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# Closed-form fair strikes for innovation-based variance-type and volatility-type contracts under a trending Ornstein–Uhlenbeck model

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**ABSTRACT**: Discrete sampling is fundamental in the valuation of variance-type and volatility-type contracts, but explicit formulas are often lost when sampled log returns are serially dependent. This paper studies this issue in a trending Ornstein–Uhlenbeck model for the log price with deterministic coefficients. In this setting, the usual realized variance based on raw log returns is a quadratic form in correlated Gaussian variables and does not admit the standard chi-square reduction. Exact tractability is recovered by working with the exact one-step innovations of the sampled dynamics. These innovations are independent Gaussian variables on disjoint sampling intervals with explicit variances, so the associated standardized quadratic sum has an exact chi-square distribution and its square root has the corresponding chi distribution. This yields explicit formulas for the density, distribution function, Laplace transform, and moments of the standardized innovation statistic. In particular, the fair strike of the innovation-based variance-type contract is explicit for deterministic coefficients and an arbitrary discrete grid, while closed-form fair strikes for both the innovation-based variance-type and innovation-based volatility-type contracts are obtained in the constant-coefficient case on a uniform grid. Numerical results support the theory and illustrate the resulting fair strikes under several trend and coefficient specifications.

**KEYWORDS**: trending Ornstein–Uhlenbeck model, discrete sampling, standardized innovation statistic, chi-square distribution, variance-type contracts

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## INTRODUCTION

Variance swaps and volatility swaps are discretely sampled volatility derivatives. Their valuation depends not only on the continuous-time dynamics of the underlying price process, but also on the dependence structure of the sampled terms that enter the payoff; see Carr and Lee [1], Demeterfi et al [2], Hull [3]. In settings where the relevant sampled variables can be reduced to independent Gaussian variables, the associated quadratic sum admits a chi-square representation, which in turn leads to explicit formulas for distribution functions, densities, Laplace transforms, moments, and fair strikes. When such independence is absent, this reduction is no longer available, and explicit valuation becomes more difficult. More broadly, issues of discrete sampling and tractable valuation play a central role in variance-type and volatility-type contracts; see Carr and Madan [4], Broadie and Jain [5], Lian et al [6], and Duangpan et al [7].

The present paper studies this problem in a trending Ornstein–Uhlenbeck (OU) specification for the log price, Thamrongrat et al [8],

$$X_t = \ln S_t = \theta(t) + Y_t, \qquad t \in [0, T],$$

where $\theta(t)$ is a deterministic trend and $Y_t$ is a Gaussian mean-reverting component with deterministic coefficients. This decomposition separates a prescribed trend from a linear Gaussian state variable. Since $Y_0$ is deterministic, the process $Y_t$ has a deterministic mean through the transition law, namely $\Phi(t,0)Y_0$, while the mean-reversion speed and the volatility level are allowed to vary with time. This model class is closely related to several classical mean-reverting specifications in finance and commodity modeling. In the constant-trend case, it reduces to the classical OU model used, for example, by Vasicek [9]. In commodity applications, related OU-type and mean-reverting models have been used for spot prices and convenience yields; see Gibson and Schwartz [11], Schwartz [12], and Zhang et al [15]. More generally, extensions with nonconstant deterministic trends have appeared in several forms in the literature. Lo and Wang [10] considered a linear trend in equity applications. In energy applications, deterministic trends or seasonal components, including trigonometric combinations of sine and cosine functions, have been used by Lucia and Schwartz [13] and Wilhelm and Winter [14]. Deng et al [16] considered a nonlinear deterministic trend in a time-dependent OU model for degradation. Recently, Mongkolsin et al [17] considered the time-dependent OU process via a time-fractional Cauchy problem. These examples show that the present method covers the classical OU case and accommodates several deterministic trend specifications used in applications. Hence, the results developed below are not tied to a single choice of $\theta(t)$.

The main difficulty appears after discrete sampling. Standard references on the OU process note that, unlike Wiener motion, its increments are not independent and are in general correlated; see Horsthemke and Lefever [18]. On a discrete grid $0 = t_0 < t_1 < \cdots < t_n = T$, the exact one-step transition of the Gaussian component has the form

$$Y_{t_i} = \Phi(t_i, t_{i-1}) Y_{t_{i-1}} + \eta_i, \qquad i = 1, 2, \ldots, n,$$

where each $\eta_i$ is a Gaussian transition error with explicit variance. Hence, the raw log return over $[t_{i-1}, t_i]$ is

$$\begin{aligned} R_i &= X_{t_i} - X_{t_{i-1}} \\ &= \theta(t_i) - \theta(t_{i-1}) + \big(\Phi(t_i, t_{i-1}) - 1\big) Y_{t_{i-1}} + \eta_i. \end{aligned}$$

This already reveals the source of the difficulty. The raw return is not driven only by the current Gaussian transition error, but also contains the previous state $Y_{t_{i-1}}$, which carries the accumulated effect of earlier shocks. Since $R_i$ depends on $Y_{t_{i-1}}$ and $\eta_i$, whereas $R_{i+1}$ depends on $Y_{t_i}$, the one-step transition above shows that $R_i$ and $R_{i+1}$ share random terms and are therefore correlated in general. Equivalently, the vector of sampled raw returns is jointly Gaussian, but its covariance matrix is typically not diagonal in the natural sampling basis. The usual realized variance based on raw returns is therefore a quadratic form in correlated Gaussian variables, namely

$$\mathrm{RV}_n = \frac{100^2}{T} \sum_{i=1}^{n} R_i^2.$$

This is the quantity that underlies the standard discrete variance swap construction. Such quadratic forms can still be analyzed, but the simple chi-square reduction that supports many closed-form arguments for variance-type and volatility-type payoffs is lost when raw returns are used. This is the precise limitation that motivates the alternative construction developed below. See also Maller et al [19] for the autoregressive viewpoint associated with discretely sampled OU-type processes.

The approach adopted here is to work instead with the exact one-step innovations of the sampled dynamics. These are obtained by subtracting, on each sampling interval, the exact conditional mean implied by the model. In the present setting, the resulting innovations are Gaussian, independent across disjoint sampling intervals, and have explicit interval variances. After normalization, they become independent standard normal variables. Consequently, the associated quadratic sum has an exact chi-square distribution, and its square root has the corresponding chi distribution. This restores tractability on a general discrete grid and yields explicit formulas for the density, distribution function, Laplace transform, and moments of the standardized innovation statistic. The reduction follows directly from the exact transition structure of the model and does not require diagonalization of the covariance matrix of the raw sampled returns.

These probabilistic results have direct valuation consequences. For the innovation-based quadratic variation, the fair strike under the chosen pricing measure $\mathbb{Q}$ is explicit for deterministic coefficients and an arbitrary discrete grid, because the expectation depends only on the sum of the interval innovation variances. For the corresponding innovation-based volatility-type contract, a similarly explicit fair strike is obtained in the constant-coefficient case on a uniform grid, where the innovation variances are equal and the unnormalized innovation-based quadratic variation becomes a scaled chi-square random variable. Thus, the tractability recovered at the probabilistic level translates directly into explicit fair-strike formulas for innovation-based variance-type and volatility-type contracts. The payoff considered in this paper is interpreted within the model as a variance-type quantity built from the exact one-step innovations of the discretely sampled log-price dynamics. Since each innovation is the forecast error obtained after removing the deterministic trend and the conditional mean implied by the previous state, the associated quadratic variation aggregates the contribution of new interval-by-interval shocks over the sampling grid, rather than the total realized variance of raw log returns. Under the chosen pricing measure $\mathbb{Q}$, the corresponding fair strike is therefore interpreted as the model-implied expectation of the unstandardized cumulative innovation-based variation. This interpretation is consistent with the model structure and provides a tractable alternative to the usual raw-return construction in the presence of serial dependence.

The main contribution of the paper is twofold. First, it identifies the exact one-step innovations of the Gaussian component as the discrete-time observables that restore exact tractability in the trending OU log-price model. The standard realized variance based on raw returns does not possess the required independence structure, whereas the exact innovations do through the one-step transition of the model. Second, this innovation construction leads to explicit fair-strike formulas for innovation-based variance-type and volatility-type contracts. In particular, it yields an exact fair strike for the innovation-based variance-type contract under general deterministic coefficients and an arbitrary sampling grid, and closed-form fair strikes for the innovation-based variance-type and innovation-based volatility-type contracts in the constant-coefficient case on a uniform grid.

The remainder of the paper is organized as follows. After introducing the trending OU model and its exact one-step transition on a discrete grid, we construct the innovation-based statistic and prove the exact chi-square representation. We then develop the corresponding distributional results and derive the fair strike of the innovation-based variance-type contract under general deterministic coefficients. The constant-coefficient case on a uniform grid is studied separately, where closed-form fair strikes are obtained for the innovation-based variance-type and volatility-type contracts. Finally, numerical results illustrate the theory under several trend and coefficient specifications, and concluding remarks are given at the end.

