RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2026.032

ScienceAsia 52 (2): 2026: ID 2026032: **1–8**

An $l^{p,q/2}$-singular value interval with application to determine the positive definiteness of a real partially symmetric rectangular tensor

Jianxing Zhao*, **Qiuhua Shi**

School of Data Science and Information Engineering, Guizhou Minzu University, Guiyang 550025 China

*Corresponding author, e-mail: zhaojianxing@gzmu.edu.cn

Received 19 Dec 2024, Accepted 8 Apr 2026
Available online 16 Apr 2026

ABSTRACT: The positive definiteness of a (p,q)-th order $m\times n$ dimensional real partially symmetric rectangular tensor $\mathscr{A}$ (with p,q positive even and $m,n\geqslant 2$) is crucial in real applications such as the strong ellipticity condition and quantum entanglement. A key characterization is that $\mathscr{A}$ is positive definite if and only if all of its $l^{k,s}$-singular values are positive for positive even numbers k and s. First, an interval with two parameter vectors is constructed to localize all $l^{p,q/2}$-singular values of $\mathscr{A}$. Then, by optimizing these two parameter vectors, the optimal parameter interval is derived. Subsequently, a concise criterion for the positive definiteness of $\mathscr{A}$ is derived directly from this optimal parameter interval. Finally, the effectiveness of the proposed interval and the resulting criterion is demonstrated through a numerical example.

KEYWORDS: rectangular tensor, partially symmetric tensor, $l^{k,s}$-singular value, $l^{p,q/2}$-singular value, positive definiteness

MSC2020: 15A18 15A69 15A72

INTRODUCTION

Let m and n be two positive integers with $m,n\geqslant 2$, $[n]:=\{1,2,\dots,n\}$, p and q be two positive even numbers, $\mathbb{R}$ be the set of all real numbers, $\mathbb{R}^n$ be the set of all n-dimensional real vectors, $\mathbb{S}^{[p,q;m,n]}$ be the set of all (p,q)-th order $m\times n$ dimensional real partially symmetric rectangular tensors, $\mathrm{x}=(x_1,x_2,\dots,x_m)^\top\in\mathbb{R}^m$, and $\mathrm{y}=(y_1,y_2,\dots,y_n)^\top\in\mathbb{R}^n$, where $\mathscr{A}=(a_{i_1\cdots i_p j_1\cdots j_q})\in\mathbb{S}^{[p,q;m,n]}$ if

$$a_{i_1\cdots i_p j_1\cdots j_q}\in\mathbb{R},\qquad i_1,\dots,i_p\in[m],\ j_1,\dots,j_q\in[n]$$

with the partially symmetric property

$$a_{\pi(i_1,\dots,i_p)\tau(j_1,\dots,j_q)}=a_{i_1\cdots i_p j_1\cdots j_q},\qquad \forall\pi\in S_p,\ \forall\tau\in S_q,$$

S_p (resp., S_q) is the permutation group of p (resp., q) indices and π (resp., τ) is any permutation of indices among $i_1,\dots,i_p$ (resp., $j_1,\dots,j_q$). The positive definiteness of the following multivariate homogeneous polynomial

$$\mathscr{A}\mathrm{x}^p\mathrm{y}^q:=\sum_{\substack{i_1,\dots,i_p\in[m],\\ j_1,\dots,j_q\in[n]}} a_{i_1\cdots i_p j_1\cdots j_q}x_{i_1}\cdots x_{i_p}y_{j_1}\cdots y_{j_q}\tag{1}$$

has important applications in the strong ellipticity condition problem in solid mechanics [1,2] and the entanglement problem in quantum physics [3–5], where $\mathscr{A}\mathrm{x}^p\mathrm{y}^q$ is said to be positive definite if $\mathscr{A}\mathrm{x}^p\mathrm{y}^q>0$ for all $\mathrm{x}\in\mathbb{R}^m\backslash\{0\}$ and $\mathrm{y}\in\mathbb{R}^n\backslash\{0\}$. Furthermore, if $\mathscr{A}\mathrm{x}^p\mathrm{y}^q$ is positive definite, then $\mathscr{A}\in\mathbb{S}^{[p,q;m,n]}$ is called positive definite.

In order to determine the positive definiteness of $\mathscr{A}\in\mathbb{S}^{[p,q;m,n]}$, the definition of $l^{k,s}$-singular values of $\mathscr{A}$ is introduced in [6]. Before reviewing this definition, we first introduce some notations. For a real number $g\in\mathbb{R}$, $\mathrm{sign}(g)=1$ if $g>0$, $\mathrm{sign}(g)=0$ if $g=0$, and $\mathrm{sign}(g)=-1$ if $g<0$. For a vector $\mathrm{z}=(z_1,z_2,\dots,z_n)\in\mathbb{R}^n$ and a positive integer k, denote

$$\mathrm{z}^{[k]}:=(z_1^k,\dots,z_n^k)^\top,$$
$$\|\mathrm{z}\|_k:=(|z_1|^k+\cdots+|z_n|^k)^{1/k},$$
$$\varphi_k^{(n)}(\mathrm{z}):=(\mathrm{sign}(z_1)|z_1|^k,\dots,\mathrm{sign}(z_n)|z_n|^k)^\top.$$

Definition 1 ([6, Def. 2.1]) Let $\mathscr{A}=(a_{i_1\cdots i_p j_1\cdots j_q})\in\mathbb{S}^{[p,q;m,n]}$ and $k,s\geqslant 2$ be two positive integers. If there exists a triplet $(\lambda,\mathrm{x},\mathrm{y})\in\mathbb{R}\times(\mathbb{R}^m\backslash\{0\})\times(\mathbb{R}^n\backslash\{0\})$ such that

$$\begin{cases}\mathscr{A}\mathrm{x}^{p-1}\mathrm{y}^q=\lambda\varphi_{k-1}^{(m)}(\mathrm{x}),\\ \mathscr{A}\mathrm{x}^p\mathrm{y}^{q-1}=\lambda\varphi_{s-1}^{(n)}(\mathrm{y}),\\ \|\mathrm{x}\|_k=\|\mathrm{y}\|_s=1,\end{cases}$$

then λ is called an $l^{k,s}$-singular value of $\mathscr{A}$, and x and y are called the left and right $l^{k,s}$-singular vectors of $\mathscr{A}$ associated with λ, where $\mathscr{A}\mathrm{x}^{p-1}\mathrm{y}^q\in\mathbb{R}^m$ with its i-th component

$$(\mathscr{A}\mathrm{x}^{p-1}\mathrm{y}^q)_i:=\sum_{\substack{i_2,\dots,i_p\in[m],\\ j_1,\dots,j_q\in[n]}} a_{ii_2\cdots i_p j_1\cdots j_q}x_{i_2}\cdots x_{i_p}y_{j_1}\cdots y_{j_q},$$

and $\mathscr{A}\mathrm{x}^p\mathrm{y}^{q-1} \in \mathbb{R}^n$ with its j-th component

$$(\mathscr{A}\mathrm{x}^p\mathrm{y}^{q-1})_j := \sum_{\substack{i_1,\ldots,i_p\in[m],\\ j_2,\ldots,j_q\in[n]}} a_{i_1\cdots i_p j j_2\cdots j_q} x_{i_1}\cdots x_{i_p} y_{j_2}\cdots y_{j_q}.$$

For simplicity, we call $(\lambda,\mathrm{x},\mathrm{y})$ an $l^{k,s}$-singular triplet of $\mathscr{A}$.

