RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2026.045

ScienceAsia 52 (3): 2026: ID 2026045: **1–8**

Non-global nonlinear mixed Jordan triple derivations on $*$-algebras

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Received 21 Jul 2025, Accepted 6 May 2026
Available online 29 May 2026

ABSTRACT: Let $\mathscr{A}$ be a unital $*$-algebra with a nontrivial projection P satisfying $X\mathscr{A}P = 0$ implies $X = 0$ and $X\mathscr{A}(I-P) = 0$ implies $X = 0$. In this paper, it is shown that if a map $\delta : \mathscr{A} \to \mathscr{A}$ satisfies $\delta(A \circ B \bullet C) = \delta(A) \circ B \bullet C + A \circ \delta(B) \bullet C + A \circ B \bullet \delta(C)$ for any $A, B, C \in \mathscr{A}$ with $ABC^* = 0$, then δ is an additive $*$-derivation on $\mathscr{A}$.

KEYWORDS: mixed Jordan triple derivation, additive $*$-derivation, $*$-algebras

MSC2020: 47L30 47B47 16W25

INTRODUCTION

Let $\mathscr{A}$ be an $*$-algebra over the complex field $\mathbb{C}$. For $A, B \in \mathscr{A}$, the Jordan product and skew Jordan product are defined by $A \circ B = AB + BA$ and $A \bullet B = AB + BA^*$, respectively.A map δ: $\mathscr{A} \to \mathscr{A}$ is called an additive derivation if it satisfies

$$\delta(A+B) = \delta(A) + \delta(B)$$

and

$$\delta(AB) = \delta(A)B + B\delta(A)$$

for all $A, B \in \mathscr{A}$. Furthermore, δ is called an additive $*$-derivation if δ also satisfies $\delta(A^*) = \delta(A)^*$ for all $A \in \mathscr{A}$. A map δ is called a global nonlinear skew Jordan triple derivation if

$$\delta(A \bullet B \bullet C) = \delta(A) \bullet B \bullet C + A \bullet \delta(B) \bullet C + A \bullet B \bullet \delta(C)$$

for all $A, B, C \in \mathscr{A}$. Zhao and Li [1] proved that every global nonlinear skew Jordan triple derivation on von Neumann algebras is an additive $*$-derivation. Darvish et al [2] extended this study to global nonlinear skew Jordan triple derivations on prime $*$-algebras. In recent years, derivations related to various new mixed products [3–6] have attracted many authors' attention. δ is called a global nonlinear mixed Jordan triple derivation if

$$\delta(A \circ B \bullet C) = \delta(A) \circ B \bullet C + A \circ \delta(B) \bullet C + A \circ B \bullet \delta(C)$$

for all $A, B, C \in \mathscr{A}$. Nadeem et al [7] proved that every global nonlinear mixed Jordan triple derivation on $*$-algebras is an additive $*$-derivation.

Recently, the local action on some proper subsets of operator algebras [8, 9] can completely determined the structure of maps on operator algebras. Liu [10] investigated any linear map δ from a factor $\mathscr{M}$ into to itself satisfying $\delta([[A, B], C]) = [[\delta(A), B], C] + [[A, \delta(B)], C] + [[A, B], \delta(C)]$ for any $A, B, C \in \mathscr{A}$ with $AB = 0$ (resp., $AB = P$, where P is a fixed non-trivial projection of $\mathscr{M}$). Zhao and Hao [11] gave the concrete structure of a map δ from a finite von Neumann algebra $\mathscr{M}$ with no central summands of type I_1 to itself satisfying $\delta([[A, B], C]) = [[\delta(A), B], C] + [[A, \delta(B)], C] + [[A, B], \delta(C)]$ for all $A, B, C \in \mathscr{A}$ with $ABC = 0$. Let $F : \mathscr{A} \times \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ be map and $\mathscr{Q}$ be proper subset $\mathscr{A}$. If δ satisfies

$$\delta(A \circ B \bullet C) = \delta(A) \circ B \bullet C + A \circ \delta(B) \bullet C + A \circ B \bullet \delta(C)$$

for all $A, B, C \in \mathscr{A}$ with $F(A, B, C) \in \mathscr{Q}$, then δ is called a non-global nonlinear mixed Jordan triple derivation. Motivated by the above works, this paper will investigate a kind of non-global nonlinear mixed Jordan triple derivation on $*$-algebras satisfying

$$\delta(A \circ B \bullet C) = \delta(A) \circ B \bullet C + A \circ \delta(B) \bullet C + A \circ B \bullet \delta(C)$$

for any $A, B, C \in \mathscr{A}$ with $ABC^* = 0$.

MAIN RESULT

In this section, we will prove the following theorem.

Theorem 1 *Let $\mathscr{A}$ be a unital $*$-algebra with unit I with a nontrivial projection P satisfying*

$$X\mathscr{A}P = 0 \Rightarrow X = 0 \qquad (\blacktriangle)$$

and

$$X\mathscr{A}(I-P) = 0 \Rightarrow X = 0. \qquad (\blacktriangledown)$$

If a map $\delta : \mathscr{A} \to \mathscr{A}$ satisfies

$$\delta(A \circ B \bullet C) = \delta(A) \circ B \bullet C + A \circ \delta(B) \bullet C + A \circ B \bullet \delta(C)$$

for any $A, B, C \in \mathscr{A}$ with $ABC^* = 0$, then δ is an additive $*$-derivation.

Write $P_1 = P$ and $P_2 = I - P_1$. Put $A_{ij} = P_i\mathscr{A}P_j$, $i, j = 1, 2$. Then $\mathscr{A} = \mathscr{A}_{11} + \mathscr{A}_{12} + \mathscr{A}_{21} + \mathscr{A}_{22}$. For any $A \in \mathscr{A}$, we have $A = A_{11} + A_{12} + A_{21} + A_{22}$, where $A_{ij} \in \mathscr{A}_{ij}$.

Lemma 1 $\delta(0)=0$.

Proof: It is clear $000^*=0$, and so

$$\begin{aligned}\delta(0)&=\delta(0\circ 0\bullet 0)\\&=\delta(0)\circ 0\bullet 0+0\circ\delta(0)\bullet 0+0\circ 0\bullet\delta(0)=0.\end{aligned}$$

Lemma 2 *For every $A_{12}\in\mathscr{A}_{12}$, $B_{21}\in\mathscr{A}_{21}$, we have*

$$\delta(A_{12}+B_{21})=\delta(A_{12})+\delta(B_{21}).$$

Proof: Let $T=\delta(A_{12}+B_{21})-\delta(A_{12})-\delta(B_{21})$. We next prove that $T=0$. Since $(A_{12}+B_{21})P_2P_1^*=A_{12}P_2P_1^*=B_{21}P_2P_1^*=0$ and $A_{12}\circ P_2\bullet P_1=0$, we have from Lemma 1 that

$$\begin{aligned}&\delta(A_{12}+B_{21})\circ P_2\bullet P_1+(A_{12}+B_{21})\circ\delta(P_2)\bullet P_1\\&\qquad+(A_{12}+B_{21})\circ P_2\bullet\delta(P_1)\\&=\delta((A_{12}+B_{21})\circ P_2\bullet P_1)\\&=\delta(A_{12}\circ P_2\bullet P_1)+\delta(B_{21}\circ P_2\bullet P_1)\\&=\big(\delta(A_{12})+\delta(B_{21})\big)\circ P_2\bullet P_1\\&\quad+(A_{12}+B_{21})\circ\delta(P_2)\bullet P_1+(A_{12}+B_{21})\circ P_2\bullet\delta(P_1),\end{aligned}$$

which implies that $T\circ P_2\bullet P_1=0$, and so

$$P_2TP_1+P_1T^*P_2=0. \tag{1}$$

Multiplying Eq. (1) by P_1 from the right, we get $T_{21}=0$. Similarly, we can show $T_{12}=0$.