## TRENDING ORNSTEIN–UHLENBECK MODEL

Throughout the paper, all probabilistic statements and expectations are taken under a fixed probability measure $\mathbb{Q}$. This measure is the chosen pricing measure used to define the model-implied fair strikes of the innovation-based payoffs considered below. If the model is embedded in a no-arbitrage pricing model, then $\mathbb{Q}$ may be interpreted as a risk-neutral pricing measure. In the present paper, however, no additional martingale restriction on the discounted spot price is imposed, since the exact transition formula, the independence of the innovations, and the chi-square representation do not require such a restriction. Accordingly, the fair strikes derived below are interpreted as model-implied fair strikes under the chosen measure $\mathbb{Q}$.

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbb{Q})$ be a filtered probability space satisfying the usual conditions and supporting a standard Brownian motion $\{W_t^{\mathbb{Q}}\}_{t\in[0,T]}$. Let $\{S_t\}_{t\in[0,T]}$ be a strictly positive process, and define its log price by

$$X_t = \ln S_t = \theta(t) + Y_t, \qquad t \in [0,T]. \tag{1}$$

Here $\theta : [0,T] \to \mathbb{R}$ is a deterministic continuous function, and $Y$ satisfies

$$\mathrm{d}Y_t = -\kappa(t) Y_t \,\mathrm{d}t + \sigma(t)\,\mathrm{d}W_t^{\mathbb{Q}}, \qquad Y_0 = y_0. \tag{2}$$

The functions $\kappa : [0,T] \to \mathbb{R}$ and $\sigma : [0,T] \to \mathbb{R}$ are deterministic, and it is assumed that $\kappa \in L^1([0,T])$ and $\sigma \in L^2([0,T])$. Under these conditions, (2) has a unique strong solution, and $Y$ is a Gaussian process. When $Y$ is interpreted as a mean-reverting component, it is further assumed that $\kappa(t) \geqslant 0$ for almost every $t \in [0,T]$.

The representation (1) separates the deterministic trend from the Gaussian fluctuation. The function $\theta(t)$ specifies the deterministic trend of the log price, while the linear Gaussian dynamics are carried by $Y_t$. This decomposition is convenient because the serial dependence in sampled log returns is generated by the state variable $Y$. For this reason, the later discrete-time analysis is based on the exact transition of $Y$ rather than on the raw increments of $X$.

For $0 \leqslant s \leqslant t \leqslant T$, define

$$\Phi(t,s) = \exp\Big(-\int_s^t \kappa(u)\,\mathrm{d}u\Big). \tag{3}$$

This propagator determines the exact conditional mean and conditional variance of $Y$ over each time interval.

The next lemma gives the exact transition formula and the corresponding conditional law.

**Lemma 1** *For every $0 \leqslant s \leqslant t \leqslant T$,*

$$Y_t = \Phi(t,s) Y_s + \int_s^t \Phi(t,u)\sigma(u)\,\mathrm{d}W_u^{\mathbb{Q}}. \tag{4}$$

*Consequently, conditional on $\mathscr{F}_s$, the random variable $Y_t$ is Gaussian with*

$$\mathbb{E}^{\mathbb{Q}}[Y_t \mid \mathscr{F}_s] = \Phi(t,s) Y_s,$$
$$\mathrm{Var}^{\mathbb{Q}}(Y_t \mid \mathscr{F}_s) = \int_s^t \Phi(t,u)^2 \sigma(u)^2\,\mathrm{d}u.$$

*Equivalently,*

$$Y_t \mid \mathscr{F}_s \sim N\Big(\Phi(t,s)Y_s, \int_s^t \Phi(t,u)^2\sigma(u)^2\,\mathrm{d}u\Big).$$

*Proof*: Define

$$K(t) = \int_0^t \kappa(u)\,\mathrm{d}u, \qquad t \in [0,T].$$

Since $\kappa \in L^1([0,T])$, the function $K$ is absolutely continuous on $[0,T]$. Applying Itô's formula to $\mathrm{e}^{K(t)}Y_t$ gives

$$\mathrm{d}\big(\mathrm{e}^{K(t)}Y_t\big) = \mathrm{e}^{K(t)}\,\mathrm{d}Y_t + \kappa(t)\,\mathrm{e}^{K(t)}Y_t\,\mathrm{d}t.$$

Substituting (2) yields

$$\begin{aligned}\mathrm{d}(\mathrm{e}^{K(t)}Y_t) &= \mathrm{e}^{K(t)}(-\kappa(t)Y_t\mathrm{d}t + \sigma(t)\mathrm{d}W_t^{\mathbb{Q}}) + \kappa(t)\,\mathrm{e}^{K(t)}Y_t\mathrm{d}t \\ &= \mathrm{e}^{K(t)}\sigma(t)\,\mathrm{d}W_t^{\mathbb{Q}}.\end{aligned}$$

Integrating from $s$ to $t$ gives

$$\mathrm{e}^{K(t)}Y_t - \mathrm{e}^{K(s)}Y_s = \int_s^t \mathrm{e}^{K(u)}\sigma(u)\,\mathrm{d}W_u^{\mathbb{Q}}.$$

Multiplying by $\mathrm{e}^{-K(t)}$, one obtains

$$Y_t = \mathrm{e}^{-(K(t)-K(s))}Y_s + \int_s^t \mathrm{e}^{-(K(t)-K(u))}\sigma(u)\,\mathrm{d}W_u^{\mathbb{Q}}.$$

Using (3), this becomes

$$Y_t = \Phi(t,s)Y_s + \int_s^t \Phi(t,u)\sigma(u)\,\mathrm{d}W_u^{\mathbb{Q}},$$

which proves (4). Let

$$I_{s,t} = \int_s^t \Phi(t,u)\sigma(u)\,\mathrm{d}W_u^{\mathbb{Q}}.$$

Since $\Phi(t,\cdot)$ is deterministic and bounded on $[s,t]$, and $\sigma \in L^2([0,T])$, the stochastic integral $I_{s,t}$ is well defined in $L^2$. Moreover, $I_{s,t}$ is Gaussian with mean zero and depends only on the Brownian increments over $[s,t]$. Hence, $I_{s,t}$ is independent of $\mathscr{F}_s$. Therefore, by (4), the conditional law of $Y_t$ given $\mathscr{F}_s$ is Gaussian.

Taking conditional expectation in (4) gives

$$\mathbb{E}^{\mathbb{Q}}[Y_t \mid \mathscr{F}_s] = \Phi(t,s)Y_s.$$

Since $\Phi(t,s)Y_s$ is $\mathscr{F}_s$-measurable,

$$\mathrm{Var}^{\mathbb{Q}}(Y_t \mid \mathscr{F}_s) = \mathrm{Var}^{\mathbb{Q}}(I_{s,t} \mid \mathscr{F}_s).$$

Because $I_{s,t}$ is independent of $\mathscr{F}_s$, its conditional variance equals its unconditional variance. By the Itô isometry,

$$\mathrm{Var}^{\mathbb{Q}}(I_{s,t}) = \mathbb{E}^{\mathbb{Q}}[I_{s,t}^2] = \int_s^t \Phi(t,u)^2\sigma(u)^2\,\mathrm{d}u.$$

Thus,

$$\mathrm{Var}^{\mathbb{Q}}(Y_t \mid \mathscr{F}_s) = \int_s^t \Phi(t,u)^2\sigma(u)^2\,\mathrm{d}u.$$

This gives the stated conditional variance, and the conditional Gaussian law follows immediately. □

A discrete sampling grid is now fixed as $0 = t_0 < t_1 < \cdots < t_n = T$, and we write $\Delta_i = t_i - t_{i-1}$ for $i = 1,2,\dots,n$. Applying Lemma 1 with $s = t_{i-1}$ and $t = t_i$ yields the exact one-step transition on this grid. For later use, define

$$a_i = \Phi(t_i,t_{i-1}) = \exp\left(-\int_{t_{i-1}}^{t_i} \kappa(u)\,\mathrm{d}u\right), \qquad (5)$$

and

$$v_i = \int_{t_{i-1}}^{t_i} \Phi(t_i,u)^2\sigma(u)^2\,\mathrm{d}u, \qquad (6)$$

for $i = 1,2,\dots,n$. The quantity $v_i$ is the variance of the exact one-step innovation on the interval $[t_{i-1},t_i]$. Since the later standardization requires positivity, assume from this point onward that $v_i > 0$ for $i = 1,2,\dots,n$.

With these notations, the exact grid transition takes the form

$$Y_{t_i} = a_iY_{t_{i-1}} + \eta_i, \quad \eta_i = \int_{t_{i-1}}^{t_i} \Phi(t_i,u)\sigma(u)\,\mathrm{d}W_u^{\mathbb{Q}}, \quad (7)$$

for $i = 1,2,\dots,n$. By Lemma 1, each $\eta_i$ is Gaussian with mean zero and variance $v_i$.

The representation (7) shows that the exact conditional mean of $Y_{t_i}$ given $\mathscr{F}_{t_{i-1}}$ is $a_iY_{t_{i-1}}$, while the random deviation from this conditional mean is precisely $\eta_i$. Since $\eta_i$ is defined by a stochastic integral over $[t_{i-1},t_i]$, the collection $\{\eta_i\}_{i=1}^n$ is built from disjoint Brownian increments. This is the basic mechanism that later yields independence.