The existence of $l^{k,s}$-singular triplets of $\mathscr{A}$ is confirmed by the following theorem.

Theorem 1 ([6, Theorem 2.1]) *Let $\mathscr{A} \in \mathbb{S}^{[p,q;m,n]}$ and $k,s \geqslant 2$ be two positive integers. For each pair of k and s, the $l^{k,s}$-singular triplets of $\mathscr{A}$ always exist.*

Subsequently, a method was proposed for determining the positive definiteness of an even-order tensor based on its $l^{k,s}$-singular values.

Theorem 2 ([7, Theorem 2]) *Let $\mathscr{A} \in \mathbb{S}^{[p,q;m,n]}$ with p and q even, $k,s \geqslant 2$ be even. Then $\mathscr{A}$ is positive definite if and only if all of its $l^{k,s}$-singular values are positive.*

Let $\mathscr{A} \in \mathbb{S}^{[p,q;m,n]}$ with p and q even. For any given pair of positive even numbers k and s, Theorem 1 ensures the existence of $l^{k,s}$-singular values of $\mathscr{A}$, and Theorem 2 ensures the positive definiteness of $\mathscr{A}$ if all of its $l^{k,s}$-singular values are positive. To more easily determine the positive definiteness of $\mathscr{A}$, it is necessary in practical applications to specify concrete values for k and s. When k and s are taken to be $k = s = p+q$, the $l^{k,s}$-singular value is referred to as a singular value in [8]. In the special case where $k = p$ and $s = q$, the $l^{k,s}$-singular value is called a V-singular value in [9] and later is named an $l^{p,q}$-singular value in [10]. Setting $k = p/2$ (or, $k = p$) and $s = q/2$ yields the $l^{p/2,q/2}$-singular value in [11] (or, the $l^{p,q/2}$-singular value in [12]) as a special case of the $l^{k,s}$-singular value. Corresponding to these specific $l^{k,s}$-singular values, a number of methods for locating all $l^{k,s}$-singular values of $\mathscr{A}$ in [9, 13, 14], computing or estimating all $l^{k,s}$-singular values of $\mathscr{A}$ in [15–17] and determining the positive definiteness of $\mathscr{A}$ in [10–12] have been developed.

Next, we review the definition of $l^{p,q/2}$-singular values from [12], as well as the criterion for determining the positive definiteness of $\mathscr{A}$ that is based on the $l^{p,q/2}$-singular values.

Definition 2 ([12, p. 969]) Let $\mathscr{A} \in \mathbb{S}^{[p,q;m,n]}$ and $k,s \geqslant 2$ be two positive integers. If there exists a triplet $(\lambda,\mathrm{x},\mathrm{y}) \in \mathbb{R}\times(\mathbb{R}^m\backslash\{0\})\times(\mathbb{R}^n\backslash\{0\})$ such that

$$\begin{cases} \mathscr{A}\mathrm{x}^{p-1}\mathrm{y}^q = \lambda\mathrm{x}^{[p-1]}, & \text{(2a)}\\ \mathscr{A}\mathrm{x}^{p}\mathrm{y}^{q-1} = \lambda\mathrm{y}^{[q/2-1]}, & \text{(2b)}\\ \|\mathrm{x}\|_p = \|\mathrm{y}\|_{q/2} = 1, & \text{(2c)} \end{cases}$$

then we call $(\lambda,\mathrm{x},\mathrm{y})$ an $l^{p,q/2}$-singular triplet of $\mathscr{A}$ and denote by $\sigma(\mathscr{A})$ the set of all $l^{p,q/2}$-singular values of $\mathscr{A}$.

To derive a criterion for positive definiteness of $\mathscr{A}$, it is necessary to construct an interval that contains all $l^{p,q/2}$-singular values. If the left endpoint of this interval is greater than zero, then all $l^{p,q/2}$-singular values are positive, and we thus obtain a criterion for the positive definiteness of $\mathscr{A}$.

Theorem 3 ([12, Theorem 2.6]) *Let $\mathscr{A} = (a_{i_1\cdots i_p j_1\cdots j_q}) \in \mathbb{S}^{[p,q;m,n]}$ with p and $q/2$ even and $m,n \geqslant 2$. Then*

$$\sigma(\mathscr{A}) \subseteq \Gamma(\mathscr{A}) := \Bigg(\bigcup_{i\in[m]}[\tilde{l}_i,\tilde{u}_i]\Bigg)\cap\Bigg(\bigcup_{j\in[n]}[\hat{l}_j,\hat{u}_j]\Bigg),$$

where $\tilde{l}_i$ and $\tilde{u}_i$ are selected by Method 1 in [12, pp. 980–983], and

$$\begin{aligned}
\hat{l}_j &:= \min_{t\in[m],k\in[n]}\Big\{\eta_2 a_{\underbrace{t\cdots t}_{p}\underbrace{j\cdots j}_{q/2}\underbrace{k\cdots k}_{q/2}}\Big\} - c_j(\mathscr{A}),\\
\hat{u}_j &:= \max_{t\in[m],k\in[n]}\Big\{\eta_2 a_{\underbrace{t\cdots t}_{p}\underbrace{j\cdots j}_{q/2}\underbrace{k\cdots k}_{q/2}}\Big\} + c_j(\mathscr{A}),\\
c_j(\mathscr{A}) &:= \sum_{i_1,\ldots,i_p\in[m],j_2,\ldots,j_q\in[n]} |a_{i_1\cdots i_p j j_2\cdots j_q}| \\
&\quad - \sum_{t\in[m],k\in[n]}\Big|\eta_2 a_{\underbrace{t\cdots t}_{p}\underbrace{j\cdots j}_{q/2}\underbrace{k\cdots k}_{q/2}}\Big|,\\
\eta_2 &:= \begin{cases} 1, & j = k,\\ \frac{(q-1)!}{(\frac{q}{2}-1)!\frac{q}{2}!}, & j \neq k. \end{cases}
\end{aligned} \tag{3}$$