From $P_1(A_{12}+B_{21})P_1^*=P_1A_{12}P_1^*=P_1B_{21}P_1^*=0$ and $P_1\circ A_{12}\bullet P_1=0$, we have

$$\begin{aligned}&\delta(P_1)\circ(A_{12}+B_{21})\bullet P_1+P_1\circ\delta(A_{12}+B_{21})\bullet P_1\\&\qquad+P_1\circ(A_{12}+B_{21})\bullet\delta(P_1)\\&=\delta(P_1\circ(A_{12}+B_{21})\bullet P_1)\\&=\delta(P_1\circ A_{12}\bullet P_1)+\delta(P_1\circ B_{21}\bullet P_1)\\&=\delta(P_1)\circ(A_{12}+B_{21})\bullet P_1+P_1\circ(\delta(A_{12})\\&\quad+\delta(B_{21}))\bullet P_1+P_1\circ(A_{12}+B_{21})\bullet\delta(P_1).\end{aligned}$$

It follows that $P_1\circ T\bullet P_1=0$, and so

$$P_1TP_1+TP_1+P_1T^*P_1+P_1T^*=0. \tag{2}$$

Since $(iP_1)(A_{12}+B_{21})P_1^*=(iP_1)A_{12}P_1^*=(iP_1)B_{21}P_1^*=0$ and $(iP_1)\circ A_{12}\bullet P_1=0$, we have

$$\begin{aligned}&\delta(iP_1)\circ(A_{12}+B_{21})\bullet P_1+(iP_1)\circ\delta(A_{12}+B_{21})\bullet P_1\\&\qquad+(iP_1)\circ(A_{12}+B_{21})\bullet\delta(P_1)\\&=\delta((iP_1\circ(A_{12}+B_{21})\bullet P_1)\\&=\delta((iP_1)\circ A_{12}\bullet P_1)+\delta((iP_1)\circ B_{21}\bullet P_1)\\&=\delta(iP_1)\circ(A_{12}+B_{21})\bullet P_1+(iP_1)\circ(\delta(A_{12})\\&\quad+\delta(B_{21}))\bullet P_1+(iP_1)\circ(A_{12}+B_{21})\bullet\delta(P_1).\end{aligned}$$

This implies that $(iP_1)\circ T\bullet P_1=0$, and so

$$P_1TP_1+TP_1-P_1T^*P_1-P_1T^*=0. \tag{3}$$

Summing Eqs. (2) and (3), we obtain

$$P_1TP_1+TP_1=0. \tag{4}$$

Multiplying Eq. (4) by P_1 from the left, we get $T_{11}=0$. Similarly, we can show $T_{22}=0$. Hence $T=0$. □

Lemma 3 *For every $A_{ii}\in\mathscr{A}_{ii}$, $B_{ij}\in\mathscr{A}_{ij}$, $C_{ji}\in\mathscr{A}_{ji}$ $(1\leqslant i\neq j\leqslant 2)$, we have*
(i) $\delta(A_{ii}+B_{ij})=\delta(A_{ii})+\delta(B_{ij})$.
(ii) $\delta(A_{ii}+C_{ji})=\delta(A_{ii})+\delta(C_{ji})$.

Proof: *(i)*: Let $T=\delta(A_{11}+B_{12})-\delta(A_{11})-\delta(B_{12})$. Since $P_2(A_{11}+B_{12})P_1^*=P_2A_{11}P_1^*=P_2B_{12}P_1^*=0$ and $P_2\circ A_{11}\bullet P_1=0$, we have

$$\begin{aligned}&\delta(P_2)\circ(A_{11}+B_{12})\bullet P_1+P_2\circ\delta(A_{11}+B_{12})\bullet P_1\\&\qquad+P_2\circ(A_{11}+B_{12})\bullet\delta(P_1)\\&=\delta(P_2\circ(A_{11}+B_{12})\bullet P_1)\\&=\delta(P_2\circ A_{11}\bullet P_1)+\delta(P_2\circ B_{12}\bullet P_1)\\&=\delta(P_2)\circ(A_{11}+B_{12})\bullet P_1+P_2\circ(\delta(A_{11})\\&\quad+\delta(B_{12}))\bullet P_1+P_2\circ(A_{11}+B_{12})\bullet\delta(P_1).\end{aligned}$$

This implies that $P_2\circ T\bullet P_1=0$, and so

$$P_2TP_1+P_1T^*P_2=0. \tag{5}$$

Multiplying Eq. (5) by P_1 from the right, we get $T_{21}=0$.

Since $P_2(A_{11}+B_{12})P_2^*=P_2A_{11}P_2^*=P_2B_{12}P_2^*=0$ and $P_2\circ A_{11}\bullet P_2=0$, we have

$$\begin{aligned}&\delta(P_2)\circ(A_{11}+B_{12})\bullet P_2+P_2\circ\delta(A_{11}+B_{12})\bullet P_2\\&\qquad+P_2\circ(A_{11}+B_{12})\bullet\delta(P_2)\\&=\delta(P_2\circ(A_{11}+B_{12})\bullet P_2)\\&=\delta(P_2\circ A_{11}\bullet P_2)+\delta(P_2\circ B_{12}\bullet P_2)\\&=\delta(P_2)\circ(A_{11}+B_{12})\bullet P_2+P_2\circ(\delta(A_{11})\\&\quad+\delta(B_{12}))\bullet P_2+P_2\circ(A_{11}+B_{12})\bullet\delta(P_2).\end{aligned}$$

It follows that $P_2\circ T\bullet P_2=0$, and so

$$P_2TP_2+TP_2+P_2T^*P_2+P_2T^*=0. \tag{6}$$

Multiplying Eq. (6) by P_1 from the left, we get $T_{12}=0$.

From $(iP_2)(A_{11}+B_{12})P_2^*=(iP_2)A_{11}P_2^*=(iP_2)B_{12}P_2^*=0$ and $(iP_2)\circ A_{11}\bullet P_2=0$, we have

$$\begin{aligned}&\delta(iP_2)\circ(A_{11}+B_{12})\bullet P_2+(iP_2)\circ\delta(A_{11}+B_{12})\bullet P_2\\&\qquad+(iP_2)\circ(A_{11}+B_{12})\bullet\delta(P_2)\\&=\delta\big((iP_2)\circ(A_{11}+B_{12})\bullet P_2\big)\\&=\delta\big((iP_2)\circ A_{11}\bullet P_2\big)+\delta\big((iP_2)\circ B_{12}\bullet P_2\big)\\&=\delta(iP_2)\circ(A_{11}+B_{12})\bullet P_2+(iP_2)\circ(\delta(A_{11})\\&\quad+\delta(B_{12}))\bullet P_2+(iP_2)\circ(A_{11}+B_{12})\bullet\delta(P_2),\end{aligned}$$

which implies that $(iP_2)\circ T\bullet P_2=0$, and so

$$P_2TP_2+TP_2-P_2T^*P_2-P_2T^*=0. \tag{7}$$

Summing Eqs. (6) and (7), we obtain $T_{22}=0$.

Let $V_{A_{11},B_{12}}=T_{11}$. Then $V_{A_{11},B_{12}}\in\mathscr{A}_{11}$, and so

$$\delta(A_{11}+B_{12})=\delta(A_{11})+\delta(B_{12})+V_{A_{11},B_{12}}, \quad (8)$$

for every $A_{11}\in\mathscr{A}_{11}, B_{12}\in\mathscr{A}_{12}$.

Since $(A_{11}+B_{12})P_1X_{12}^*=A_{11}P_1X_{12}^*=B_{12}P_1X_{12}^*=0$ and $A_{11}\circ P_1\bullet X_{12}=2A_{11}X_{12}, B_{12}\circ P_1\bullet X_{12}=X_{12}B_{12}^*$, we have from Eq. (8) that there exists $V_{X_{12}B_{12}^*,2A_{11}X_{12}}\in\mathscr{A}_{11}$ such that

$$\begin{aligned}
&\delta(A_{11}+B_{12})\circ P_1\bullet X_{12}+(A_{11}+B_{12})\circ\delta(P_1)\bullet X_{12}\\
&\qquad+(A_{11}+B_{12})\circ P_1\bullet\delta(X_{12})\\
&=\delta((A_{11}+B_{12})\circ P_1\bullet X_{12})\\
&=\delta(A_{11}\circ P_1\bullet X_{12}+B_{12}\circ P_1\bullet X_{12})\\
&=\delta(2A_{11}X_{12}+X_{12}B_{12}^*)\\
&=\delta(X_{12}B_{12}^*)+\delta(2A_{11}X_{12})+V_{X_{12}B_{12}^*,2A_{11}X_{12}}\\
&=\delta(A_{11}\circ P_1\bullet X_{12})+\delta(B_{12}\circ P_1\bullet X_{12})+V_{X_{12}B_{12}^*,2A_{11}X_{12}}\\
&=(\delta(A_{11})+\delta(B_{12}))\circ P_1\bullet X_{12}+(A_{11}+B_{12})\circ\delta(P_1)\bullet X_{12}\\
&\quad+(A_{11}+B_{12})\circ P_1\bullet\delta(X_{12})+V_{X_{12}B_{12}^*,2A_{11}X_{12}}.
\end{aligned}$$

It follows that $T\circ P_1\bullet X_{12}=V_{X_{12}B_{12}^*,2A_{11}X_{12}}$, and so

$$TX_{12}+P_1TX_{12}+X_{12}T^*P_1=V_{X_{12}}B^*_{12,2}A_{111}X_{12}. \quad (9)$$

Multiplying Eq. (9) by P_1 from the left and by P_2 from the right, and using $V_{X_{12}B_{12}^*,2A_{11}X_{12}}\in\mathscr{A}_{11}$, we obtain $P_1TX_{12}=0$, and so $T_{11}=0$ by (▼).