It is also useful to rewrite the transition in terms of the observable log price. Since $Y_t = X_t - \theta(t)$, relation (7) becomes

$$X_{t_i} - \theta(t_i) = a_i\big(X_{t_{i-1}} - \theta(t_{i-1})\big) + \eta_i, \qquad (8)$$

for $i = 1,2,\dots,n$. This identity shows that the exact one-step innovation on the interval $[t_{i-1},t_i]$ is the deviation from the exact conditional mean in (8). In the next section, this quantity is denoted by $\varepsilon_i$ and is studied as the basic discrete-time observable that yields the exact chi-square reduction.

## EXACT CHI-SQUARE REPRESENTATION

This section introduces the discrete-time statistic associated with the exact one-step innovations of the Gaussian component. By the transition formula in the previous section, these innovations are obtained by subtracting the exact conditional mean on each sampling interval. This removes the dependence present in the raw log returns and leads to an exact chi-square representation. In linear Gaussian analysis, one-step prediction errors, often called innovations, are standard objects associated with conditional means and play an important role in forecasting and state-space methods; see Harvey [20], Durbin and Koopman [21]. In the present model, however, they arise directly from the exact discrete-time transition of the Gaussian component rather than from a separate filtering problem.

For each $i = 1,2,\dots,n$, define

$$\varepsilon_i = Y_{t_i} - a_iY_{t_{i-1}},$$

where $a_i$ is given by (5). Since $Y_t = X_t - \theta(t)$, the same quantity can be written in terms of the observed log prices as

$$\varepsilon_i = X_{t_i} - \theta(t_i) - a_i\left(X_{t_{i-1}} - \theta(t_{i-1})\right). \qquad (9)$$

By (7), it follows that

$$\varepsilon_i = \int_{t_{i-1}}^{t_i} \Phi(t_i,u)\sigma(u)\,\mathrm{d}W_u^{\mathbb{Q}}. \qquad (10)$$

Thus, $\varepsilon_i$ coincides with the exact one-step innovation introduced in the previous section; the notation $\varepsilon_i$ is adopted here to emphasize its role in the chi-square reduction. Hence, $\varepsilon_i \sim N(0,v_i)$ for $i = 1,2,\dots,n$, where $v_i$ is given by (6). Since $v_i > 0$ for $i = 1,2,\dots,n$, each innovation can be standardized.

The next theorem gives the exact chi-square representation by reducing the problem to independent standard normal random variables.

**Theorem 1** *For each $i = 1, 2, \ldots, n$, define*

$$Z_i = \frac{\varepsilon_i}{\sqrt{v_i}}. \tag{11}$$

*Then, $Z_1, Z_2, \ldots, Z_n$ are independent $N(0,1)$ random variables. Consequently, the statistic*

$$Q_n = \sum_{i=1}^{n} Z_i^2 = \sum_{i=1}^{n} \frac{\varepsilon_i^2}{v_i} \tag{12}$$

*has the chi-square distribution with n degrees of freedom. Equivalently,*

$$Q_n = \sum_{i=1}^{n} \frac{\left[X_{t_i} - \theta(t_i) - a_i\big(X_{t_{i-1}} - \theta(t_{i-1})\big)\right]^2}{\displaystyle\int_{t_{i-1}}^{t_i} \Phi(t_i,u)^2 \sigma(u)^2 \,\mathrm{d}u} \sim \chi_n^2.$$

*Proof*: From the representation (10), for each $i = 1, 2, \ldots, n$,

$$\varepsilon_i = \int_{t_{i-1}}^{t_i} \Phi(t_i,u)\sigma(u)\,\mathrm{d}W_u^{\mathbb{Q}}.$$

The function $u \mapsto \Phi(t_i,u)\sigma(u)$ is deterministic and square-integrable on $[t_{i-1}, t_i]$. Therefore, the stochastic integral is well defined. Hence, $\varepsilon_i$ is a Gaussian random variable with mean zero. By Ito's isometry,

$$\mathrm{Var}^{\mathbb{Q}}(\varepsilon_i) = \int_{t_{i-1}}^{t_i} \Phi(t_i,u)^2 \sigma(u)^2 \,\mathrm{d}u = v_i.$$

Therefore, $\varepsilon_i \sim N(0, v_i)$.

It remains to prove independence. For $u \in [0, T]$, define

$$f_i(u) = \Phi(t_i,u)\sigma(u)\mathbf{1}_{[t_{i-1},t_i]}(u).$$

Then

$$\varepsilon_i = \int_0^T f_i(u)\,\mathrm{d}W_u^{\mathbb{Q}}.$$

Since $f_1, f_2, \ldots, f_n$ are deterministic functions, the vector $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ is jointly Gaussian. If $i \neq j$, then the supports of $f_i$ and $f_j$ are disjoint because the intervals $[t_{i-1}, t_i]$ and $[t_{j-1}, t_j]$ are disjoint, and hence

$$\mathrm{Cov}^{\mathbb{Q}}(\varepsilon_i, \varepsilon_j) = \int_0^T f_i(u) f_j(u)\,\mathrm{d}u = 0.$$

Thus, the covariance matrix of $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ is diagonal. Since a jointly Gaussian vector with diagonal covariance matrix has independent components, it follows that $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are independent.

Now define $Z_i$ by (11). Since $v_i > 0$ for $i = 1, 2, \ldots, n$, it follows that $Z_i \sim N(0,1)$ for $i = 1, 2, \ldots, n$. Because each $Z_i$ is obtained from $\varepsilon_i$ by deterministic scaling, the random variables $Z_1, Z_2, \ldots, Z_n$ are also independent. Therefore,

$$Q_n = \sum_{i=1}^{n} Z_i^2$$

has the chi-square distribution with $n$ degrees of freedom. Finally, substituting (9) and (6) into (12) gives the stated observable representation. This completes the proof. □

Theorem 1 identifies the quadratic statistic that is tractable in the present model. The relevant object is not the sum of squared raw log returns, but the sum of squared standardized innovations. This distinction is essential. The raw increments of the log price are correlated, so the corresponding realized variance is a quadratic form in correlated Gaussian variables. By contrast, the statistic $Q_n$ has an exact chi-square distribution on an arbitrary grid under general deterministic coefficients.

Its square-root counterpart is $R_n = \sqrt{Q_n}$, and therefore $R_n \sim \chi_n$. The normalization by $v_i$ in (12) is necessary in the general deterministic setting, since the innovation variances need not be equal across sampling intervals. This produces a dimensionless statistic whose distribution depends only on the number of sampling intervals. In the constant-coefficient case on a uniform grid, all $v_i$ are equal, and the construction becomes simpler. Before turning to the constant-coefficient case, the next section develops the distributional consequences of Theorem 1 and derives the fair strike of the innovation-based variance-type contract under general deterministic coefficients.

## DISTRIBUTIONAL RESULTS AND VARIANCE-TYPE VALUATION

This section develops the consequences of the exact chi-square representation obtained in the previous section. Since the standardized innovation statistic $Q_n$ has a chi-square distribution and its square-root $R_n$ has the corresponding chi distribution, the main distributional formulas follow directly from standard identities. These formulas are useful in their own right and also prepare the valuation results for claims written on the innovation-based quadratic variation. The discussion begins with the standardized statistic, whose distribution is universal under the present assumptions, and then turns to the unstandardized innovation-based quadratic variation, which retains the model-implied scale and is therefore the relevant object for valuation.

The first result records the basic formulas for $Q_n$.

**Theorem 2** *The random variable $Q_n$ has density*

$$f_{Q_n}(q) = \frac{1}{2^{n/2}\Gamma(n/2)}\, q^{n/2-1}\, \mathrm{e}^{-q/2}, \qquad q > 0, \tag{13}$$

*distribution function*

$$F_{Q_n}(q)=\frac{\gamma\left(\frac{n}{2},\frac{q}{2}\right)}{\Gamma(n/2)},\qquad q\geqslant 0,$$

*and Laplace transform*

$$\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-sQ_n}\right]=(1+2s)^{-n/2},\qquad s\geqslant 0.$$

*Moreover, for every $p>-n/2$,*

$$\mathbb{E}^{\mathbb{Q}}[Q_n^p]=2^p\,\frac{\Gamma\left(\frac{n}{2}+p\right)}{\Gamma(n/2)},\ \mathbb{E}^{\mathbb{Q}}[Q_n]=n,\ \mathrm{Var}^{\mathbb{Q}}(Q_n)=2n.$$

*Proof*: By Theorem 1, $Q_n\sim\chi_n^2$. Hence, the formulas above are the standard formulas for the density, distribution function, Laplace transform, mean, variance, and moments of the chi-square distribution with $n$ degrees of freedom. The Laplace transform also follows from the gamma representation $Q_n\sim\Gamma\left(\frac{n}{2},2\right)$, where the first parameter is the shape and the second parameter is the scale. This completes the proof. □

In the present setting, the moment formulas are obtained from exact distributional representations. Related analytical approaches for deriving moments of diffusion-type models based on explicit probability density representations and chi-square structures were investigated in [22], while alternative methodologies based on the Feynman–Kac theorem and associated partial differential equations were studied in [23–25].

The corresponding formulas for the square-root statistic $R_n$ follow immediately.