Theorem 4 ([12, Theorem 2.7]) *Let $\mathscr{A} \in \mathbb{S}^{[p,q;m,n]}$ with p and $q/2$ even and $m,n \geqslant 2$. If $\tilde{l}_i > 0$ for $i \in [m]$ or $\hat{l}_j > 0$ for $j \in [n]$, then $\mathscr{A}$ and $\mathscr{A}\mathrm{x}^p\mathrm{y}^q$ in (1) are positive definite.*

Theorem 3 and Theorem 4 show that if they are to be used to obtain the interval $\Gamma(\mathscr{A})$ for locating all $l^{p,q/2}$-singular values of $\mathscr{A}$ or to determine the positive definiteness of $\mathscr{A}$, then the computation of $\tilde{l}_i$ and $\tilde{u}_i$ for $i \in [n]$ is required. However, as can be seen from Method 1 in [12, pp. 980–983], the method for selecting $\tilde{l}_i$ and $\tilde{u}_i$ for $i \in [n]$ is overly complex and cumbersome to use, which poses difficulties for practical application. Therefore, we aim to provide a simpler method for determining the positive definiteness of $\mathscr{A}$. The new method is required to offer two advantages over those in Theorem 3 and Theorem 4: first, operational simplicity, and second, reduced computational cost. This constitutes the motivation and objective of the present work.

AN INTERVAL AND A CRITERION FOR THE POSITIVE DEFINITENESS OF A RECTANGULAR TENSOR

Let $\mathscr{A} = (a_{i_1\cdots i_p j_1\cdots j_q}) \in \mathbb{S}^{[p,q;m,n]}$ with p and $q/2$ positive even numbers. In this section, we first construct an interval to localize all $l^{p,q/2}$-singular values of $\mathscr{A}$, and then derive from it a criterion for determining the positive definiteness of $\mathscr{A}$. Before proceeding, we first define some notations. For vectors $\alpha=(\alpha_1,\ldots,\alpha_m)^\top\in\mathbb{R}^m$ and $\beta=(\beta_1,\ldots,\beta_n)^\top\in\mathbb{R}^n$, and for $i\in[m]$ and $j\in[n]$, define

$$\begin{aligned}
\overline{\Psi}(\alpha_i)&:=\alpha_i-\max_{t,k\in[n]}\Big|\eta_1 a_{\underbrace{i\cdots i}_{p}\underbrace{t\cdots t}_{q/2}\underbrace{k\cdots k}_{q/2}}-\alpha_i\Big|-r_i(\mathscr{A}),\\
\overline{\Phi}(\alpha_i)&:=\alpha_i+\max_{t,k\in[n]}\Big|\eta_1 a_{\underbrace{i\cdots i}_{p}\underbrace{t\cdots t}_{q/2}\underbrace{k\cdots k}_{q/2}}-\alpha_i\Big|+r_i(\mathscr{A}),\\
\widehat{\Psi}(\beta_j)&:=\beta_j-\max_{t\in[m],k\in[n]}\Big|\eta_2 a_{\underbrace{t\cdots t}_{p}\underbrace{j\cdots j}_{q/2}\underbrace{k\cdots k}_{q/2}}-\beta_j\Big|-c_j(\mathscr{A}),\\
\widehat{\Phi}(\beta_j)&:=\beta_j+\max_{t\in[m],k\in[n]}\Big|\eta_2 a_{\underbrace{t\cdots t}_{p}\underbrace{j\cdots j}_{q/2}\underbrace{k\cdots k}_{q/2}}-\beta_j\Big|+c_j(\mathscr{A}),
\end{aligned}\tag{4}$$

where $c_j(\mathscr{A})$ is listed in (3) and

$$r_i(\mathscr{A}):=\sum_{i_2,\ldots,i_p\in[m],j_1,\ldots,j_q\in[n]}|a_{ii_2\cdots i_pj_1\cdots j_q}|-\sum_{t,k\in[n]}|\eta_1 a_{\underbrace{i\cdots i}_{p}\underbrace{t\cdots t}_{q/2}\underbrace{k\cdots k}_{q/2}}|,\tag{5}$$

$$\eta_1:=\begin{cases}1, & t=k;\\ \frac{q!}{\frac{q}{2}!\frac{q}{2}!}, & t\neq k.\end{cases}\tag{6}$$

Lemma 1 *Let $\mathscr{A}=(a_{i_1\cdots i_pj_1\cdots j_q})\in\mathbb{S}^{[p,q;m,n]}$ with p and $q/2$ even, $\alpha\in\mathbb{R}^m$ and $\beta\in\mathbb{R}^n$. Then*

$$\sigma(\mathscr{A})\subseteq\Omega(\mathscr{A},\alpha,\beta):=\Big(\bigcup_{i\in[m]}[\overline{\Psi}(\alpha_i),\overline{\Phi}(\alpha_i)]\Big)\cap\Big(\bigcup_{j\in[n]}[\widehat{\Psi}(\beta_j),\widehat{\Phi}(\beta_j)]\Big).\tag{7}$$

Proof: Let $(\lambda,\mathrm{x},\mathrm{y})$ be an $l^{p,q/2}$-singular triplet of $\mathscr{A}$. Then $\|\mathrm{x}\|_p=\|\mathrm{y}\|_{q/2}=1$, i.e.,

$$x_1^p+\cdots+x_m^p=y_1^{q/2}+\cdots+y_n^{q/2}=1,$$

which implies that

$$1=(y_1^{q/2}+\cdots+y_n^{q/2})^2=\sum_{t,k\in[n]}y_t^{q/2}y_k^{q/2}.\tag{8}$$

Let $|x_g|=\max_{i\in[m]}|x_i|$ and $|y_h|=\max_{j\in[n]}|y_j|$. Then $0<|x_g|\leqslant 1$ and $0<|y_h|\leqslant 1$.