Similarly, we can show that $\delta(A_{22}+B_{21})=\delta(A_{22})+\delta(B_{21})$.

(ii): Let $T=\delta(A_{11}+C_{21})-\delta(A_{11})-\delta(C_{21})$. Since $P_1(A_{11}+C_{21})P_2^*=P_1A_{11}P_2^*=P_1C_{21}P_2^*=0$ and $P_1\circ A_{11}\bullet P_2=0$, we have

$$\begin{aligned}
&\delta(P_1)\circ(A_{11}+C_{21})\bullet P_2+P_1\circ\delta(A_{11}+C_{21})\bullet P_2\\
&\qquad+P_1\circ(A_{11}+C_{21})\bullet\delta(P_2)\\
&=\delta(P_1\circ(A_{11}+C_{21})\bullet P_2)\\
&=\delta(P_1\circ A_{11}\bullet P_2)+\delta(P_1\circ C_{21}\bullet P_2)\\
&=\delta(P_1)\circ(A_{11}+C_{21})\bullet P_2+P_1\circ(\delta(A_{11})\\
&\quad+\delta(C_{21}))\bullet P_2+P_1\circ(A_{11}+C_{21})\bullet\delta(P_2).
\end{aligned}$$

It follows that $P_1\circ T\bullet P_2=0$, and so

$$P_1TP_2+P_2T^*P_1=0. \quad (10)$$

Multiplying Eq. (10) by P_2 from the right, we get $T_{12}=0$. Similarly, we can show $T_{21}=0$.

It follows from $P_2(A_{11}+C_{21})P_2^*=P_2A_{11}P_2^*=P_2C_{21}P_2^*=0$ and $P_2\circ A_{11}\bullet P_2=0$ that

$$\begin{aligned}
&\delta(P_2)\circ(A_{11}+C_{21})\bullet P_2+P_2\circ\delta(A_{11}+C_{21})\bullet P_2\\
&\qquad+P_2\circ(A_{11}+C_{21})\bullet\delta(P_2)\\
&=\delta(P_2\circ(A_{11}+C_{21})\bullet P_2)\\
&=\delta(P_2\circ A_{11}\bullet P_2)+\delta(P_2\circ C_{21}\bullet P_2)\\
&=\delta(P_2)\circ(A_{11}+C_{21})\bullet P_2+P_2\circ(\delta(A_{11})\\
&\quad+\delta(C_{21}))\bullet P_2+P_2\circ(A_{11}+C_{21})\bullet\delta(P_2).
\end{aligned}$$

This implies that $P_2\circ T\bullet P_2=0$, and so

$$P_2TP_2+TP_2+P_2T^*P_2+P_2T^*=0. \quad (11)$$

Since $(iP_2)(A_{11}+C_{21})P_2^*=(iP_2)A_{11}P_2^*=(iP_2)C_{21}P_2^*=0$ and $(iP_2)\circ A_{11}\bullet P_2=0$, we have

$$\begin{aligned}
&\delta(iP_2)\circ(A_{11}+C_{21})\bullet P_2+(iP_2)\circ\delta(A_{11}+C_{21})\bullet P_2\\
&\qquad+(iP_2)\circ(A_{11}+C_{21})\bullet\delta(P_2)\\
&=\delta((iP_2)\circ(A_{11}+C_{21})\bullet P_2)\\
&=\delta((iP_2)\circ A_{11}\bullet P_2)+\delta((iP_2)\circ C_{21}\bullet P_2)\\
&=\delta(iP_2)\circ(A_{11}+C_{21})\bullet P_2+(iP_2)\circ(\delta(A_{11})\\
&\quad+\delta(C_{21}))\bullet P_2+(iP_2)\circ(A_{11}+C_{21})\bullet\delta(P_2).
\end{aligned}$$

This implies that $(iP_2)\circ T\bullet P_2=0$. Then

$$P_2TP_2+TP_2-P_2T^*P_2-P_2T^*=0. \quad (12)$$

Summing Eqs. (11) and (12), we obtain $T_{22}=0$.

For any $X_{21}\in\mathscr{A}_{21}$, since $(A_{11}+C_{21})X_{21}P_1^*=A_{11}X_{21}P_1^*=C_{21}X_{21}P_1^*=0$ and $C_{21}\circ X_{21}\bullet P_1=0$, we have

$$\begin{aligned}
&\delta(A_{11}+C_{21})\circ X_{21}\bullet P_1+(A_{11}+C_{21})\circ\delta(X_{21})\bullet P_1\\
&\qquad+(A_{11}+C_{21})\circ X_{21}\bullet\delta(P_1)\\
&=\delta((A_{11}+C_{21})\circ X_{21}\bullet P_1)\\
&=\delta(A_{11}\circ X_{21}\bullet P_1)+\delta(C_{21}\circ X_{21}\bullet P_1)\\
&=(\delta(A_{11})+\delta(C_{21}))\circ X_{21}\bullet P_1\\
&\quad+(A_{11}+C_{21})\circ\delta(X_{21})\bullet P_1+(A_{11}+C_{21})\circ X_{21}\bullet\delta(P_1),
\end{aligned}$$

which implies that $T\circ X_{21}\bullet P_1=0$, and so

$$TX_{21}+X_{21}TP_1+X_{21}^*T^*+P_1T^*X_{21}^*=0. \quad (13)$$

Multiplying Eq. (13) by P_2 from the left and by P_1 from the right, and using the fact that $T_{22}=0$, we obtain $X_{21}TP_1=0$. Then $P_1T^*X_{21}^*=0$, and so $T_{11}=0$ by (▼).

Similarly, we can show that $\delta(A_{22}+C_{12})=\delta(A_{22})+\delta(C_{12})$. □

Lemma 4 *For every $A_{ij},B_{ij}\in\mathscr{A}_{ij}$ $(1\leqslant i\neq j\leqslant 2)$, we have*

$$\delta(A_{ij}+B_{ij})=\delta(A_{ij})+\delta(B_{ij}).$$

Proof: Let $T=\delta(A_{ij}+B_{ij})-\delta(A_{ij})-\delta(B_{ij})$. Since $P_i(A_{ij}+B_{ij})P_i^*=P_iA_{ij}P_i^*=P_iB_{ij}P_i^*=0$ and $P_i\circ(A_{ij}+B_{ij})\bullet P_i=0$, we have

$$\begin{aligned}
&\delta(P_i)\circ(A_{ij}+B_{ij})\bullet P_i+P_i\circ\delta(A_{ij}+B_{ij})\bullet P_i\\
&\qquad+P_i\circ(A_{ij}+B_{ij})\bullet\delta(P_i)\\
&=\delta(P_i\circ(A_{ij}+B_{ij})\bullet P_i)\\
&=\delta(P_i\circ A_{ij}\bullet P_i)+\delta(P_i\circ B_{ij}\bullet P_i)\\
&=\delta(P_i)\circ(A_{ij}+B_{ij})\bullet P_i+P_i\circ(\delta(A_{ij})+\delta(B_{ij}))\bullet P_i\\
&\quad+P_i\circ(A_{ij}+B_{ij})\bullet\delta(P_i),
\end{aligned}$$

which implies that $P_i \circ T \bullet P_i = 0$, and so

$$P_i T P_i + T P_i + P_i T^* P_i + P_i T^* = 0. \tag{14}$$

From $(iP_i)(A_{ij}+B_{ij})P_i^* = (iP_i)A_{ij}P_i^* = (iP_i)B_{ij}P_i^* = 0$ and $(iP_i)\circ(A_{ij}+B_{ij})\bullet P_i = 0$, we have

$$\begin{aligned}\delta(iP_i)\circ(A_{ij}+B_{ij})\bullet P_i + (iP_i)\circ\delta(A_{ij}+B_{ij})\bullet P_i \\ + (iP_i)\circ(A_{ij}+B_{ij})\bullet\delta(P_i) \\ = \delta((iP_i)\circ(A_{ij}+B_{ij})\bullet P_i) \\ = \delta((iP_i)\circ A_{ij}\bullet P_i) + \delta((iP_i)\circ B_{ij}\bullet P_i) \\ = \delta(iP_i)\circ(A_{ij}+B_{ij})\bullet P_i + (iP_i)\circ(\delta(A_{ij}) \\ + \delta(B_{ij}))\bullet P_i + (iP_i)\circ(A_{ij}+B_{ij})\bullet\delta(P_i).\end{aligned}$$

It follows that $(iP_i)\circ T\bullet P_i = 0$. Then

$$P_i T P_i + T P_i - P_i T^* P_i - P_i T^* = 0. \tag{15}$$

Summing Eqs. (14) and (15), we obtain

$$P_i T P_i + T P_i = 0. \tag{16}$$

Multiplying Eq. (16) by P_i from the left, we get $T_{ii} = 0$. Multiplying Eq. (16) by P_j from the left, we get $T_{ji} = 0$.