**Corollary 1** *The random variable $R_n$ has density*

$$f_{R_n}(r)=\frac{2^{1-n/2}}{\Gamma(n/2)}\,r^{n-1}\,\mathrm{e}^{-r^2/2},\qquad r>0,\tag{14}$$

*distribution function*

$$F_{R_n}(r)=\frac{\gamma\left(\frac{n}{2},\frac{r^2}{2}\right)}{\Gamma(n/2)},\qquad r\geqslant 0.$$

*Moreover, for every $p>-n$,*

$$\mathbb{E}^{\mathbb{Q}}[R_n^p]=2^{p/2}\frac{\Gamma\left(\frac{n+p}{2}\right)}{\Gamma(n/2)},\qquad \mathbb{E}^{\mathbb{Q}}[R_n]=\sqrt{2}\,\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n/2)},$$
$$\mathrm{Var}^{\mathbb{Q}}(R_n)=n-2\left(\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n/2)}\right)^2.$$

*Proof*: Since $R_n=\sqrt{Q_n}$ and Theorem 1 gives $Q_n\sim\chi_n^2$, one has $R_n\sim\chi_n$. Hence, the formulas above for the density and distribution function are the standard formulas for the chi distribution with $n$ degrees of freedom. The moment formula also follows from the identity $R_n^p=Q_n^{p/2}$. The expressions for the mean and variance are obtained from the cases $p=1$ and $p=2$. This completes the proof. □

The formulas above describe the standardized statistic and remain exact under general deterministic coefficients and an arbitrary sampling grid. For valuation, however, the relevant object is the unstandardized innovation-based quadratic variation, because it retains the scale implied by the model. Define

$$\mathrm{IV}_n=\frac{100^2}{T}\sum_{i=1}^{n}\varepsilon_i^2.$$

Using (11), this may be rewritten as

$$\mathrm{IV}_n=\frac{100^2}{T}\sum_{i=1}^{n}v_iZ_i^2.\tag{15}$$

When the variances $v_i$ are not all equal, $\mathrm{IV}_n$ is a weighted sum of independent chi-square random variables with one degree of freedom. Its distribution is therefore not, in general, a simple scaled chi-square distribution. Even so, its Laplace transform and first two moments remain explicit, and these are sufficient to determine the fair strike of the associated innovation-based variance-type contract.

It is important to distinguish $\mathrm{IV}_n$ from the standard realized variance based on raw log returns. The latter is given by

$$\mathrm{RV}_n=\frac{100^2}{T}\sum_{i=1}^{n}\left(X_{t_i}-X_{t_{i-1}}\right)^2.$$

The quantity $\mathrm{RV}_n$ is the usual discretely sampled realized variance. By contrast, $\mathrm{IV}_n$ is built from the exact one-step innovations after removing the deterministic trend and the conditional mean implied by the previous state. Thus, $\mathrm{IV}_n$ measures the cumulative contribution of new shocks over the sampling grid within the proposed model. The payoffs considered in this paper are therefore innovation-based variance-type and volatility-type payoffs, rather than standard market payoffs written directly on raw realized variance. The next theorem records the basic identities.

**Theorem 3** *The Laplace transform of $\mathrm{IV}_n$ is*

$$\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-s\,\mathrm{IV}_n}\right]=\prod_{i=1}^{n}\left(1+\frac{2\cdot 100^2}{T}v_is\right)^{-1/2},\qquad s\geqslant 0.\tag{16}$$

*Moreover,*

$$\mathbb{E}^{\mathbb{Q}}[\mathrm{IV}_n]=\frac{100^2}{T}\sum_{i=1}^{n}v_i,\quad \mathrm{Var}^{\mathbb{Q}}(\mathrm{IV}_n)=\frac{100^4}{T^2}2\sum_{i=1}^{n}v_i^2.\tag{17}$$

*Proof*: By (15),

$$\mathrm{IV}_n=\frac{100^2}{T}\sum_{i=1}^{n}v_iZ_i^2,$$

where $Z_1,Z_2,\dots,Z_n$ are independent $N(0,1)$ random variables by Theorem 1. Hence, $Z_i^2\sim\chi_1^2$ for each

$i=1,2,\dots,n$, and the random variables $(100^2/T)\,v_i Z_i^2$ for $i=1,2,\dots,n$ are independent. For $s\geqslant 0$, independence gives

$$\mathbb{E}^{\mathbb{Q}}\big[\mathrm{e}^{-s\,\mathrm{IV}_n}\big]=\prod_{i=1}^{n}\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\frac{100^2}{T}s\,v_i Z_i^2\Big)\Big].$$

Since $Z_i^2\sim\chi_1^2$, its Laplace transform is

$$\mathbb{E}^{\mathbb{Q}}\big[\mathrm{e}^{-uZ_i^2}\big]=(1+2u)^{-1/2},\qquad u\geqslant 0.$$

Applying this with $u=(100^2/T)\,v_i s$ gives the stated Laplace transform in (16). Next, since $\mathbb{E}^{\mathbb{Q}}[Z_i^2]=1$,

$$\mathbb{E}^{\mathbb{Q}}[\mathrm{IV}_n]=\frac{100^2}{T}\sum_{i=1}^{n}v_i\,\mathbb{E}^{\mathbb{Q}}[Z_i^2]=\frac{100^2}{T}\sum_{i=1}^{n}v_i.$$

Finally, by independence,

$$\begin{aligned}\mathrm{Var}^{\mathbb{Q}}(\mathrm{IV}_n)&=\mathrm{Var}^{\mathbb{Q}}\Big(\frac{100^2}{T}\sum_{i=1}^{n}v_i Z_i^2\Big)\\&=\frac{100^4}{T^2}\sum_{i=1}^{n}v_i^2\,\mathrm{Var}^{\mathbb{Q}}(Z_i^2).\end{aligned}$$

Since $Z_i^2\sim\chi_1^2$, one has $\mathrm{Var}^{\mathbb{Q}}(Z_i^2)=2$. This gives the stated variance in (17). This completes the proof. □

These formulas show that the full law of $\mathrm{IV}_n$ depends on the collection of interval variances $v_1,v_2,\dots,v_n$. In particular, unless all $v_i$ are equal, the distribution is not a simple scaled chi-square law. Nevertheless, its expectation remains explicit, and this is exactly the quantity needed for the fair strike of the corresponding innovation-based variance-type contract.

Consider the terminal payoff

$$N_{\mathrm{var}}\big(\mathrm{IV}_n-K_{\mathrm{var}}\big),$$

where $N_{\mathrm{var}}>0$ is the notional and $K_{\mathrm{var}}$ is the strike. Assume that discounting over $[0,T]$ is deterministic, with discount factor $P(0,T)$. Under the pricing measure $\mathbb{Q}$, the model-implied fair strike is defined by the zero-value condition

$$0=P(0,T)\,N_{\mathrm{var}}\,\mathbb{E}^{\mathbb{Q}}\big[\mathrm{IV}_n-K_{\mathrm{var}}\big],$$

or equivalently by

$$K_{\mathrm{var}}=\mathbb{E}^{\mathbb{Q}}[\mathrm{IV}_n].$$

The next result gives the exact formula.

**Theorem 4** *The fair strike of the innovation-based variance-type contract is*

$$\begin{aligned}K_{\mathrm{var}}&=\frac{100^2}{T}\sum_{i=1}^{n}v_i\\&=\frac{100^2}{T}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\Phi(t_i,u)^2\sigma(u)^2\,\mathrm{d}u.\end{aligned}\qquad(18)$$

*If the claim is discounted by a deterministic discount factor $P(0,T)$, then its time-0 value is*

$$V_0^{\mathrm{var}}=P(0,T)\,N_{\mathrm{var}}\Big(\frac{100^2}{T}\sum_{i=1}^{n}v_i-K_{\mathrm{var}}\Big).$$

*Proof*: By Theorem 3, we have $\mathbb{E}^{\mathbb{Q}}[\mathrm{IV}_n]$ in (17). Hence, the fair-strike identity above immediately gives (18). The integral representation follows from the definition of $v_i$ in (6). Under deterministic discounting, the stated time-0 value is the discounted expectation of the terminal payoff. This completes the proof. □

Theorem 4 gives an exact fair strike on a fixed sampling grid. The following remark records its consistency with the corresponding continuous-time integrated variance when the sampling grid becomes fine.