Next, we construct an interval that captures all $l^{p/2,q}$-singular values of $\mathscr{A}$. For the g-th equation of (2a), i.e.,

$$\lambda x_g^{p-1}=\sum_{\substack{i_2,\ldots,i_p\in[m],\\ j_1,\ldots,j_q\in[n]}}a_{gi_2\cdots i_pj_1\cdots j_q}x_{i_2}\cdots x_{i_p}y_{j_1}\cdots y_{j_q},$$

and for an arbitrary real number α_g, it follows from (8) that

$$\begin{aligned}
(\lambda-\alpha_g)x_g^{p-1}&=\lambda x_g^{p-1}-\alpha_g x_g^{p-1}(y_1^{q/2}+\cdots+y_n^{q/2})^2\\
&=\sum_{\substack{i_2,\ldots,i_p\in[m],\\ j_1,\ldots,j_q\in[n]}}a_{gi_2\cdots i_pj_1\cdots j_q}x_{i_2}\cdots x_{i_p}y_{j_1}\cdots y_{j_q}-\sum_{t,k\in[n]}\alpha_g x_g^{p-1}y_t^{q/2}y_k^{q/2}\\
&=\sum_{\substack{i_2,\ldots,i_p\in[m],\\ j_1,\ldots,j_q\in[n],\\ (i_2,\ldots,i_p,j_1,\ldots,j_q)\notin\Lambda_g}}a_{gi_2\cdots i_pj_1\cdots j_q}x_{i_2}\cdots x_{i_p}y_{j_1}\cdots y_{j_q}\\
&\quad+\sum_{t,k\in[n]}(\eta_1 a_{g\underbrace{g\cdots g}_{p-1}\underbrace{t\cdots t}_{q/2}\underbrace{k\cdots k}_{q/2}}-\alpha_g)x_g^{p-1}y_t^{q/2}y_k^{q/2},
\end{aligned}\tag{9}$$

where

$$\Lambda_g:=\Big\{(i_2,\ldots,i_p,j_1,\ldots,j_q):(i_2,\ldots,i_p,j_1,\ldots,j_q)\in\bigcup_{t,k\in[n]}\Big\{(\underbrace{g,\ldots,g}_{p-1},\tau(\underbrace{t,\ldots,t}_{q/2},\underbrace{k,\ldots,k}_{q/2}))\Big\}\Big\},$$

η_1 is the number of permutations of the indices $\underbrace{t,\ldots,t}_{q/2},\underbrace{k,\ldots,k}_{q/2}$ of $a_{g\underbrace{g\cdots g}_{p-1}\underbrace{t\cdots t}_{q/2}\underbrace{k\cdots k}_{q/2}}$, which leads to (6). For the sake of convenience, we still write $a_{g\underbrace{g\cdots g}_{p-1}\underbrace{t\cdots t}_{q/2}\underbrace{k\cdots k}_{q/2}}$ as $a_{gg\cdots gt\cdots tk\cdots k}$.

Taking the modulus of (9) and using the triangle inequality gives

$$\begin{aligned}
|\lambda-\alpha_g||x_g|^{p-1}\leqslant&\sum_{\substack{i_2,\ldots,i_p\in[m],\\ j_1,\ldots,j_q\in[n],\\ (i_2,\ldots,i_p,j_1,\ldots,j_q)\notin\Lambda_g}}|a_{gi_2\cdots i_pj_1\cdots j_q}||x_{i_2}|\cdots|x_{i_p}||y_{j_1}|\cdots|y_{j_q}|\\
&+\sum_{t,k\in[n]}|\eta_1 a_{gg\cdots gt\cdots tk\cdots k}-\alpha_g||x_g|^{p-1}|y_t|^{q/2}|y_k|^{q/2}.
\end{aligned}$$

By $\|\mathrm{y}\|_{q/2}=1$, we have $0\leqslant|y_{j_1}|,\ldots,|y_{j_q}|\leqslant 1$, which implies that

$$\begin{aligned}
&\sum_{\substack{i_2,\ldots,i_p\in[m],\\ j_1,\ldots,j_q\in[n],\\ (i_2,\ldots,i_p,j_1,\ldots,j_q)\notin\Lambda_g}}|a_{gi_2\cdots i_pj_1\cdots j_q}||x_{i_2}|\cdots|x_{i_p}||y_{j_1}|\cdots|y_{j_q}|\\
&\leqslant\sum_{\substack{i_2,\ldots,i_p\in[m],\\ j_1,\ldots,j_q\in[n],\\ (i_2,\ldots,i_p,j_1,\ldots,j_q)\notin\Lambda_g}}|a_{gi_2\cdots i_pj_1\cdots j_q}|\times|x_g|^{p-1}\times\underbrace{1\times\cdots\times 1}_{q}\\
&=\Big(\sum_{i_2,\ldots,i_p\in[m],j_1,\ldots,j_q\in[n]}|a_{gi_2\cdots i_pj_1\cdots j_q}|-\sum_{t,k\in[n]}|\eta_1 a_{gg\cdots gt\cdots tk\cdots k}|\Big)|x_g|^{p-1}\\
&=r_g(\mathscr{A})|x_g|^{p-1},
\end{aligned}$$

and that

$$\begin{aligned}\sum_{t,k\in[n]} &|\eta_1 a_{gg\cdots gt\cdots tk\cdots k}-\alpha_g||y_t|^{q/2}|y_k|^{q/2}\\ &=\sum_{t\in[n]}\Big(\sum_{k\in[n]}|\eta_1 a_{gg\cdots gt\cdots tk\cdots k}-\alpha_g||y_k|^{q/2}\Big)|y_t|^{q/2}\\ &\leqslant \sum_{t\in[n]}\Big(\max_{k\in[n]}|\eta_1 a_{gg\cdots gt\cdots tk\cdots k}-\alpha_g|\sum_{k\in[n]}|y_k|^{q/2}\Big)|y_t|^{q/2}\\ &=\sum_{t\in[n]}\max_{k\in[n]}|\eta_1 a_{gg\cdots gt\cdots tk\cdots k}-\alpha_g||y_t|^{q/2}\\ &\leqslant \max_{t,k\in[n]}|\eta_1 a_{gg\cdots gt\cdots tk\cdots k}-\alpha_g|\sum_{t\in[n]}|y_t|^{q/2}\\ &=\max_{t,k\in[n]}|\eta_1 a_{gg\cdots gt\cdots tk\cdots k}-\alpha_g|.\end{aligned}$$

Then, we have

$$|\lambda-\alpha_g||x_g|^{p-1}\leqslant r_g(\mathscr{A})|x_g|^{p-1} + \max_{t,k\in[n]}|\eta_1 a_{gg\cdots gt\cdots tk\cdots k}-\alpha_g||x_g|^{p-1}.$$

By $|x_g|>0$, we have

$$|\lambda-\alpha_g|\leqslant r_g(\mathscr{A})+\max_{t,k\in[n]}|\eta_1 a_{gg\cdots gt\cdots tk\cdots k}-\alpha_g|,$$

which implies that

$$\lambda\in[\overline{\Psi}(\alpha_g),\overline{\Phi}(\alpha_g)]\subseteq\bigcup_{i\in[m]}[\overline{\Psi}(\alpha_i),\overline{\Phi}(\alpha_i)].$$

Furthermore, by the arbitrariness of $\lambda\in\sigma(\mathscr{A})$, we have $\sigma(\mathscr{A})\subseteq\bigcup_{i\in[m]}[\overline{\Psi}(\alpha_i),\overline{\Phi}(\alpha_i)]$.