By $X_{ij}(A_{ij}+B_{ij})P_j^* = X_{ij}A_{ij}P_j^* = X_{ij}B_{ij}P_j^* = 0$ and $X_{ij}\circ(A_{ij}+B_{ij})\bullet P_j = 0$, we have

$$\begin{aligned}\delta(X_{ij})\circ(A_{ij}+B_{ij})\bullet P_j + X_{ij}\circ\delta(A_{ij}+B_{ij})\bullet P_j \\ + X_{ij}\circ(A_{ij}+B_{ij})\bullet\delta(P_j) \\ = \delta(X_{ij}\circ(A_{ij}+B_{ij})\bullet P_j) \\ = \delta(X_{ij}\circ A_{ij}\bullet P_j) + \delta(X_{ij}\circ B_{ij}\bullet P_j) \\ = \delta(X_{ij})\circ(A_{ij}+B_{ij})\bullet P_j + X_{ij}\circ(\delta(A_{ij}) \\ + \delta(B_{ij}))\bullet P_j + X_{ij}\circ(A_{ij}+B_{ij})\bullet\delta(P_j)\end{aligned}$$

Then $X_{ij}\circ T\bullet P_j = 0$, and so

$$X_{ij} T P_j + T X_{ij} + P_j T^* X_{ij}^* + X_{ij}^* T^* = 0. \tag{17}$$

Multiplying Eq. (17) by P_i from the left, we obtain $X_{ij}TP_j + P_iTX_{ij} = 0$. Then by the fact $T_{ii} = 0$, we have $X_{ij}TP_j = 0$, and so $T_{jj} = 0$.

From $(P_i + A_{ij}^*)(P_j + B_{ij}^*)P_i^* = 0$ and $(P_i + A_{ij}^*)\circ(P_j + B_{ij}^*)\bullet P_i = A_{ij}^* + B_{ij}^* + A_{ij} + B_{ij}$, and using Lemmas 2 and 3, we have

$$\begin{aligned}&\delta(A_{ij}^*+B_{ij}^*) + \delta(A_{ij}+B_{ij}) = \delta((P_i+A_{ij}^*)\circ(P_j+B_{ij}^*)\bullet P_i) \\ &= \delta(P_i+A_{ij}^*)\circ(P_j+B_{ij}^*)\bullet P_i + (P_i+A_{ij}^*)\circ\delta(P_j \\ &\quad + B_{ij}^*)\bullet P_i + (P_i+A_{ij}^*)\circ(P_j+B_{ij}^*)\bullet\delta(P_i) \\ &= (\delta(P_i)+\delta(A_{ij}^*))\circ(P_j+B_{ij}^*)\bullet P_i + (P_i+A_{ij}^*)\circ(\delta(P_j) \\ &\quad + \delta(B_{ij}^*))\bullet P_i + (P_i+A_{ij}^*)\circ(P_j+B_{ij}^*)\bullet\delta(P_i) \\ &= \delta(P_i\circ P_j\bullet P_i) + \delta(P_i\circ B_{ij}^*\bullet P_i) \\ &\quad + \delta(A_{ij}^*\circ P_j\bullet P_i) + \delta(A_{ij}^*\circ B_{ij}^*\bullet P_i) \\ &= \delta(B_{ij}+B_{ij}^*) + \delta(A_{ij}+A_{ij}^*) \\ &= \delta(B_{ij}^*) + \delta(A_{ij}^*) + \delta(A_{ij}) + \delta(B_{ij}).\end{aligned} \tag{18}$$

Let $U_{A_{ij},B_{ij}} = T_{ij}$. Then $U_{A_{ij},B_{ij}} \in \mathscr{A}_{ij}$, and so

$$\delta(A_{ij}+B_{ij}) = \delta(A_{ij}) + \delta(B_{ij}) + U_{A_{ij},B_{ij}}, \tag{19}$$

for every $A_{ij}, B_{ij} \in \mathscr{A}_{ij}$.

Similarly, there exists $W_{A_{ij}^*,B_{ij}^*} \in \mathscr{A}_{ji}$ such that

$$\delta(A_{ij}^*+B_{ij}^*) = \delta(A_{ij}^*) + \delta(B_{ij}^*) + W_{A_{ij}^*,B_{ij}^*}, \tag{20}$$

for every $A_{ij}, B_{ij} \in \mathscr{A}_{ij}$.

From Eqs. (18)–(20), we have

$$W_{A_{ij}^*,B_{ij}^*} + U_{A_{ij},B_{ij}} = 0. \tag{21}$$

Multiplying Eq. (21) by P_j from the right, then by the fact that $W_{A_{ij}^*,B_{ij}^*} \in \mathscr{A}_{ji}$, we obtain $T_{ij} = U_{A_{ij},B_{ij}} = 0$. Therefore, $\delta(A_{ij}+B_{ij}) = \delta(A_{ij}) + \delta(B_{ij})$. □

Lemma 5 *For every $A_{ii}, B_{ii} \in \mathscr{A}_{ii}$ $(i = 1, 2)$, we have*

$$\delta(A_{ii}+B_{ii}) = \delta(A_{ii}) + \delta(B_{ii}).$$

Proof: Let $T = \delta(A_{ii}+B_{ii}) - \delta(A_{ii}) - \delta(B_{ii})$. Since $P_jP_j(A_{ii}+B_{ii})^* = P_jP_jA_{ii}^* = P_jP_jB_{ii}^* = 0$ and $P_j\circ P_j\bullet A_{ii} = 0$, we have

$$\begin{aligned}\delta(P_j)\circ P_j\bullet(A_{ii}+B_{ii}) + P_j\circ\delta(P_j)\bullet(A_{ii}+B_{ii}) \\ + P_j\circ P_j\bullet\delta(A_{ii}+B_{ii}) \\ = \delta(P_j\circ P_j\bullet(A_{ii}+B_{ii})) \\ = \delta(P_j\circ P_j\bullet A_{ii}) + \delta(P_j\circ P_j\bullet B_{ii}) \\ = \delta(P_j)\circ P_j\bullet(A_{ii}+B_{ii}) + P_j\circ\delta(P_j)\bullet(A_{ii}+B_{ii}) \\ + P_j\circ P_j\bullet(\delta(A_{ii}) + \delta(B_{ii})),\end{aligned}$$

which implies that $P_j\circ P_j\bullet T = 0$, and so

$$P_j T + T P_j = 0. \tag{22}$$

Multiplying Eq. (22) by P_j from both sides, we get $T_{jj} = 0$. Multiplying Eq. (22) by P_i from the right, we get $T_{ji} = 0$. Multiplying Eq. (22) by P_i from the left, we obtain $T_{ij} = 0$.

For any $X_{ij} \in \mathscr{A}_{ij}$ with $1 \leqslant i \neq j \leqslant 2$, since $P_i(A_{ii}+B_{ii})X_{ij}^* = P_iA_{ii}X_{ij}^* = P_iB_{ii}X_{ij}^* = 0$, we have from Lemma 4 that

$$\begin{aligned}\delta(P_i)\circ(A_{ii}+B_{ii})\bullet X_{ij} + P_i\circ\delta(A_{ii}+B_{ii})\bullet X_{ij} \\ + P_i\circ(A_{ii}+B_{ii})\bullet\delta(X_{ij}) \\ = \delta(P_i\circ(A_{ii}+B_{ii})\bullet X_{ij}) \\ = \delta(2A_{ii}X_{ij} + 2B_{ii}X_{ij}) \\ = \delta(2A_{ii}X_{ij}) + \delta(2B_{ii}X_{ij}) \\ = \delta(P_i\circ A_{ii}\bullet X_{ij}) + \delta(P_i\circ B_{ii}\bullet X_{ij}) \\ = \delta(P_i)\circ(A_{ii}+B_{ii})\bullet X_{ij} + P_i\circ(\delta(A_{ii}) \\ + \delta(B_{ii}))\bullet X_{ij} + P_i\circ(A_{ii}+B_{ii})\bullet\delta(X_{ij}).\end{aligned}$$

It follows that $P_i \circ T \bullet X_{ij} = 0$, and so

$$P_i T X_{ij} + T X_{ij} + X_{ij} T^* P_i = 0. \tag{23}$$

Multiplying Eq. (23) by P_i from the left and by P_j from the right, we get $P_i T X_{ij} = 0$, and so $T_{ii} = 0$. Hence, $\delta(A_{ii} + B_{ii}) = \delta(A_{ii}) + \delta(B_{ii})$. □

Lemma 6 *For every $A_{11} \in \mathscr{A}_{11}$, $B_{12} \in \mathscr{A}_{12}$, $C_{21} \in \mathscr{A}_{21}$, $D_{22} \in \mathscr{A}_{22}$, we have*
(i) $\delta(A_{11} + B_{12} + C_{21}) = \delta(A_{11}) + \delta(B_{12}) + \delta(C_{21})$;
(ii) $\delta(B_{12} + C_{21} + D_{22}) = \delta(B_{12}) + \delta(C_{21}) + \delta(D_{22})$.