**Remark 1** Let $|\pi_n|$ denote the mesh of the sampling grid, defined by

$$|\pi_n|=\max_{1\leqslant i\leqslant n}(t_i-t_{i-1}).$$

Suppose that $|\pi_n|\to 0$. In the uniform-grid case, this condition is equivalent to $n\to\infty$. Since $\kappa\in L^1([0,T])$, one has

$$\sup_{1\leqslant i\leqslant n}\ \sup_{u\in[t_{i-1},t_i]}\Big|\int_u^{t_i}\kappa(r)\,\mathrm{d}r\Big|\longrightarrow 0.$$

Hence, $\Phi(t_i,u)^2\to 1$ uniformly for $u\in[t_{i-1},t_i]$. Using the definition of $v_i$, it follows that

$$\sum_{i=1}^{n}v_i=\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\Phi(t_i,u)^2\sigma(u)^2\,\mathrm{d}u\longrightarrow\int_0^T\sigma(u)^2\,\mathrm{d}u.$$

Moreover, the innovations are independent Gaussian random variables with mean zero and variances $v_i$. Since $|\pi_n|\to 0$ implies that $\max_{1\leqslant i\leqslant n}v_i\to 0$, the preceding convergence of $\sum_{i=1}^{n}v_i$ gives

$$\mathrm{Var}^{\mathbb{Q}}\Big(\sum_{i=1}^{n}\varepsilon_i^2\Big)=2\sum_{i=1}^{n}v_i^2\leqslant 2\Big(\max_{1\leqslant i\leqslant n}v_i\Big)\sum_{i=1}^{n}v_i\longrightarrow 0.$$

Consequently,

$$\sum_{i=1}^{n}\varepsilon_i^2\xrightarrow{L^2(\mathbb{Q})}\int_0^T\sigma(u)^2\,\mathrm{d}u.$$

It follows from the definition of $\mathrm{IV}_n$ that

$$\mathrm{IV}_n\xrightarrow{L^2(\mathbb{Q})}\frac{100^2}{T}\int_0^T\sigma(u)^2\,\mathrm{d}u.$$

Thus, although $\mathrm{IV}_n$ is different from the raw-return realized variance on a fixed grid, its high-frequency limit agrees with the continuous-time integrated variance generated by the diffusion coefficient $\sigma(t)$.

The result above gives an exact closed-form strike for the innovation-based quadratic payoff under general deterministic coefficients and an arbitrary sampling grid. No diagonalization of a covariance matrix is needed, because the model already yields an independent innovation representation. This is the main valuation consequence of the innovation construction in the general case.

A square-root payoff may also be defined from the same innovation-based quadratic variation by $\mathrm{VOL}_n = \sqrt{\mathrm{IV}_n}$. When the variances $v_i$ are unequal, $\mathrm{IV}_n$ is a weighted sum of independent chi-square random variables with unequal weights. Its expectation remains explicit, which yields the variance-type strike. By contrast, the expectation of $\mathrm{VOL}_n$ no longer reduces to a simple chi moment formula. For this reason, the corresponding innovation-based volatility-type strike does not, in general, admit an equally simple closed-form expression under general deterministic coefficients. The constant-coefficient case on a uniform grid is therefore treated separately in the next section. In that setting, all interval innovation variances are equal, the weighted sum reduces to a scaled chi-square law, and both the innovation-based variance-type and innovation-based volatility-type fair strikes become explicit.

## CONSTANT-COEFFICIENT CASE ON A UNIFORM GRID

This section studies the special case in which the mean-reversion coefficient and the volatility coefficient are constant and the sampling grid is uniform. In this setting, the exact one-step coefficient is the same on every sampling interval and all innovation variances coincide. The general results of the previous sections then reduce to scaled chi-square and scaled chi laws. Consequently, explicit formulas are available not only for the innovation-based quadratic variation, but also for its square root and the associated fair strikes.

Assume that $\kappa(t) \equiv \kappa > 0$ and $\sigma(t) \equiv \sigma > 0$ for $t \in [0, T]$, and let the sampling grid be uniform, $t_i = i\Delta$ for $i = 0, 1, \dots, n$, where $\Delta = T/n$. Then

$$\Phi(t,s) = \mathrm{e}^{-\kappa(t-s)}, \qquad 0 \leqslant s \leqslant t \leqslant T,$$

so the one-step coefficient in (5) is constant over the grid and satisfies

$$a_i = \mathrm{e}^{-\kappa\Delta} = a,$$

for $i = 1, 2, \dots, n$. The innovation variance in (6) is also constant and satisfies

$$v_i = \sigma^2 \int_{t_{i-1}}^{t_i} \mathrm{e}^{-2\kappa(t_i - u)}\,\mathrm{d}u = \frac{\sigma^2}{2\kappa}\left(1 - \mathrm{e}^{-2\kappa\Delta}\right) = v \quad (19)$$

for $i = 1, 2, \dots, n$. Since $\sigma > 0$, $\kappa > 0$, and $\Delta > 0$, one has $v > 0$. The exact grid transition (7) now becomes

$$\begin{aligned} Y_{t_i} &= aY_{t_{i-1}} + \varepsilon_i, \\ \varepsilon_i &= Y_{t_i} - aY_{t_{i-1}} = X_{t_i} - \theta(t_i) - a\big(X_{t_{i-1}} - \theta(t_{i-1})\big), \end{aligned}$$

for $i = 1, 2, \dots, n$. By Theorem 1, the random variables $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ are independent and satisfy $\varepsilon_i \sim N(0, v)$ for $i = 1, 2, \dots, n$.

The next result gives the exact law of the innovation-based quadratic variation in this setting.

**Theorem 5** *In the constant-coefficient case on a uniform grid,*

$$\mathrm{IV}_n = \frac{100^2}{T}\sum_{i=1}^{n} \varepsilon_i^2 = \frac{100^2 v}{T} Q_n \sim \frac{100^2 v}{T}\chi_n^2, \quad (20)$$

*where $v$ is given by* (19). *The random variable $\mathrm{IV}_n$ has density, for $x > 0$,*

$$f_{\mathrm{IV}_n}(x) = \frac{1}{(2 \cdot 100^2 v/T)^{\frac{n}{2}}\,\Gamma(\frac{n}{2})}\, x^{\frac{n}{2}-1} \exp\left(\frac{-Tx}{2 \cdot 100^2 v}\right).$$

*In particular,*

$$\mathbb{E}^{\mathbb{Q}}[\mathrm{IV}_n] = \frac{100^2}{T} nv, \qquad \mathrm{Var}^{\mathbb{Q}}(\mathrm{IV}_n) = \frac{100^4}{T^2}\, 2nv^2.$$

*Proof*: Since all innovation variances are equal, (12) reduces to

$$Q_n = \frac{1}{v}\sum_{i=1}^{n} \varepsilon_i^2.$$

Hence,

$$\mathrm{IV}_n = \frac{100^2}{T}\sum_{i=1}^{n} \varepsilon_i^2 = \frac{100^2 v}{T} Q_n.$$

Let $c = 100^2 v/T$. Then $\mathrm{IV}_n = cQ_n$, which gives (20) by Theorem 1. Since $Q_n$ has density (13) by Theorem 2, the change-of-variables formula gives, for $x > 0$, $f_{\mathrm{IV}_n}(x) = (1/c) f_{Q_n}(x/c)$. Substituting $c = 100^2 v/T$ gives the stated density. Moreover, $\mathbb{E}^{\mathbb{Q}}[\mathrm{IV}_n] = c\,\mathbb{E}^{\mathbb{Q}}[Q_n]$, and $\mathrm{Var}^{\mathbb{Q}}(\mathrm{IV}_n) = c^2\,\mathrm{Var}^{\mathbb{Q}}(Q_n)$. This completes the proof. □

The square-root payoff is equally tractable in the present setting. Since $\mathrm{IV}_n$ is a positive scalar multiple of a chi-square random variable, its square root is a positive scalar multiple of a chi random variable. This yields an equally explicit law for the corresponding innovation-based volatility-type quantity.

**Theorem 6** *In the constant-coefficient case on a uniform grid,*

$$\mathrm{VOL}_n = \sqrt{\mathrm{IV}_n} = \frac{100\sqrt{v}}{\sqrt{T}} R_n \sim \frac{100\sqrt{v}}{\sqrt{T}}\chi_n, \quad (21)$$

*where $v$ is given by* (19). *The random variable $\mathrm{VOL}_n$ has density, for $x > 0$,*

$$f_{\mathrm{VOL}_n}(x) = \frac{2^{1-n/2}}{\Gamma(\frac{n}{2})}\left(\frac{\sqrt{T}}{100\sqrt{v}}\right)^n x^{n-1} \exp\left(-\frac{Tx^2}{2 \cdot 100^2 v}\right).$$

*In particular,*

$$\mathbb{E}^{\mathbb{Q}}[\mathrm{VOL}_n] = \frac{100}{\sqrt{T}}\sqrt{2\nu}\,\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n/2)},$$
$$\mathrm{Var}^{\mathbb{Q}}(\mathrm{VOL}_n) = \frac{100^2}{T}\,\nu\left[n - 2\left(\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n/2)}\right)^2\right].$$

*Proof*: By (20),

$$\mathrm{IV}_n = \frac{100^2\nu}{T}Q_n.$$

Therefore,

$$\mathrm{VOL}_n = \sqrt{\mathrm{IV}_n} = \frac{100\sqrt{\nu}}{\sqrt{T}}\sqrt{Q_n} = \frac{100\sqrt{\nu}}{\sqrt{T}}R_n.$$

Let $c := 100\sqrt{\nu}/\sqrt{T}$. Then $\mathrm{VOL}_n = cR_n$, so (21) follows from Corollary 1. Since $R_n$ has density (14) by Corollary 1, the change-of-variables formula gives, for $x > 0$, $f_{\mathrm{VOL}_n}(x) = (1/c)f_{R_n}(x/c)$. Moreover, $\mathbb{E}^{\mathbb{Q}}[\mathrm{VOL}_n] = c\,\mathbb{E}^{\mathbb{Q}}[R_n]$, and $\mathrm{Var}^{\mathbb{Q}}(\mathrm{VOL}_n) = c^2\,\mathrm{Var}^{\mathbb{Q}}(R_n)$. This completes the proof. □

The model-implied fair strikes of the innovation-based variance-type and volatility-type contracts now take particularly simple forms.