Finally, by [12, Theorem 2.2], we have

$$\sigma(\mathscr{A})\subseteq\bigcup_{j\in[n]}[\widehat{\Psi}(\beta_j),\widehat{\Phi}(\beta_j)].$$

Hence, (7) holds. □

Since the interval $\Omega(\mathscr{A},\alpha,\beta)$ involves two parameter vectors $\alpha\in\mathbb{R}^m$ and $\beta\in\mathbb{R}^n$, a natural question arises: how can we minimize $\Omega(\mathscr{A},\alpha,\beta)$ by finding the optimal parameter vectors? This problem is addressed by the following lemma.

Lemma 2 ([18, Lemma 3]) *Let*

$$\varphi(x)=x-\max_{i\in[n]}|x-b_i|-c$$

and

$$\phi(x)=x+\max_{i\in[n]}|x-b_i|+c$$

be two real valued function of $x\in\mathbb{R}$, where $b_i\in\mathbb{R}$ with $b_1\leqslant b_2\leqslant\cdots\leqslant b_n$ and $c\in\mathbb{R}$. Then

(a) $\max_{x\in\mathbb{R}}\varphi(x)=b_1-c$ and this maximum is reached when $x\geqslant\frac{b_1+b_n}{2}$;

(b) $\min_{x\in\mathbb{R}}\phi(x)=b_n+c$ and this minimum is reached when $x\leqslant\frac{b_1+b_n}{2}$.

Theorem 5 *Let $\mathscr{A}=(a_{i_1\cdots i_p j_1\cdots j_q})\in\mathbb{S}^{[p,q;m,n]}$ with p and $q/2$ even and $m,n\geqslant 2$. Then*

$$\sigma(\mathscr{A})\subseteq\Omega(\mathscr{A}):=\Big(\bigcup_{i\in[m]}[\bar{l}_i,\bar{u}_i]\Big)\cap\Big(\bigcup_{j\in[n]}[\hat{l}_j,\hat{u}_j]\Big),$$

where $\hat{l}_j$ and $\hat{u}_j$ is given in (3), η_1 is given in (6) and

$$\begin{aligned}\bar{l}_i&=\min_{t,k\in[n]}\Big\{\eta_1 a_{\underbrace{i\cdots i}_{p}\underbrace{t\cdots t}_{q/2}\underbrace{k\cdots k}_{q/2}}\Big\}-r_i(\mathscr{A}),\\ \bar{u}_i&=\max_{t,k\in[n]}\Big\{\eta_1 a_{\underbrace{i\cdots i}_{p}\underbrace{t\cdots t}_{q/2}\underbrace{k\cdots k}_{q/2}}\Big\}+r_i(\mathscr{A}).\end{aligned}\tag{10}$$

Proof: Let $\lambda\in\sigma(\mathscr{A})$. Lemma 1 shows that $\lambda\in\Omega(\mathscr{A},\alpha,\beta)$ for $\alpha\in\mathbb{R}^m$ and $\beta\in\mathbb{R}^n$, which implies that there is an index $i\in[m]$ and an index $j\in[n]$ such that $\lambda\in[\overline{\Psi}(\alpha_i),\overline{\Phi}(\alpha_i)]$ and $\lambda\in[\widehat{\Psi}(\beta_j),\widehat{\Phi}(\beta_j)]$, where $\overline{\Psi}(\alpha_i)$, $\overline{\Phi}(\alpha_i)$, $\widehat{\Psi}(\beta_j)$ and $\widehat{\Phi}(\beta_j)$ are given in (4).

The result in [12, Theorem 2.6] shows that, for $\widehat{\Psi}(\beta_j)$ and $\widehat{\Phi}(\beta_j)$, if we take

$$\beta_j=0.5\Big(\min_{t\in[m],k\in[n]}\{\eta_2 a_{\underbrace{t\cdots t}_{p}\underbrace{j\cdots j}_{q/2}\underbrace{k\cdots k}_{q/2}}\} + \max_{t\in[m],k\in[n]}\{\eta_2 a_{\underbrace{t\cdots t}_{p}\underbrace{j\cdots j}_{q/2}\underbrace{k\cdots k}_{q/2}}\}\Big),$$

then the optimal interval $[\hat{l}_j,\hat{u}_j]$ of $[\widehat{\Psi}(\beta_j),\widehat{\Phi}(\beta_j)]$ can be derived. Consequently, we have

$$\lambda\in[\hat{l}_j,\hat{u}_j]\subseteq\bigcup_{j\in[n]}[\hat{l}_j,\hat{u}_j].$$

Next, we give the optimal parameter α_i^* such that $[\overline{\Psi}(\alpha_i),\overline{\Phi}(\alpha_i)]$ is minimized. Let

$$\alpha_i^*=0.5\Big(\min_{t,k\in[n]}\{\eta_1 a_{\underbrace{i\cdots i}_{p}\underbrace{t\cdots t}_{q/2}\underbrace{k\cdots k}_{q/2}}\} + \max_{t,k\in[n]}\{\eta_1 a_{\underbrace{i\cdots i}_{p}\underbrace{t\cdots t}_{q/2}\underbrace{k\cdots k}_{q/2}}\}\Big).$$

By Lemma 2, we have

$$\begin{aligned}\max_{\alpha_i\in\mathbb{R}}\overline{\Psi}(\alpha_i)&=\overline{\Psi}(\alpha_i^*)\\ &=\min_{t,k\in[n]}\Big\{\eta_1 a_{\underbrace{i\cdots i}_{p}\underbrace{t\cdots t}_{q/2}\underbrace{k\cdots k}_{q/2}}\Big\}-r_i(\mathscr{A})=\bar{l}_i,\\ \min_{\alpha_i\in\mathbb{R}}\overline{\Phi}(\alpha_i)&=\overline{\Phi}(\alpha_i^*)\\ &=\max_{t,k\in[n]}\Big\{\eta_1 a_{\underbrace{i\cdots i}_{p}\underbrace{t\cdots t}_{q/2}\underbrace{k\cdots k}_{q/2}}\Big\}+r_i(\mathscr{A})=\bar{u}_i,\end{aligned}$$

which implies that the minimum interval of $[\overline{\Psi}(\alpha_i), \overline{\Phi}(\alpha_i)]$ is $[\bar{l}_i, \bar{u}_i]$. Hence,

$$\lambda \in [\bar{l}_i, \bar{u}_i] \subseteq \bigcup_{i\in[m]} [\bar{l}_i, \bar{u}_i].$$

Consequently, we have

$$\lambda \in \Omega(\mathscr{A}) = \Big(\bigcup_{i\in[m]} [\bar{l}_i, \bar{u}_i]\Big) \cap \Big(\bigcup_{j\in[n]} [\hat{l}_j, \hat{u}_j]\Big).$$

Finally, by the arbitrariness of $\lambda \in \sigma(\mathscr{A})$, we have

$$\sigma(\mathscr{A}) \subseteq \Omega(\mathscr{A}).$$

Therefore, the desired conclusion holds. □

By Theorem 2 and the $l^{p,q/2}$-singular value interval $\Omega(\mathscr{A})$ in Theorem 5, we can derive a criterion for judging the positive definiteness of $\mathscr{A}$.