Proof: *(i)*: Let $T = \delta(A_{11}+B_{12}+C_{21}) - \delta(A_{11}) - \delta(B_{12}) - \delta(C_{21})$. Since $(A_{11} + B_{12} + C_{21})P_1P_2^* = A_{11}P_1P_2^* = B_{12}P_1P_2^* = C_{21}P_1P_2^* = 0$ and $A_{11} \circ P_1 \bullet P_2 = C_{21} \circ P_1 \bullet P_2 = 0$, we have

$$\begin{aligned}
&\delta(A_{11}+B_{12}+C_{21}) \circ P_1 \bullet P_2 + (A_{11}+B_{12}+C_{21}) \circ \delta(P_1) \bullet P_2 \\
&\qquad + (A_{11}+B_{12}+C_{21}) \circ P_1 \bullet \delta(P_2) \\
&= \delta((A_{11}+B_{12}+C_{21}) \circ P_1 \bullet P_2) \\
&= \delta(A_{11} \circ P_1 \bullet P_2) + \delta(B_{12} \circ P_1 \bullet P_2) + \delta(C_{21} \circ P_1 \bullet P_2) \\
&= (\delta(A_{11}) + \delta(B_{12}) + \delta(C_{21})) \circ P_1 \bullet P_2 + (A_{11} + B_{12} \\
&\quad + C_{21}) \circ \delta(P_1) \bullet P_2 + (A_{11}+B_{12}+C_{21}) \circ P_1 \bullet \delta(P_2).
\end{aligned}$$

This implies that $T \circ P_1 \bullet P_2 = 0$, and so

$$P_1 T P_2 + P_2 T^* P_1 = 0. \tag{24}$$

Multiplying Eq. (24) by P_2 from the right, we get $T_{12} = 0$. Similarly, we can show that $T_{21} = 0$.

Since $P_2X_{12}(A_{11} + B_{12} + C_{21})^* = P_2X_{12}A_{11}^* = P_2X_{12}B_{12}^* = P_2X_{12}C_{21}^* = 0$ and $P_2 \circ X_{12} \bullet (A_{11} + B_{12} + C_{21}) = X_{12}C_{21} + B_{12}X_{12}^*$, we have from Lemma 5 that

$$\begin{aligned}
&\delta(P_2) \circ X_{12} \bullet (A_{11}+B_{12}+C_{21}) + P_2 \circ \delta(X_{12}) \bullet (A_{11} \\
&\qquad + B_{12}+C_{21}) + P_2 \circ X_{12} \bullet \delta(A_{11}+B_{12}+C_{21}) \\
&= \delta(P_2 \circ X_{12} \bullet (A_{11}+B_{12}+C_{21})) \\
&= \delta(P_2 \circ X_{12} \bullet C_{21}) + \delta(P_2 \circ X_{12} \bullet B_{12}) + \delta(P_2 \circ X_{12} \bullet A_{11}) \\
&= \delta(P_2) \circ X_{12} \bullet (A_{11}+B_{12}+C_{21}) \\
&\quad + P_2 \circ \delta(X_{12}) \bullet (A_{11}+B_{12}+C_{21}) \\
&\quad + P_2 \circ X_{12} \bullet (\delta(A_{11}) + \delta(B_{12}) + \delta(C_{21})).
\end{aligned}$$

It follows that $P_2 \circ X_{12} \bullet T = 0$, and so

$$X_{12}T + TX_{12}^* = 0. \tag{25}$$

Multiplying Eq. (25) by P_2 from the right, we obtain $T_{22} = 0$.

Let $S_{A_{11},B_{12},C_{21}} = T_{11}$. Then $S_{A_{11},B_{12},C_{21}} \in \mathscr{A}_{11}$ and $\delta(A_{11}+B_{12}+C_{21}) = \delta(A_{11}) + \delta(B_{12}) + \delta(C_{21}) + S_{A_{11},B_{12},C_{21}}$, for every $A_{11} \in \mathscr{A}_{11}, B_{12} \in \mathscr{A}_{12}, C_{21} \in \mathscr{A}_{21}$.

Similarly, there exists $R_{B_{12},C_{21},D_{22}} \in \mathscr{A}_{22}$ such that

$$\begin{aligned}
&\delta(B_{12} + C_{21} + D_{22}) \\
&\qquad = \delta(B_{12}) + \delta(C_{21}) + \delta(D_{22}) + R_{B_{12},C_{21},D_{22}},
\end{aligned} \tag{26}$$

for every $B_{12} \in \mathscr{A}_{12}, C_{21} \in \mathscr{A}_{21}, D_{22} \in \mathscr{D}_{22}$.

By $P_1X_{21}(A_{11} + B_{12} + C_{21})^* = P_1X_{21}A_{11}^* = P_1X_{21}B_{12}^* = P_1X_{21}C_{21}^* = 0$ and $P_1 \circ X_{21} \bullet (A_{11} + B_{12} + C_{21}) = X_{21}A_{11} + A_{11}X_{21}^* + X_{21}B_{12} + C_{21}X_{21}^*$, we have from Eq. (26) that there exists $R_{A_{11}X_{21}^*,X_{21}A_{11},X_{21}B_{12}+C_{21}X_{21}^*} \in \mathscr{A}_{22}$ such that

$$\begin{aligned}
&\delta(P_1) \circ X_{21} \bullet (A_{11}+B_{12}+C_{21}) + P_1 \circ \delta(X_{21}) \bullet (A_{11} \\
&\qquad + B_{12}+C_{21}) + P_1 \circ X_{21} \bullet \delta(A_{11}+B_{12}+C_{21}) \\
&= \delta(P_1 \circ X_{21} \bullet (A_{11}+B_{12}+C_{21})) \\
&= \delta(X_{21}A_{11}) + \delta(A_{11}X_{21}^*) + \delta(X_{21}B_{12} + C_{21}X_{21}^*) \\
&\quad + R_{A_{11}X_{21}^*,X_{21}A_{11},X_{21}B_{12}+C_{21}X_{21}^*} \\
&= \delta(A_{11}X_{21}^* + X_{21}A_{11}) + \delta(X_{21}B_{12}) \\
&\quad + \delta(C_{21}X_{21}^*) + R_{A_{11}X_{21}^*,X_{21}A_{11},X_{21}B_{12}+C_{21}X_{21}^*} \\
&= \delta(P_1 \circ X_{21} \bullet A_{11}) + \delta(P_1 \circ X_{21} \bullet B_{12}) \\
&\quad + \delta(P_1 \circ X_{21} \bullet C_{21}) + R_{A_{11}X_{21}^*,X_{21}A_{11},X_{21}B_{12}+C_{21}X_{21}^*} \\
&= \delta(P_1) \circ X_{21} \bullet (A_{11}+B_{12}+C_{21}) \\
&\quad + P_1 \circ \delta(X_{21}) \bullet (A_{11}+B_{12}+C_{21}) + P_1 \circ X_{21} \bullet (\delta(A_{11}) \\
&\quad + \delta(B_{12}) + \delta(C_{21})) + R_{A_{11}X_{21}^*,X_{21}A_{11},X_{21}B_{12}+C_{21}X_{21}^*}.
\end{aligned}$$

This implies that

$$P_1 \circ X_{21} \bullet T = R_{A_{11}X_{21}^*,X_{21}A_{11},X_{21}B_{12}+C_{21}X_{21}^*},$$

and so

$$X_{21}T + TX_{21}^* = R_{A_{11}X_{21}^*,X_{21}A_{11},X_{21}B_{12}+C_{21}X_{21}^*}. \tag{27}$$

Multiplying Eq. (27) by P_2 from the left and by P_1 from the right, and using the fact that $R_{A_{11}X_{21}^*,X_{21}A_{11},X_{21}B_{12}+C_{21}X_{21}^*} \in \mathscr{A}_{22}$, we obtain $X_{21}TP_1 = 0$, and so $T_{11} = 0$.