**Corollary 2** *In the constant-coefficient case on a uniform grid,*

$$K_{\mathrm{var}} = \frac{100^2}{T}\,n\nu = \frac{100^2}{T}\,n\frac{\sigma^2}{2\kappa}\left(1 - \mathrm{e}^{-2\kappa T/n}\right), \qquad (22)$$

*and*

$$K_{\mathrm{vol}} = \frac{100}{\sqrt{T}}\sqrt{2\nu}\,\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n/2)}$$
$$= \frac{100}{\sqrt{T}}\sqrt{\frac{\sigma^2}{\kappa}\left(1 - \mathrm{e}^{-2\kappa T/n}\right)}\,\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n/2)}. \qquad (23)$$

*Proof*: The formula for $K_{\mathrm{var}}$ follows from Theorem 5 and (19). The formula for $K_{\mathrm{vol}}$ follows from Theorem 6 and (19). □

This special case shows most clearly how the innovation construction enters the valuation problem. Under constant coefficients and uniform sampling, the innovation-based quadratic variation is exactly a scaled chi-square random variable, and its square root is exactly a scaled chi random variable. Therefore, both model-implied fair strikes are available in closed form. In the general deterministic setting, the variance-type strike remains explicit, but the volatility-type strike no longer has the same simple form because the innovation variances need not be equal. The present section isolates the case in which both payoffs admit the same degree of tractability.

## NUMERICAL VERIFICATION AND SENSITIVITY ANALYSIS

This section reports numerical results for the formulas derived in the previous sections. All sample paths are generated from the exact one-step transition in Lemma 1. Therefore, the numerical comparisons are not affected by any additional time-discretization error. All exact strike curves shown below are the model-implied fair strikes derived under the chosen pricing measure $\mathbb{Q}$.

For $t \in [0, T]$, the deterministic trend used throughout the numerical study is

$$\theta(t) = 0.20t + 0.08\sin(2\pi t/T).$$

In the general deterministic case, the baseline setting is $T = 1$, $y_0 = 0.15$, and $n = 40$, with the uniform grid $t_i = iT/n$ for $i = 0, 1, \ldots, n$, together with

$$\kappa(t) = 4.0 + 0.6\cos(2\pi t/T),$$
$$\sigma(t) = 0.28 + 0.07\sin(2\pi t/T + \pi/6).$$

This setting is used in Fig. 1, Fig. 2, Fig. 3, Fig. 4, Fig. 6, and in the periodic example of Fig. 7. In the constant-coefficient case, the same values of $T$, $y_0$, $n$, and $\theta(t)$ are used together with $\kappa = 2.5$ and $\sigma = 0.30$. This setting is used in Fig. 5. Monte Carlo estimates are computed from $M = 50{,}000$ simulated paths, and the interval coefficients $a_i$ and $\nu_i$ are evaluated with $N_{\mathrm{quad}} = 2001$ quadrature points per interval. In each parameter study, only the parameter shown on the horizontal axis is varied, while all other inputs are kept at their baseline values.

In the plots with error bars labeled Raw RV $\pm 1.96$ SE or $\mathrm{IV}_n \pm 1.96$ SE, the plotted value is the Monte Carlo sample mean, where $\mathrm{SE} = s/\sqrt{M}$ and $s$ is the sample standard deviation of the simulated values. Hence, the bars represent approximate 95% confidence intervals for the corresponding expectation. They measure Monte Carlo uncertainty in the estimated mean and do not describe the spread of individual paths.

Fig. 1 and Fig. 2 compare raw returns with exact innovations on the sampling grid. In Fig. 1(a), the raw-return correlation matrix shows a clear off-diagonal pattern. In Fig. 2(a), the mean absolute correlation is largest at short lags and then decreases gradually. This is consistent with the exact one-step transition in (7), because the raw increments of $X_t = \theta(t) + Y_t$ inherit the serial dependence of the Gaussian state variable. By contrast, Fig. 1(b) shows that the empirical correlation matrix of the innovations is nearly diagonal, and Fig. 2(a) shows that the lag correlations of the innovations remain close to the Monte Carlo noise level. The QQ plot in Fig. 2(b) is also close to the standard normal reference line. These observations agree with Theorem 1, in particular with the standardized innovation defined in (11).

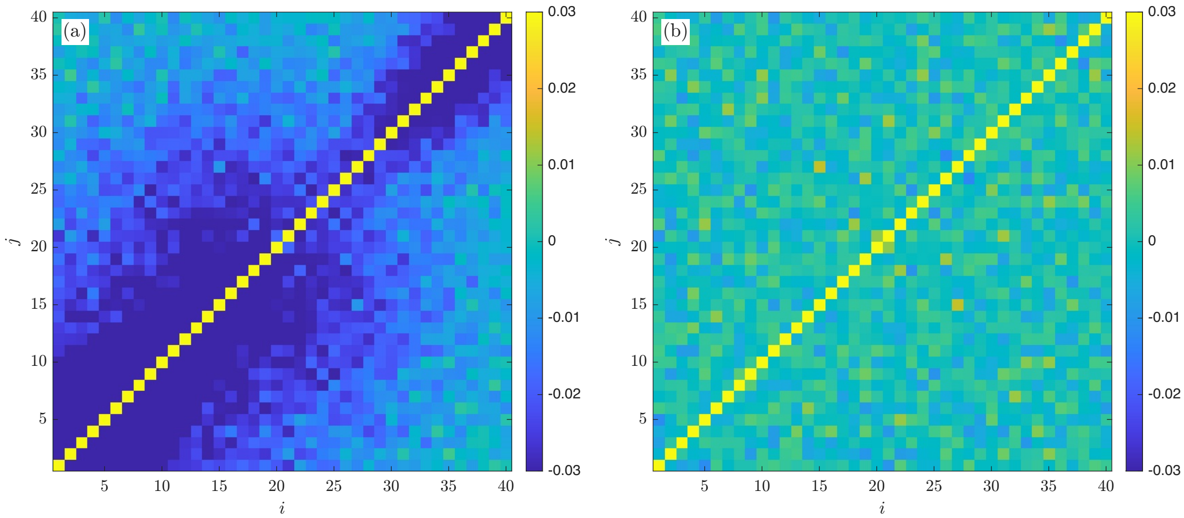


**Fig. 1** Empirical correlation matrices for raw returns and exact innovations.

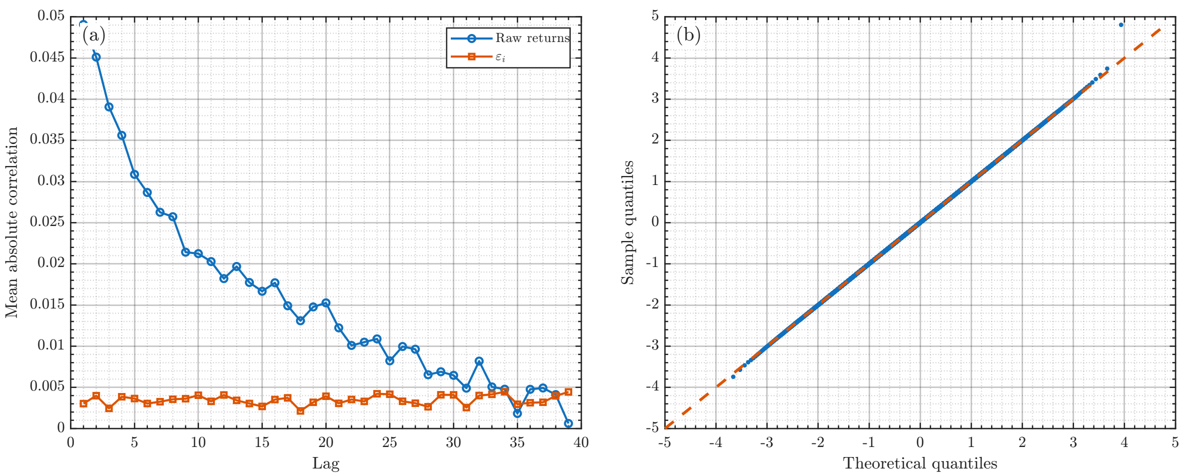


**Fig. 2** Correlation by lag for raw returns and innovations, and QQ plot of standardized innovations.

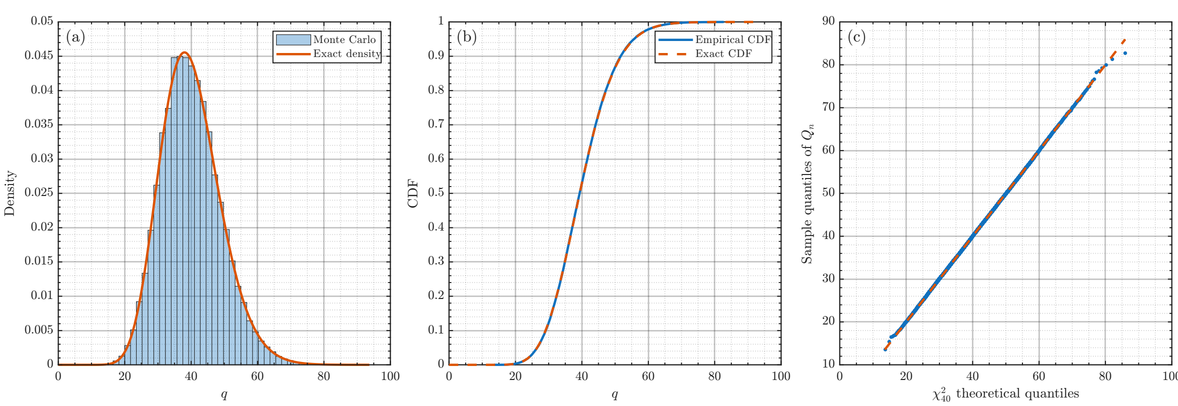


**Fig. 3** Verification of the law of $Q_n$.

Fig. 3 examines the distributional statement in Theorem 1 and the formulas in Theorem 2. In Fig. 3(a), the simulated histogram agrees closely with the $\chi_n^2$ density. In Fig. 3(b), the empirical distribution function follows the exact $\chi_n^2$ distribution function. In Fig. 3(c), the QQ plot stays close to the reference line. The numerical evidence is therefore consistent with the definition of $Q_n$ in (12) and with its chi-square law. Since $R_n = \sqrt{Q_n}$, this also agrees with Corollary 1.

