overall, these findings validate the robustness and computational efficiency of the closed-form formulas. Together with the visual evidence from Fig. 2, the results confirm that the analytical method achieves both theoretical tractability and numerical reliability in practice.

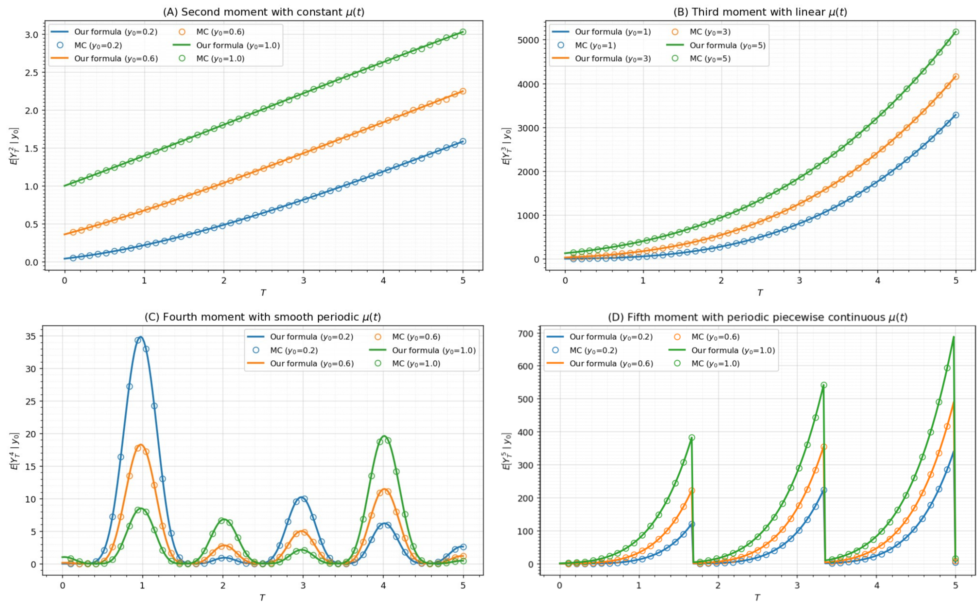

Fig. 1 Comparison of the conditional moments of the trending log-price process obtained from closed-form formulas and MC simulations under various specifications of $\mu(t)$.

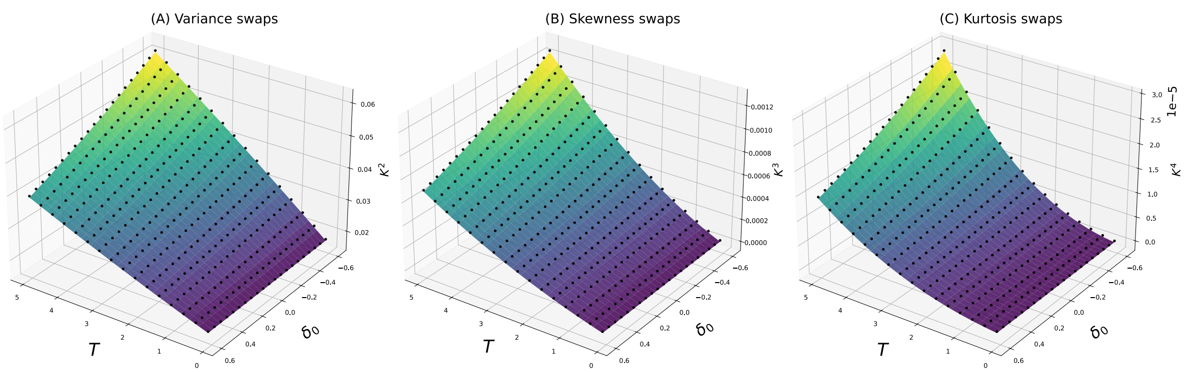

Fig. 2 Fair delivery prices of moment swaps under the linear trend function.

CONCLUSION

This paper presents a closed-form approach for pricing discretely sampled moment swaps in commodity markets under a time-dependent mean-reverting model. The log-price process is formulated as a trending O–U process, and a centered auxiliary O–U process is introduced to derive the required conditional moments in a consistent manner. We first derive the MGF of the standard O–U process and employ complete Bell polynomials together with recurrence relations to obtain closed-form expressions for polynomial-exponential expectations and conditional moments. These results are then transformed back to the trending log-price process, which is the state variable used in the realized log-return payoff.

Based on this formulation, we derive conditional central moments, conditional mixed moments, covariance, and correlation for the trending log-price process. The mixed moment formula is written with the correct binomial expansion and time arguments, which allows the dependence structure between different observation dates to be handled consistently. For moment swap pricing, the fair delivery prices are rederived directly from the realized log-return payoff. In particular, the variance, skewness, and kurtosis swap formulas are expressed through the one-step innovation representation of the log-price increment, ensuring that the conditioning information at each sampling date is used correctly.