(ii): Similarly, we can show $\delta(B_{12} + C_{21} + D_{22}) = \delta(B_{12}) + \delta(C_{21}) + \delta(D_{22})$. □

Lemma 7 *For every $A_{11} \in \mathscr{A}_{11}$, $B_{12} \in \mathscr{A}_{12}$, $C_{21} \in \mathscr{A}_{21}$, $D_{22} \in \mathscr{A}_{22}$, we have*

$$\delta(A_{11}+B_{12}+C_{21}+D_{22}) = \delta(A_{12}) + \delta(B_{12}) + \delta(C_{21}) + \delta(D_{22}).$$

Proof: Let $T = \delta(A_{11} + B_{12} + C_{21} + D_{22}) - \delta(A_{11}) - \delta(B_{12}) - \delta(C_{21}) - \delta(D_{22})$. Since $(A_{11} + B_{12} + C_{21} + D_{22})P_1P_2^* = A_{11}P_1P_2^* = B_{12}P_1P_2^* = C_{21}P_1P_2^* = D_{22}P_1P_2^* = 0$ and $(A_{11} + B_{12} + C_{21} + D_{22}) \circ P_1 \bullet P_2 = B_{12} + B_{12}^*$, we have from Lemma 6 that

$$\begin{aligned}
&\delta(A_{11}+B_{12}+C_{21}+D_{22}) \circ P_1 \bullet P_2 \\
&\qquad + (A_{11}+B_{12}+C_{21}+D_{22}) \circ \delta(P_1) \bullet P_2 \\
&\qquad\qquad + (A_{11}+B_{12}+C_{21}+D_{22}) \circ P_1 \bullet \delta(P_2) \\
&= \delta((A_{11}+B_{12}+C_{21}+D_{22}) \circ P_1 \bullet P_2) \\
&= \delta(B_{12} + B_{12}^*) \\
&= \delta(A_{11} \circ P_1 \bullet P_2) + \delta(B_{12} \circ P_1 \bullet P_2) \\
&\quad + \delta(C_{21} \circ P_1 \bullet P_2) + \delta(D_{22} \circ P_1 \bullet P_2) \\
&= (\delta(A_{11}) + \delta(B_{12}) + \delta(C_{21}) + \delta(D_{22})) \circ P_1 \bullet P_2 \\
&\quad + (A_{11}+B_{12}+C_{21}+D_{22}) \circ \delta(P_1) \bullet P_2 \\
&\quad + (A_{11}+B_{12}+C_{21}+D_{22}) \circ P_1 \bullet \delta(P_2).
\end{aligned}$$

This implies that $T\circ P_1\bullet P_2=0$, and so

$$P_1TP_2+P_2T^*P_1=0. \tag{28}$$

Multiplying Eq. (28) by P_2 from the right, we get $T_{12}=0$. Similarly, we obtain $T_{21}=0$.

Since $P_1X_{21}(A_{11}+B_{12}+C_{21}+D_{22})^*=P_1X_{21}A_{11}^*=P_1X_{21}B_{12}^*=P_1X_{21}C_{21}^*=P_1X_{21}D_{22}^*=0$ and $P_1\circ X_{21}\bullet(A_{11}+B_{12}+C_{21}+D_{22})=X_{21}A_{11}+X_{21}B_{12}+A_{11}X_{21}^*+C_{21}X_{21}^*$, we have from Lemma 6 that

$$\begin{aligned}
&\delta(P_1)\circ X_{21}\bullet(A_{11}+B_{12}+C_{21}+D_{22})\\
&\quad+P_1\circ\delta(X_{21})\bullet(A_{11}+B_{12}+C_{21}+D_{22})\\
&\quad+P_1\circ X_{21}\bullet\delta(A_{11}+B_{12}+C_{21}+D_{22})\\
&=\delta(P_1\circ X_{21}\bullet(A_{11}+B_{12}+C_{21}+D_{22}))\\
&=\delta(X_{21}A_{11}+A_{11}X_{21}^*)+\delta(X_{21}B_{12})+\delta(C_{21}X_{21}^*)\\
&=\delta(P_1\circ X_{21}\bullet A_{11})+\delta(P_1\circ X_{21}\bullet B_{12})\\
&\quad+\delta(P_1\circ X_{21}\bullet C_{21})+\delta(P_1\circ X_{21}\bullet D_{22})\\
&=\delta(P_1)\circ X_{21}\bullet(A_{11}+B_{12}+C_{21}+D_{22})\\
&\quad+P_1\circ\delta(X_{21})\bullet(A_{11}+B_{12}+C_{21}+D_{22})\\
&\quad+P_1\circ X_{21}\bullet(\delta(A_{11})+\delta(B_{12})+\delta(C_{21})+\delta(D_{22})),
\end{aligned}$$

which implies that $P_1\circ X_{21}\bullet T=0$, and so

$$X_{21}T+TX_{21}^*=0. \tag{29}$$

Multiplying Eq. (29) by P_1 from right, we get $T_{11}=0$. Similarly, $T_{22}=0$. □

Lemma 8 *δ is additive on $\mathscr{A}$.*

Proof: For every $A,B\in\mathscr{A}$, we have $A=A_{11}+A_{12}+A_{21}+A_{22}$ and $B=B_{11}+B_{12}+B_{21}+B_{22}$. By Lemmas 4, 5 and 7, we have

$$\begin{aligned}
\delta(A+B)&=\delta(A_{11}+B_{11})+\delta(A_{12}+B_{12})\\
&\quad+\delta(A_{21}+B_{21})+\delta(A_{22}+B_{22})\\
&=\delta(A_{11})+\delta(B_{11})+\delta(A_{12})+\delta(B_{12})\\
&\quad+\delta(A_{21})+\delta(B_{21})+\delta(A_{22})+\delta(B_{22})\\
&=\delta(A_{11}+A_{12}+A_{21}+A_{22})\\
&\quad+\delta(B_{11}+B_{12}+B_{21}+B_{22})\\
&=\delta(A)+\delta(B).
\end{aligned}$$

Lemma 9 (i) *$P_i\delta(P_i)P_j=-P_i\delta(P_j)P_j$ ($1\leqslant i\neq j\leqslant 2$);*
(ii) *$P_i\delta(P_j)P_i=0$ ($1\leqslant i\neq j\leqslant 2$);*
(iii) *$P_i\delta(P_i)P_i=0$ ($i=1,2$).*

Proof: *(i)*: Since $P_1P_1P_2^*=0$ and $P_1\circ P_1\bullet P_2=0$, we have

$$\begin{aligned}
0&=\delta(P_1\circ P_1\bullet P_2)\\
&=\delta(P_1)\circ P_1\bullet P_2+P_1\circ\delta(P_1)\bullet P_2+P_1\circ P_1\bullet\delta(P_2)\\
&=2P_1\delta(P_1)P_2+2P_2\delta(P_1)^*P_1+2P_1\delta(P_2)+2\delta(P_2)P_1.
\end{aligned} \tag{30}$$

Multiplying Eq. (30) by P_1 from the left and by P_2 from the right, we obtain

$$P_1\delta(P_1)P_2=-P_1\delta(P_2)P_2.$$

Similarly, we can show $P_2\delta(P_1)P_1=-P_2\delta(P_2)P_1$.

(ii): Multiplying Eq. (30) by P_1 from both sides, we obtain $P_1\delta(P_2)P_1=0$. Similarly, we can show that $P_2\delta(P_1)P_2=0$.

(iii): For any $A_{12}\in\mathscr{A}_{12}$, since $P_1P_1A_{12}^*=0$, we have

$$\begin{aligned}
2\delta(A_{12})&=\delta(P_1\circ P_1\bullet A_{12})\\
&=\delta(P_1)\circ P_1\bullet A_{12}+P_1\circ\delta(P_1)\bullet A_{12}+P_1\circ P_1\bullet\delta(A_{12})\\
&=2\delta(P_1)A_{12}+2P_1\delta(P_1)A_{12}\\
&\quad+2A_{12}\delta(P_1)^*P_1+2P_1\delta(A_{12})+2\delta(A_{12})P_1.
\end{aligned} \tag{31}$$

Multiplying Eq. (31) by P_1 from the left and by P_2 from the right, we obtain $P_1\delta(P_1)A_{12}=0$. It follows from (▼) that $P_1\delta(P_1)P_1=0$. Similarly, we can prove that $P_2\delta(P_2)P_2=0$. □

Lemma 10 $\delta(P_i)^*=\delta(P_i)$.