Fig. 4 examines the variance-type strike in the general deterministic case. In Fig. 4(a), the exact values and Monte Carlo estimates agree over all tested choices of $n$. In Fig. 4(b), they again agree when the amplitude of $\kappa(t)$ varies. In Fig. 4(c), the same agreement is observed when the level of $\sigma(t)$ varies. The remaining differences are of the same order as the Monte Carlo uncertainty indicated by the error bars. This agrees with Theorem 4, or equivalently with the explicit formula (18), and with Theorem 3. The parameter dependence is explained by the interval variance formula (6). In Fig. 4(a), the exact strike increases with $n$. In this experiment, a finer sampling grid captures the local variance structure more accurately, and the discrete sum $\sum_{i=1}^{n} v_i$ moves upward toward its continuous-time limit $100^2 \int_0^T \sigma(u)^2 \,\mathrm{d}u$. In Fig. 4(b), when the amplitude of $\kappa(t)$ increases, the damping factor in (6) becomes smaller on average, so the interval variances decrease and $K_{\mathrm{var}}$ decreases as well. In Fig. 4(c), when the level of $\sigma(t)$ increases, the factor $\sigma(u)^2$ in (6) increases, so both $v_i$ and $K_{\mathrm{var}}$ increase.

Fig. 5 examines the constant-coefficient formula for the volatility-type strike. In Fig. 5(a), the exact values and Monte Carlo estimates agree over all tested choices of $n$. In Fig. 5(b), they remain in close agreement as $\kappa$ varies. In Fig. 5(c), the same agreement is observed as $\sigma$ varies. This is consistent with Theorem 5, which shows that $\mathrm{IV}_n$ has a scaled chi-square law in the constant-coefficient case on a uniform grid, Theorem 6, which gives the corresponding scaled chi law of $\mathrm{VOL}_n$, and Corollary 2, which gives the closed-form fair strikes. Since the same $\theta(t)$ is used here as in the general deterministic case, the figure also shows that the constant-case formulas depend on the common innovation variance (19) and not on the particular choice of deterministic trend. By (22) and (23), the decrease of $v$ with respect to $\kappa$ in Fig. 5(b) implies that both $K_{\mathrm{var}}$ and $K_{\mathrm{vol}}$ decrease as the mean-reversion speed increases. Their dependence on $\sigma$ is also immediate from (19). In particular, Fig. 5(c) shows that $K_{\mathrm{var}}$ is proportional to $\sigma^2$, whereas $K_{\mathrm{vol}}$ is proportional to $\sigma$. This explains why the variance-type strike rises more steeply in $\sigma$, while the volatility-type strike is close to linear in $\sigma$. As $n \to \infty$, one has $v \sim \sigma^2 T/n$, and therefore $K_{\mathrm{var}} \to 100^2\sigma^2$ and $K_{\mathrm{vol}} \to 100\sigma$, which explains the asymptotic limit lines in Fig. 4(a) and Fig. 5(a).

Fig. 6 examines the effect of the deterministic trend $\theta(t)$. In these experiments, the trend specification is varied while the remaining inputs are fixed at the general deterministic baseline. The three one-parameter variations are chosen to represent simple linear, seasonal, and nonlinear deterministic trend effects, consistent with deterministic trend specifications used in related applications; see Deng et al [16], Lo and Wang [10], Lucia and Schwartz [13]. Fig. 6(a) shows the effect of the linear trend parameter, Fig. 6(b) shows the effect of the sinusoidal amplitude, and Fig. 6(c) shows the effect of the quadratic coefficient. Across all three panels, the raw-return statistic changes visibly under these variations. By contrast, the innovation-based statistic changes very little and remains close to the exact level. The error bars show that the remaining variation in the innovation-based means is of the same order as the Monte Carlo uncertainty. This is explained by the observable innovation formula (9). The deterministic trend enters the observed log price $X_t$, but it is removed when the exact one-step conditional mean is subtracted. It is also important that the interval variances $v_i$ in (6) depend on $\kappa(t)$, $\sigma(t)$, and the grid, but not on $\theta(t)$. Therefore, Theorem 1 and Theorem 4 are consistent with the numerical observation in Fig. 6(a–c) that the innovation-based statistic is essentially insensitive to the choice of trend, whereas the raw-return statistic is not.

Fig. 7 examines the effect of time-varying coefficients. The periodic example shown in Fig. 7(a–c) uses the same baseline functions $\kappa(t)$ and $\sigma(t)$ as above. A piecewise-constant example, shown in Fig. 7(d–f), is also included in order to generate stronger variation in the interval variances. In that case,

$$\kappa(t) = \begin{cases} 1.2, & 0 \leqslant t < T/3, \\ 4.8, & T/3 \leqslant t < 2T/3, \\ 2.0, & 2T/3 \leqslant t \leqslant T, \end{cases}$$

$$\sigma(t) = \begin{cases} 0.18, & 0 \leqslant t < T/3, \\ 0.40, & T/3 \leqslant t < 2T/3, \\ 0.24, & 2T/3 \leqslant t \leqslant T. \end{cases}$$

Fig. 7(b) and Fig. 7(e) show that the interval variances $v_1, v_2, \ldots, v_n$ given by (6) are clearly unequal in both examples. Hence, $\mathrm{IV}_n$ takes the weighted form (15) and is therefore a weighted sum of independent chi-square terms. This agrees with Theorem 3. In particular, the laws displayed in Fig. 7(c) and Fig. 7(f) are no longer the simple scaled chi-square laws obtained in Theorem 5, because the equal-variance condition fails.

Fig. 7(a–f) also explain the different shapes of the histograms. In the periodic case, Fig. 7(a) and Fig. 7(b) show that the smooth oscillation of $\kappa(t)$ and $\sigma(t)$ produces a smooth variation in $v_i$, and Fig. 7(c) shows the resulting law of $\mathrm{IV}_n$. In the piecewise-constant case, Fig. 7(d) and Fig. 7(e) show that the abrupt changes in the coefficients generate