The resulting formulas are explicit, interpretable, and computationally efficient. They avoid repeated symbolic expansion while retaining the effects of mean reversion, volatility, and deterministic trends. Monte Carlo simulations based on the same log-price formulation are used to validate the conditional moment formulas and the fair delivery prices of variance, skewness, and kurtosis swaps. The numerical results show strong agreement between the closed-form values and the simulation estimates, supporting the accuracy and practical usefulness of the proposed method.

Overall, the proposed approach provides a consistent analytical tool for evaluating higher-order realized moment risks in commodity markets. It can be applied to variance, skewness, and kurtosis swaps under deterministic trends and may serve as a basis for further developments in commodity risk management, volatility forecasting, and quantitative financial modeling.

Acknowledgements: We gratefully acknowledge the insightful comments from the anonymous referees, which have significantly improved both the quality and clarity of the manuscript. Any remaining errors are solely the responsibility of the authors.

REFERENCES

1. Schoutens W (2005) Moment swaps. *Quant Finance* **5**, 525–530.
2. Rompolis LS, Tzavalis E (2017) Pricing and hedging contingent claims using variance and higher order moment swaps. *Quant Finance* **17**, 531–550.
3. Zhu SP, Lian GH (2012) On the valuation of variance swaps with stochastic volatility. *Appl Math Comput* **219**, 1654–1669.
4. Neuberger A (2012) Realized skewness. *Rev Financ Stud* **25**, 3423–3455.
5. Chumpong K, Mekchay K, Rujivan S, Thamrongrat N (2022) Simple analytical formulas for pricing and hedging moment swaps. *Thai J Math* **20**, 693–713.
6. Weraprasertsakun A, Rujivan S (2017) A closed-form formula for pricing variance swaps on commodities. *Vietnam J Math* **45**, 255–264.
7. Chumpong K, Mekchay K, Thamrongrat N (2021) Analytical formulas for pricing discretely-sampled skewness and kurtosis swaps based on Schwartz's one-factor model. *Songklanakarin J Sci Technol* **43**, 1–6.
8. Chumpong K, Mekchay K, Rujivan S (2020) A simple closed-form formula for the conditional moments of the Ornstein–Uhlenbeck process. *Songklanakarin J Sci Technol* **42**, 836–845.
9. Rujivan S (2021) Analytically pricing variance swaps in commodity derivative markets under stochastic convenience yields. *Commun Math Sci* **19**, 111–146.
10. Duangpan A, Boonklurb R, Rakwongwan U, Sutthimat P (2022) Analytical formulas using affine transformation for pricing generalized swaps in commodity markets with stochastic convenience yields. *Symmetry* **14**, 2385.
11. Schwartz ES (1997) The stochastic behavior of commodity prices: Implications for valuation and hedging. *J Finance* **52**, 922–973.
12. Deng Y, Barros A, Grall A (2013) Residual useful life estimation based on a time-dependent Ornstein–Uhlenbeck process. *Chem Eng Trans* **33**, 325–330.
13. Lo AW, Wang J (1995) Implementing option pricing models when asset returns are predictable. *J Finance* **50**, 87–129.
14. Lucia JJ, Schwartz ES (2002) Electricity prices and power derivatives: Evidence from the Nordic power exchange. *Rev Deriv Res* **5**, 5–50.
15. Zhang B, Grzelak LA, Oosterlee CW (2012) Efficient pricing of commodity options with early-exercise under the Ornstein–Uhlenbeck process. *Appl Numer Math* **62**, 91–111.
16. Chumpong K, Mekchay K, Nualsri F, Sutthimat P (2024) Closed-form formula for the conditional moment-generating function under a regime-switching, nonlinear drift CEV Process, with applications to option pricing. *Mathematics* **12**, 2667.
17. Chumpong K, Mekchay K, Nualsri F, Sutthimat P (2026) Analytical formula for fractional-order conditional moments of nonlinear drift CEV process with regime switching: hybrid approach with applications. *AIMS Math* **11**, 2816–2851.
18. Laohakosol V, Meesa R, Sutthimat P (2023) Closed-form solutions of general second order linear recurrences and applications. *Discrete Math* **346**, 113206.
19. Sutthimat P, Laohakosol V, Meesa R (2024) A unified approach to derive explicit solutions of generalized second-order linear recurrences and applications. *Discrete Math* **347**, 113757.
20. Thamrongrat N, Kanjanasopon K, Rujivan S (2023) An application of solutions of linear difference equations for obtaining the conditional moments of the trending Ornstein–Uhlenbeck processes. *ScienceAsia* **49S**, 59–67.