Proof: Let $1\leqslant i\neq j\leqslant 2$. Since $P_jP_iP_i^*=0$ and $P_j\circ P_i\bullet P_i=0$, we have

$$\begin{aligned}
0&=\delta(P_j\circ P_i\bullet P_i)\\
&=\delta(P_j)\circ P_i\bullet P_i+P_j\circ\delta(P_i)\bullet P_i+P_j\circ P_i\bullet\delta(P_i)\\
&=\delta(P_j)P_i+P_i\delta(P_j)P_i+P_i\delta(P_j)^*\\
&\quad+P_i\delta(P_j)^*P_i+P_j\delta(P_i)P_i+P_i\delta(P_i)^*P_j.
\end{aligned} \tag{32}$$

Multiplying Eq. (32) by P_j from the left and by P_i from the right, we have

$$P_j\delta(P_j)P_i+P_j\delta(P_i)P_i=0. \tag{33}$$

Then

$$P_i\delta(P_j)^*P_j+P_i\delta(P_i)^*P_j=0. \tag{34}$$

Since $P_iP_iP_j^*=0$ and $P_i\circ P_i\bullet P_j=0$, we have

$$\begin{aligned}
0&=\delta(P_i\circ P_i\bullet P_j)\\
&=\delta(P_i)\circ P_i\bullet P_j+P_i\circ\delta(P_i)\bullet P_j+P_i\circ P_i\bullet\delta(P_j)\\
&=P_i\delta(P_i)P_j+P_j\delta(P_i)^*P_i+P_i\delta(P_i)P_j\\
&\quad+P_j\delta(P_i)^*P_i+2P_i\delta(P_j)+2\delta(P_j)P_i.
\end{aligned} \tag{35}$$

Multiplying Eq. (35) by P_i from the left and by P_j from the right, we obtain

$$P_i\delta(P_i)P_j+P_i\delta(P_j)P_j=0. \tag{36}$$

Multiplying Eq. (35) by P_j from the left and by P_i from the right, we obtain

$$P_j\delta(P_i)^*P_i+P_j\delta(P_j)P_i=0. \tag{37}$$

It follows from Lemma 9(i) that

$$P_j\delta(P_i)P_i+P_j\delta(P_j)P_i=0. \tag{38}$$

From Eqs. (37) and (38), we obtain $P_j\delta(P_i)^*P_i = P_j\delta(P_i)P_i$. It follows from Lemma 9(ii) that $P_j\delta(P_i)^*P_j = 0$, and so $P_j\delta(P_i)^*P_j = P_j\delta(P_i)P_j$.

Since $P_jP_jP_i^* = 0$ and $P_j \circ P_j \bullet P_i = 0$, we have

$$\begin{aligned}0 &= \delta(P_j \circ P_j \bullet P_i)\\ &= \delta(P_j)\circ P_j \bullet P_i + P_j \circ \delta(P_j)\bullet P_i + P_j \circ P_j \bullet \delta(P_i)\\ &= P_j\delta(P_j)P_i + P_i\delta(P_j)^*P_j + P_j\delta(P_j)P_i\\ &\quad + P_i\delta(P_j)^*P_j + 2P_j\delta(P_i) + 2\delta(P_i)P_j. \qquad (39)\end{aligned}$$

Multiplying Eq. (39) by P_i from the left and by P_j from the right, we have

$$P_i\delta(P_j)^*P_j + P_i\delta(P_i)P_j = 0. \qquad (40)$$

Comparing Eqs. (34) and (40), we obtain that $P_i\delta(P_i)^*P_j = P_i\delta(P_i)P_j$. From Lemma 9(iii), we have $P_i\delta(P_i)^*P_i = P_i\delta(P_i)P_i$. Hence, $\delta(P_i)^* = \delta(P_i)$. □

Remark 1 Let $M = P_1\delta(P_1)P_2 - P_2\delta(P_1)P_1$. Then $M = -M^*$ by Lemma 10. Define a map $\triangle : \mathscr{A} \to \mathscr{A}$ by

$$\triangle(A) = \delta(A) - [A, M]$$

for all $A \in \mathscr{A}$. It is easy to verify that $\triangle$ also satisfies

$$\triangle(A\circ B\bullet C) = \triangle(A)\circ B\bullet C + A\circ\triangle(B)\bullet C + A\circ B\bullet\triangle(C)$$

for any $A, B, C \in \mathscr{A}$ with $ABC^* = 0$. By Lemmas 8 and 9, $\triangle$ has the following properties.
(i) $\triangle$ is additive on $\mathscr{A}$;
(ii) $\triangle(P_i) = 0$.

Lemma 11 $\triangle(\mathscr{A}_{ij}) \subseteq \mathscr{A}_{ij}$ $(i, j = 1, 2)$.

Proof: Let $1 \leqslant i \neq j \leqslant 2$. Since $P_1P_1A_{12}^* = 0$ and $\triangle(P_1) = 0$, we have

$$\begin{aligned}2\triangle(A_{12}) &= \triangle(P_1 \circ P_1 \bullet A_{12})\\ &= P_1 \circ P_1 \bullet \triangle(A_{12})\\ &= 2P_1\triangle(A_{12}) + 2\triangle(A_{12})P_1,\end{aligned}$$

which implies that $P_1\triangle(A_{12})P_1 = P_2\triangle(A_{12})P_2 = 0$.

From $P_1A_{12}P_1^* = 0$, $P_1 \circ A_{12} \bullet P_1 = 0$ and $\triangle(P_1) = 0$, we have

$$\begin{aligned}0 &= \triangle(P_1 \circ A_{12} \bullet P_1)\\ &= P_1 \circ \triangle(A_{12}) \bullet P_1\\ &= P_1\triangle(A_{12})P_1 + \triangle(A_{12})P_1 + P_1\triangle(A_{12})^*P_1 + P_1(A_{12})^*.\end{aligned}$$

It follows that $P_2\triangle(A_{12})P_1 = 0$ and thus $\triangle(\mathscr{A}_{12}) \subseteq \mathscr{A}_{12}$. Similarly, we can show $\triangle(\mathscr{A}_{21}) \subseteq \mathscr{A}_{21}$.

It follows from $P_2P_2A_{11}^* = 0$, $P_2 \circ P_2 \bullet A_{11} = 0$ and $\triangle(P_2) = 0$ that

$$\begin{aligned}0 &= \triangle(P_2 \circ P_2 \bullet A_{11})\\ &= P_2 \circ P_2 \bullet \triangle(A_{11})\\ &= 2P_2\triangle(A_{11}) + 2\triangle(A_{11})P_2.\end{aligned}$$

This implies that $P_2\triangle(A_{11})P_1 = P_1\triangle(A_{11})P_2 = P_2\triangle(A_{11})P_2 = 0$. Hence $\triangle(\mathscr{A}_{11}) \subseteq \mathscr{A}_{11}$.

Similarly, we can show $\triangle(\mathscr{A}_{22}) \subseteq \mathscr{A}_{22}$. □

Lemma 12 *For every* $A_{ij}, B_{ij} \in \mathscr{A}_{ij}$ $(1 \leqslant i \neq j \leqslant 2)$*, we have*
(i) $\triangle(A_{ii}B_{ij}) = \triangle(A_{ii})B_{ij} + A_{ii}\triangle(B_{ij})$*;*
(ii) $\triangle(A_{ij}B_{ji}) = \triangle(A_{ij})B_{ji} + A_{ij}\triangle(B_{ji})$*;*
(iii) $\triangle(A_{ii}B_{ii}) = \triangle(A_{ii})B_{ii} + A_{ii}\triangle(B_{ii})$*;*
(iv) $\triangle(A_{ij}B_{jj}) = \triangle(A_{ij})B_{jj} + A_{ij}\triangle(B_{jj})$*.*

Proof: *(i)*: Since $P_iA_{ii}B_{ij}^* = 0$ and $P_i \circ A_{ii} \bullet B_{ij} = 2A_{ii}B_{ij}$, we have from Lemma 11 and $\triangle(P_i) = 0$ that

$$\begin{aligned}2\triangle(A_{ii}B_{ij}) &= \triangle(P_i \circ A_{ii} \bullet B_{ij})\\ &= P_i \circ \triangle(A_{ii}) \bullet B_{ij} + P_i \circ A_{ii} \bullet \triangle(B_{ij})\\ &= 2\triangle(A_{ii})B_{ij} + 2A_{ii}\triangle(B_{ij}).\end{aligned}$$

Hence $\triangle(A_{ii}B_{ij}) = \triangle(A_{ii})B_{ij} + A_{ii}\triangle(B_{ij})$.