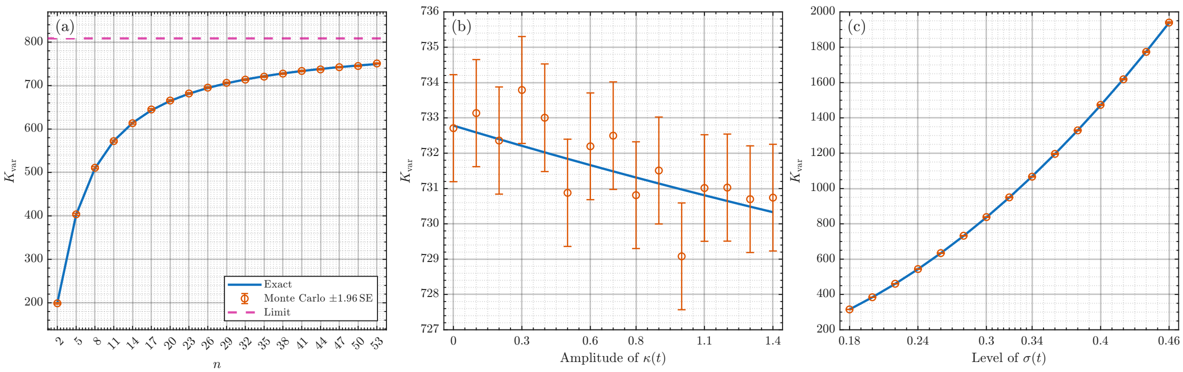


**Fig. 4** Verification of $K_{\mathrm{var}}$ in the general deterministic case.

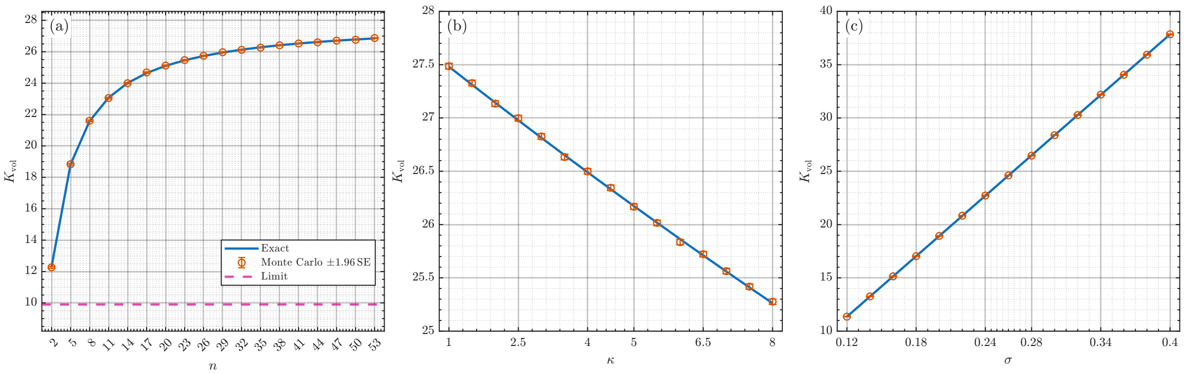


**Fig. 5** Verification of $K_{\mathrm{vol}}$ in the constant-coefficient case.

larger contrasts among the interval variances, and Fig. 7(f) shows the resulting law of $\mathrm{IV}_n$. This leads to a more heterogeneous weighted sum in $\mathrm{IV}_n$. The moment-matched scaled chi-square curve provides a useful approximation near the center, but it does not reproduce the full law exactly. This agrees with Theorem 3, which gives the exact mean and variance of $\mathrm{IV}_n$, but does not reduce the unequal-variance case to a single scaled chi-square law.

Table 1 supports the formulas for the mean and variance in Theorem 3. In both examples, the empirical mean and variance remain close to the exact values. Thus, the unequal-variance case does not preserve a simple closed form for the full distribution, but it still preserves exact formulas for the first two moments. This is precisely the level of tractability stated in Theorem 3 and Theorem 4.

Taken together, the numerical results are consistent with the main theoretical results of the paper. Fig. 1(a) and Fig. 2(a) agree with the serial dependence of raw returns, whereas Fig. 1(b) and Fig. 2(a,b) agree with the independence and standardization re-

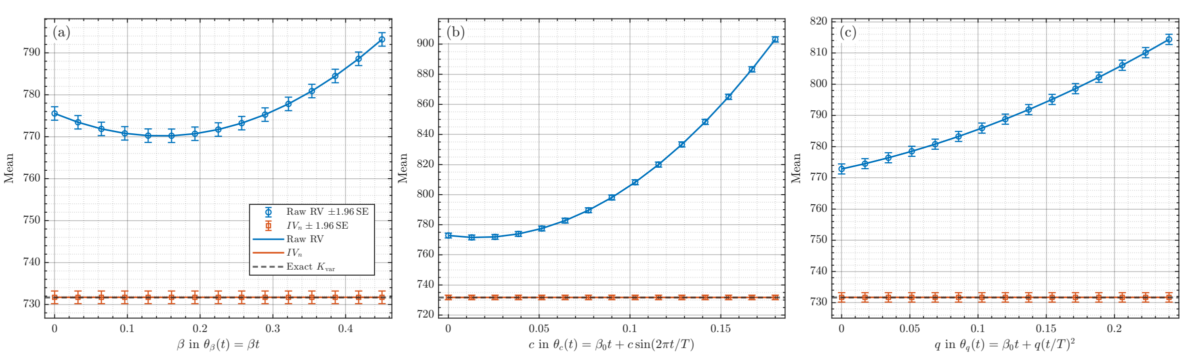


**Fig. 6** Effect of the trend parameters on raw RV and $\mathrm{IV}_n$.

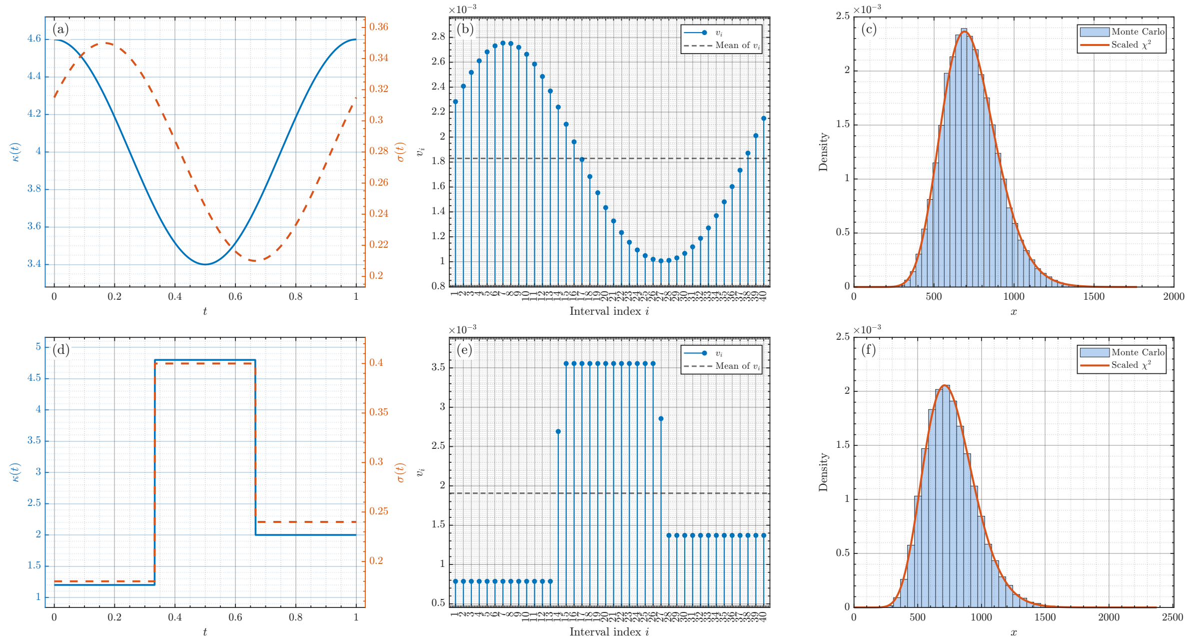


**Fig. 7** Time-varying coefficients, interval variances, and the law of $\mathrm{IV}_n$.

**Table 1** Exact and empirical moments of $\mathrm{IV}_n$ under the periodic and piecewise-constant coefficient choices.

| Scenario | Moment | Exact | Empirical | Relative error (%) |
|---|---|---|---|---|
| Periodic coefficients | Mean | 731.66 | 731.70 | 0.0056 |
| | Variance | 29838.30 | 29623.45 | 0.7201 |
| Piecewise-constant coefficients | Mean | 762.57 | 763.09 | 0.0677 |
| | Variance | 39921.89 | 40160.57 | 0.5979 |

sults in Theorem 1. Fig. 3(a–c) agree with Theorem 1, Corollary 1, and Theorem 2. Fig. 4(a–c) and Fig. 5(a–c) agree with Theorem 4, Corollary 2, Theorem 5, and Theorem 6. Fig. 6(a–c) and Fig. 7(a–f) clarify the roles of $\theta(t)$, $\kappa(t)$, and $\sigma(t)$. The deterministic trend affects the raw-return statistic, but not the innovation-based one. Time variation in $\kappa(t)$ and $\sigma(t)$ changes the interval variances and therefore changes the law of $\mathrm{IV}_n$. From a financial viewpoint, these results show that the innovation-based strike is driven by the model-implied variance of new shocks rather than by deterministic trend movements or by the serial dependence contained in raw returns. The sensitivity to $\kappa(t)$ and $\sigma(t)$ therefore has a direct interpretation through the interval innovation variances $v_i$: stronger mean reversion reduces the contribution of past shocks, while a higher volatility level increases the innovation variance and hence the fair strike.

## CONCLUSION

This paper studies a trending OU model for the log price under discrete sampling. The main difficulty is that raw log returns are correlated, so the usual realized variance based on raw returns does not admit the standard chi-square reduction. This difficulty is resolved by working with the exact one-step innovations of the Gaussian component. The resulting innovation construction restores exact tractability and leads directly to explicit valuation results. Under general deterministic coefficients and an arbitrary discrete grid, the innovation-based variance-type contract admits an exact model-implied fair strike under the chosen pricing measure $\mathbb{Q}$. In the constant-coefficient case on a uniform grid, the innovation-based quadratic variation is a scaled chi-square random variable and its square root is a scaled chi random variable. Consequently, closed-form fair strikes are obtained for both the innovation-based variance-type and innovation-based volatility-type contracts.

The payoffs considered here should be interpreted within the model rather than as standard market contracts written directly on raw log returns. Since they are built from cumulative one-step innovations of the sampled log-price dynamics, the associated quadratic variation represents innovation-based variation across the sampling grid rather than the total realized variance of the observed path. This distinction clarifies both the meaning of the resulting fair strikes and the source of the exact tractability established in this paper. Several directions remain for future work. One is to study the same innovation-based construction when the deterministic coefficients are estimated from data rather than treated as known inputs. Another direction is to analyze weighted innovation sums and related payoffs under more general sampling schemes. It would be interesting to investigate whether similar tractable innovation-based constructions can be developed for richer OU-type models, such as stochastic-volatility OU models or Lévy-driven OU dynamics, together with calibration procedures based on market data.

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