Theorem 6 *Let $\mathscr{A} \in \mathbb{S}^{[p,q;m,n]}$ with p and $q/2$ even and $m, n \geqslant 2$. If $\bar{l}_i > 0$ for $i \in [m]$ or $\hat{l}_j > 0$ for $j \in [n]$, then $\mathscr{A}$ and $\mathscr{A}\mathrm{x}^p\mathrm{y}^q$ in (1) is positive definite.*

Proof: Let $\lambda \in \sigma(\mathscr{A})$. By Theorem 5, we have $\lambda \in \bigcup_{i\in[m]} [\bar{l}_i, \bar{u}_i]$ and $\lambda \in \bigcup_{j\in[n]} [\hat{l}_j, \hat{u}_j]$. Thus, there exist indices $g \in [m]$ and $h \in [n]$ such that $\lambda \in [\bar{l}_g, \bar{u}_g] \cap [\hat{l}_h, \hat{u}_h]$. Assume that $\lambda \leqslant 0$. Then, we must have $\bar{l}_g \leqslant 0$ and $\hat{l}_h \leqslant 0$. However, it contradicts this criterion: $\bar{l}_i > 0$ for $i \in [m]$ or $\hat{l}_j > 0$ for $j \in [n]$. Therefore, $\lambda > 0$. Since λ is an arbitrary $l^{p,q/2}$-singular value, all $l^{p,q/2}$-singular values of $\mathscr{A}$ are positive. By Theorem 2, this implies that $\mathscr{A}$ is positive definite and, consequently, the polynomial $\mathscr{A}\mathrm{x}^p\mathrm{y}^q$ is positive definite. □

Finally, we provide a concluding remark that compares the criteria in Theorem 4 and Theorem 6. We highlight the advantages of the criterion in Theorem 6 over that in Theorem 4. Similarly, a comparison of the intervals in Theorem 3 and Theorem 5 leads to the same conclusion.

Remark 1 Let $\mathscr{A} \in \mathbb{S}^{[p,q;m,n]}$ with p and $q/2$ even. We compare the criteria in Theorem 4 and Theorem 6 in terms of computational cost and performance in determining the positive definiteness of $\mathscr{A}$.

(a) Comparison of computational costs.

To determine the positive definiteness of $\mathscr{A}$ using the criterion in Theorem 6, one needs to verify $\bar{l}_i > 0$ for $i \in [m]$. This requires computing $\eta_1 a_{\underbrace{i\cdots i}_{p}\underbrace{t\cdots t}_{q/2}\underbrace{k\cdots k}_{q/2}}$ for $t, k \in [n]$ and $r_i(\mathscr{A})$ for $i \in [m]$. In contrast, applying the criterion in Theorem 4 requires checking $\tilde{l}_i > 0$ for $i \in [m]$. However, as demonstrated in Method 1 in [12, pp. 980–983], the computation of $\tilde{l}_i$ is notably cumbersome-spanning approximately four pages. This process not only involves the aforementioned computations ($\eta_1 a_{\underbrace{i\cdots i}_{p}\underbrace{t\cdots t}_{q/2}\underbrace{k\cdots k}_{q/2}}$ for $t, k \in [n]$ and $r_i(\mathscr{A})$ for $i \in [m]$), but also requires additional computations for $L_1^i, L_2^i, \ldots, L_7^i, L(i, s_0)$. Furthermore, substantial time overhead is incurred from computing and comparing the values $\frac{a_{i,1}+a_{i,n}}{2}$, $c_{i,1}, c_{i,2}, \ldots, c_{i,\frac{n(n-1)}{2}}$ for $i \in [m]$, where $a_{i,1} \leqslant a_{i,2} \leqslant \cdots \leqslant a_{i,n}$ is an arrangement in non-decreasing order of $a_{\underbrace{i\cdots i}_{p}\underbrace{t\cdots t}_{q}}$ for $t \in [n]$ and $c_{i,1} \leqslant c_{i,2} \leqslant \cdots \leqslant c_{i,\frac{n(n-1)}{2}}$ is an arrangement in non-decreasing order of $\frac{\ell_1}{2} a_{\underbrace{i\cdots i}_{p}\underbrace{t\cdots t}_{q/2}\underbrace{k\cdots k}_{q/2}}$ for $t, k \in [n]$ and $t \neq k$.

(b) Comparison of determination performance.

The criteria provided by Theorem 4 and Theorem 6 offer distinct advantages. That is, for some tensors, Theorem 6 can determine its positive definiteness while Theorem 4 cannot; for other tensors, the converse holds. This point is evident in Example 1.

A NUMERICAL EXAMPLE

In this section, a numerical example is given to show the effectiveness of the interval $\Omega(\mathscr{A})$ in Theorem 5 and the criterion in Theorem 6 in judging the positive definiteness of $\mathscr{A} \in \mathbb{S}^{[p,q;m,n]}$ and $\mathscr{A}\mathrm{x}^p\mathrm{y}^q$ in (1). This example also shows that $\Omega(\mathscr{A})$ and the interval $\Gamma(\mathscr{A})$ in Theorem 3 do not contain each other, and that the criterion in Theorem 6 and the criterion in Theorem 4 each have their own advantages.