(ii): Since $P_iA_{ij}B_{ji}^* = 0$ and $P_i \circ A_{ij} \bullet B_{ji} = A_{ij}B_{ji}$, we have from Lemma 11 and $\triangle(P_i) = 0$ that

$$\begin{aligned}\triangle(A_{ij}B_{ji}) &= \triangle(P_i \circ A_{ij} \bullet B_{ji})\\ &= P_i \circ \triangle(A_{ij}) \bullet B_{ji} + P_i \circ A_{ij} \bullet \triangle(B_{ji})\\ &= \triangle(A_{ij})B_{ji} + A_{ij}\triangle(B_{ji}).\end{aligned}$$

(iii): For every $X_{ij} \in \mathscr{A}_{ij}$, we have from Lemma 12(i) that

$$\begin{aligned}&\triangle(A_{ii}B_{ii})X_{ij} + A_{ii}B_{ii}\triangle(X_{ij})\\ &= \triangle(A_{ii}B_{ii}X_{ij})\\ &= \triangle(A_{ii})B_{ii}X_{ij} + A_{ii}\triangle(B_{ii}X_{ij})\\ &= \triangle(A_{ii})B_{ii}X_{ij} + A_{ii}\triangle(B_{ii})X_{ij} + A_{ii}B_{ii}\triangle(X_{ij}).\end{aligned}$$

Then $(\triangle(A_{ii}B_{ii}) - \triangle(A_{ii})B_{ii} - A_{ii}\triangle(B_{ii}))X_{ij} = 0$ for all $X_{ij} \in \mathscr{A}_{ij}$. It follows that $\triangle(A_{ii}B_{ii}) = \triangle(A_{ii})B_{ii} + A_{ii}\triangle(B_{ii})$.

(iv): From $P_jA_{ij}B_{jj}^* = 0$ and $P_j \circ A_{ij} \bullet B_{jj} = A_{ij}B_{jj} + B_{jj}A_{ij}^*$, Lemma 11 and $\triangle(P_j) = 0$, we have

$$\begin{aligned}\triangle(A_{ij}B_{jj}) + \triangle(B_{jj}A_{ij}^*) &= \triangle(P_j \circ A_{ij} \bullet B_{jj})\\ &= P_j \circ \triangle(A_{ij}) \bullet B_{jj} + P_j \circ A_{ij} \bullet \triangle(B_{jj})\\ &= \triangle(A_{ij})B_{jj} + B_{jj}\triangle(A_{ij})^* + A_{ij}\triangle(B_{jj}) + \triangle(B_{jj})A_{ij}^*. \qquad (41)\end{aligned}$$

Multiplying Eq. (41) by P_j from the right, we obtain $\triangle(A_{ij}B_{jj}) = \triangle(A_{ij})B_{jj} + A_{ij}\triangle(B_{jj})$. □

Lemma 13 $\triangle(A^*) = \triangle(A)^*$ *for all* $A \in \mathscr{A}$*.*

Proof: Let $1 \leqslant i \neq j \leqslant 2$. From $A_{ij}P_iP_j^* = 0$ and $A_{ij} \circ P_i \bullet P_j = A_{ij} + A_{ij}^*$ and Lemma 11, we have

$$\begin{aligned}\triangle(A_{ij}) + \triangle(A_{ij}^*) &= \triangle(A_{ij} + A_{ij}^*)\\ &= \triangle(A_{ij}) \circ P_i \bullet P_j\\ &= \triangle(A_{ij}) + \triangle(A_{ij})^*.\end{aligned}$$

Hence $\triangle(A_{ij}^*) = \triangle(A_{ij})^*$.

Since $A_{ii}X_{ji}P_i^* = 0$ and $A_{ii} \circ X_{ji} \bullet P_i = X_{ji}A_{ii} + A_{ii}^* X_{ji}^*$, we have from Lemma 12 that

$$\begin{aligned}\triangle(X_{ji})A_{ii} + X_{ji}\triangle(A_{ii}) + \triangle(A_{ii}^*)X_{ji}^* + A_{ii}^*\triangle(X_{ji}^*) \\ = \triangle(X_{ji}A_{ii} + A_{ii}^*X_{ji}^*) \\ = \triangle(A_{ii} \circ X_{ji} \bullet P_i) \\ = \triangle(A_{ii}) \circ X_{ji} \bullet P_i + A_{ii} \circ \triangle(X_{ji}) \bullet P_i \\ = X_{ji}\triangle(A_{ii}) + \triangle(A_{ii})^* X_{ji}^* + \triangle(X_{ji})A_{ii} + A_{ii}^*\triangle(X_{ji})^*,\end{aligned}$$

which together with $\triangle(X_{ji}^*) = \triangle(X_{ji})^*$ yields that $(\triangle(A_{ii}^*) - \triangle(A_{ii})^*)X_{ji}^* = 0$. Then $\triangle(A_{ii}^*) = \triangle(A_{ii})^*$.

For any $A \in \mathscr{A}$, we have $A = A_{11} + A_{12} + A_{21} + A_{22}$. It follows that

$$\Delta(A^*) = \sum_{i,j=1}^{2} \Delta(A_{ij}^*) = (\sum_{i,j=1}^{2} \Delta(A_{ij}))^* = \Delta(A)^*.$$

Hence $\triangle(A^*) = \triangle(A)^*$. □

Proof of Theorem 1

For any $A, B \in \mathscr{A}$, we have $A = A_{11} + A_{12} + A_{21} + A_{22}$ and $B = B_{11} + B_{12} + B_{21} + B_{22}$. By the additivity of $\triangle$ and Lemma 13, we next need to prove that $\triangle$ is a derivation on $\mathscr{A}$. It follows from Lemmas 11 and 12 that

$$\begin{aligned}\triangle(AB) = \triangle(A_{11}B_{11}) + \triangle(A_{11}B_{12}) + \triangle(A_{12}B_{21}) \\ + \triangle(A_{12}B_{22}) + \triangle(A_{21}B_{11}) + \triangle(A_{21}B_{12}) \\ + \triangle(A_{22}B_{21}) + \triangle(A_{22}B_{22}) \\ = \triangle(A_{11} + A_{12} + A_{21} + A_{22})(B_{11} + B_{12} + B_{21} + B_{22}) \\ + (A_{11} + A_{12} + A_{21} + A_{22})\triangle(B_{11} + B_{12} + B_{21} + B_{22}) \\ = \triangle(A)B + A\triangle(B).\end{aligned}$$

Consequently, $\triangle$ is an additive $*$-derivation, and so δ is an additive $*$-derivation by Remark 1.

Corollary 1 *Let $\mathscr{A}$ be a prime $*$-algebra with unit I containing non-trivial projection. If a map $\delta : \mathscr{A} \to \mathscr{A}$ satisfies*

$$\delta(A \circ B \bullet C) = \delta(A) \circ B \bullet C + A \circ \delta(B) \bullet C + A \circ B \bullet \delta(C)$$

for any $A, B, C \in \mathscr{A}$ with $ABC^* = 0$, then δ is an additive $*$-derivation.

Corollary 2 *Let $\mathscr{A}$ be factor on Neumann algebra acting on a complex Hilbert space $\mathscr{H}$ with dim $\mathscr{A} > 1$. If a map $\delta : \mathscr{A} \to \mathscr{A}$ satisfies*

$$\delta(A \circ B \bullet C) = \delta(A) \circ B \bullet C + A \circ \delta(B) \bullet C + A \circ B \bullet \delta(C)$$

for any $A, B, C \in \mathscr{A}$ with $ABC^* = 0$, then δ is an additive $*$-derivation.

Corollary 3 *Let $\mathscr{A}$ be a unital standard operator algebra on an infinite dimensional complex Hilbert space $\mathscr{H}$ with dim $\mathscr{A} > 1$, which is closed under adjoint operation and contains a nontrivial projection. If a map $\delta : \mathscr{A} \to \mathscr{A}$ satisfies*

$$\delta(A \circ B \bullet C) = \delta(A) \circ B \bullet C + A \circ \delta(B) \bullet C + A \circ B \bullet \delta(C)$$

for all $A, B, C \in \mathscr{R}$ with $ABC^* = 0$, then δ is an additive $*$-derivation.

Acknowledgements: This research is supported by the key Scientific Research Project of Shaanxi Provincial Department of Education (Key Laboratory Project) (No. 25JS042).

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