Let $\mathscr{A} = (a_{i_1 i_2 j_1 j_2 j_3 j_4}) \in \mathbb{S}^{[2,4;2,2]}$ with its 15 independent entries:

$$\begin{aligned} e_1 &:= a_{111111}, & e_2 &:= a_{111112}, & e_3 &:= a_{111122}, \\ e_4 &:= a_{111222}, & e_5 &:= a_{112222}, & e_6 &:= a_{121111}, \\ e_7 &:= a_{121112}, & e_8 &:= a_{121122}, & e_9 &:= a_{121222}, \\ e_{10} &:= a_{122222}, & e_{11} &:= a_{221111}, & e_{12} &:= a_{221112}, \\ e_{13} &:= a_{221122}, & e_{14} &:= a_{221222}, & e_{15} &:= a_{222222}. \end{aligned} \tag{11}$$

By (3) and (5), we have

$$\begin{aligned} r_1(\mathscr{A}) &= 4e_2 + 4e_4 + e_6 + 4e_7 + 6e_8 + 4e_9 + e_{10}, \\ r_2(\mathscr{A}) &= e_6 + 4e_7 + 6e_8 + 4e_9 + e_{10} + 4e_{12} + 4e_{14}, \\ c_1(\mathscr{A}) &= 3e_2 + e_4 + 2e_6 + 6e_7 + 6e_8 + 2e_9 + 3e_{12} + e_{14}, \\ c_2(\mathscr{A}) &= e_2 + 3e_4 + 2e_7 + 6e_8 + 6e_9 + 2e_{10} + e_{12} + 3e_{14}. \end{aligned} \tag{12}$$

By (3), we have

$$\begin{aligned} \hat{l}_1 &= \min\{e_1, e_{11}, 3e_5, 3e_{13}\} - c_1(\mathscr{A}), \\ \hat{u}_1 &= \max\{e_1, e_{11}, 3e_5, 3e_{13}\} + c_1(\mathscr{A}), \\ \hat{l}_2 &= \min\{e_5, e_{15}, 3e_3, 3e_{13}\} - c_2(\mathscr{A}), \\ \hat{u}_2 &= \max\{e_5, e_{15}, 3e_3, 3e_{13}\} + c_2(\mathscr{A}). \end{aligned} \tag{13}$$

By (10), we have

$$\begin{aligned} \bar{l}_1 &= \min\{e_1, 6e_3, e_5\} - r_1(\mathscr{A}), \\ \bar{u}_1 &= \max\{e_1, 6e_3, e_5\} + r_1(\mathscr{A}), \\ \bar{l}_2 &= \min\{e_{11}, 6e_{13}, e_{15}\} - r_2(\mathscr{A}), \\ \bar{u}_2 &= \max\{e_{11}, 6e_{13}, e_{15}\} - r_2(\mathscr{A}). \end{aligned} \tag{14}$$

Example 1 Let us determine the positive definiteness of the following multivariate homogeneous polynomial with $d_1, d_2, d_3 \in \mathbb{R}$:

$$\begin{aligned} f(\mathrm{x},\mathrm{y}) =& d_1 x_1^2 y_1^4 - 4d_3 x_1^2 y_1^3 y_2 + 6d_2 x_1^2 y_1^2 y_2^2 \\ & - 4d_3 x_1^2 y_1 y_2^3 + d_1 x_1^2 y_2^4 - 2d_1 x_1 x_2 y_1^4 \\ & - 8d_3 x_1 x_2 y_1^3 y_2 - 12 d_3 x_1 x_2 y_1^2 y_2^2 \\ & - 8d_3 x_1 x_2 y_1 y_2^3 - 2d_3 x_1 x_2 y_2^4 + d_1 x_2^2 y_1^4 \\ & - 4d_3 x_2^2 y_1^3 y_2 + 6d_2 x_2^2 y_1^2 y_2^2 - 4d_3 x_2^2 y_1 y_2^3 \\ & + d_1 x_2^2 y_2^4. \end{aligned}$$

Firstly, it is not difficult to verify that

$$f(\mathrm{x},\mathrm{y}) = \mathscr{A}\mathrm{x}^2\mathrm{y}^4,$$

where $\mathscr{A} = (a_{i_1 i_2 j_1 j_2 j_3 j_4}) \in \mathbb{S}^{[2,4;2,2]}$ in (11) with its 15 independent entries:

$$\begin{aligned} & e_1 = e_5 = e_{11} = e_{15} = d_1, \quad e_3 = e_{13} = d_2, \\ & e_2 = e_4 = e_6 = e_7 = e_8 = e_9 = e_{10} = e_{12} = e_{14} = d_3. \end{aligned}$$

Furthermore, by (12), (13) and (14), we have

$$\begin{aligned} r_1(\mathscr{A}) = r_2(\mathscr{A}) = c_1(\mathscr{A}) &= c_2(\mathscr{A}) = 24|d_3|, \\ \hat{l}_i &= \min\{d_1, 3d_2\} - 24|d_3|, \\ \hat{u}_i &= \max\{d_1, 3d_2\} + 24|d_3|, \\ \bar{l}_i &= \min\{d_1, 6d_2\} - 24|d_3|, \\ \bar{u}_i &= \max\{d_1, 6d_2\} + 24|d_3|. \end{aligned} \tag{15}$$

Case I: $d_1 = 2.7$, $d_2 = 0.5$ and $d_3 = -0.1$.

Firstly, we try to use the interval $\Gamma(\mathscr{A})$ in Theorem 3 (i.e., [12, Theorem 2.6]) and the criterion in Theorem 4 (i.e., [12, Theorem 2.7]) to determine the positive definiteness of $\mathscr{A}$. To this end, we need to employ Method 1 in [12, p. 983] to compute $\tilde{l}_i$ and $\tilde{u}_i$ for $i \in [2]$. In Method 1 of [12, p. 983], for $i \in [2]$, $a_{i,1} \leqslant a_{i,2}$ is an arrangement in non-decreasing order of a_{ii1111} and a_{ii2222}, and $c_{i,1} = 3a_{ii1122}$. Furthermore, by

$$a_{111111} = a_{112222} = a_{221111} = a_{222222} = 2.7,$$

we have $a_{1,1} = a_{1,2} = a_{2,1} = a_{2,2} = 2.7$ and $c_{1,1} = c_{2,1} = 1.5$. Subsequently, from $n = 2$,

$$\frac{a_{i,1} + a_{i,n}}{2} = 2.7 > 1.5 = c_{i,\frac{n^2-n}{2}}$$

for $i \in [2]$, $\frac{n^2-n}{2} = 1$ being odd and

$$2^{\frac{q}{2}-1} = 2 > 1 = \frac{n^2 - n + 2}{4},$$

it can be seen that $\tilde{l}_i$ and $\tilde{u}_i$ can be computed by ② of Method 1 (c) in [12, p. 983] and hence

$$\begin{aligned} \tilde{l}_i = L_6^i &= a_{i,1} - \frac{1}{4}\left(a_{i,1} + a_{i,2}\right) + \frac{1}{2} c_{i,1} - r_i(\mathscr{A}) \\ &= \frac{1}{2}(d_1 + 3d_2) - 24|d_3| = -0.3, \\ \tilde{u}_i = U_7^i &= a_{i,2} + r_i(\mathscr{A}) = 5.1, \ i \in [2]. \end{aligned}$$

By (15), we have $\hat{l}_i = -0.9$ and $\hat{u}_i = 5.1$ for $i \in [2]$.

From $\tilde{l}_i < 0$ and $\hat{l}_i < 0$ for $i \in [2]$, it can be seen that the criterion in Theorem 4 (i.e., [12, Theorem 2.7]) is not satisfied, which implies that we cannot use it to determine the positive definiteness of $\mathscr{A}$. Moreover, Theorem 3 indicates that all $l^{p,q/2}$-singular values of $\mathscr{A}$ lie in the interval

$$\begin{aligned} \Gamma(\mathscr{A}) &= ([\tilde{l}_1, \tilde{u}_1] \cup [\tilde{l}_2, \tilde{u}_2]) \cap ([\hat{l}_1, \hat{u}_1] \cup [\hat{l}_2, \hat{u}_2]) \\ &= [-0.3, 5.1]. \end{aligned}$$

From $\sigma(\mathscr{A}) \subseteq [-0.3, 5.1]$, one cannot determine whether all $l^{p,q/2}$-singular values of $\mathscr{A}$ are positive. Hence, this interval $\Gamma(\mathscr{A})$ is insufficient to determine the positive definiteness of $\mathscr{A}$.

Secondly, we try to use the interval $\Omega(\mathscr{A})$ in Theorem 5 and the criterion in Theorem 6 to determine the positive definiteness of $\mathscr{A}$. By (15), we have $\bar{l}_i = 0.3$ and $\bar{u}_i = 5.4$ for $i \in [2]$. Then, from $\bar{l}_i > 0$ for $i \in [2]$, it can be seen that the criterion in Theorem 6 is satisfied, which implies that $\mathscr{A}$ is positive definite. Moreover, Theorem 5 shows that all $l^{p,q/2}$-singular values of $\mathscr{A}$ lie in the interval

$$\begin{aligned} \Omega(\mathscr{A}) &= ([\bar{l}_1, \bar{u}_1] \cup [\bar{l}_2, \bar{u}_2]) \cap ([\hat{l}_1, \hat{u}_1] \cup [\hat{l}_2, \hat{u}_2]) \\ &= [0.3, 5.1], \end{aligned}$$

which implies that all $l^{p,q/2}$-singular values of $\mathscr{A}$ are positive and hence that $\mathscr{A}$ is positive definite. Hence, $f(\mathrm{x},\mathrm{y}) = \mathscr{A}\mathrm{x}^2\mathrm{y}^4$ is positive definite.

Finally, we compute all $l^{p,q/2}$-singular triplets of $\mathscr{A}$ by the direct method in [12, Theorem 3.1] to show the correctness and effectiveness of Theorem 5 and Theorem 6. All $l^{p,q/2}$-singular triplets of $\mathscr{A}$ are listed in Table 1 as follows.

Table 1 All $l^{p,q/2}$-singular triplets $(\lambda,\mathrm{x},\mathrm{y})$ of $\mathscr{A}$ in Case I.

No.	λ	x	y
1	2.8	$\pm(-0.7071, 0.7071)^\top$	$\pm(1, 0)^\top$
2	2.8	$\pm(-0.7071, 0.7071)^\top$	$\pm(0, 1)^\top$
3	2.6571	$\pm(0.7071, 0.7071)^\top$	$\pm(-0.1444, 0.9895)^\top$
4	2.6571	$\pm(0.7071, 0.7071)^\top$	$\pm(0.9895, -0.1444)^\top$
5	2.3	$\pm(\sqrt{1-w^2}, w)^\top, w \in \mathbb{R}$,	$\pm(0.7071, -0.7071)^\top$
6	2.3	$\pm(0.7071, 0.7071)^\top$	$\pm(0.7071, -0.7071)^\top$
7	2.3	$\pm(0.7071, -0.7071)^\top$	$\pm(0.7071, 0.7071)^\top$
8	1.5	$\pm(0.7071, 0.7071)^\top$	$\pm(0.7071, 0.7071)^\top$

In summary, we have demonstrated that for this case, Theorem 3 and Theorem 4 cannot be used to determine the positive definiteness of $\mathscr{A}$ and $\mathscr{A}\mathrm{x}^p\mathrm{y}^q$, while Theorem 5 and Theorem 6 can. This fact is clearly demonstrated by Table 2, where $i \in [2]$ and the abbreviations are defined as follows: "CS" for "Criterion Satisfied", "DR" for "Determination Result" and "PD" for "Positive Definite".

Case II: $d_1 = 4$, $d_2 = 0.3$ and $d_3 = -0.1$.

Table 2 Comparisons of Theorems 3, 4, 5, and 6 for $\mathscr{A}$ in Case I.

Method	$\tilde{l}_i$	$\hat{l}_i$	$\bar{l}_i$	Interval	CS	DR
Theorem 3	−0.3	−0.9		$[-0.3, 5.1]$		Inconclusive
Theorem 4	−0.3	−0.9			No	Inconclusive
Theorem 5		−0.9	0.3	$[0.3, 5.1]$		PD
Theorem 6		−0.9	0.3		Yes	PD

Firstly, we consider the interval $\Omega(\mathscr{A})$ in Theorem 5. By (15), we have $\bar{l}_i = -0.6$, $\hat{l}_i = -1.5$ and $\bar{u}_i = \hat{u}_i = 6.4$ for $i \in [2]$. From $\bar{l}_i = -0.6$ and $\hat{l}_i = -1.5$ for $i \in [2]$, it can be seen that the criterion in Theorem 6 is not satisfied. By Theorem 5, we know that all $l^{p,q/2}$-singular values of $\mathscr{A}$ lie in the interval

$$\begin{aligned}\Omega(\mathscr{A}) &= ([\bar{l}_1, \bar{u}_1] \cup [\bar{l}_2, \bar{u}_2]) \cap ([\hat{l}_1, \hat{u}_1] \cup [\hat{l}_2, \hat{u}_2]) \\ &= [-0.6, 6.4].\end{aligned}$$

From $\sigma(\mathscr{A}) \subseteq [-0.6, 6.4]$, one cannot determine whether all $l^{p,q/2}$-singular values of $\mathscr{A}$ are positive. Hence, we cannot use Theorem 5 and Theorem 6 to determine the positive definiteness of $\mathscr{A}$.

Secondly, we consider the interval of $\Gamma(\mathscr{A})$ in Theorem 3 and the criterion in Theorem 4. Following a computational process similar to that in Case 1, by $n = 2$, $\frac{n^2-n}{2} = 1$ and

$$a_{111111} = a_{112222} = a_{221111} = a_{222222} = 4,$$

we have $a_{1,1} = a_{1,2} = a_{2,1} = a_{2,2} = 4$ and $c_{1,1} = c_{2,1} = 0.9$. Furthermore, by

$$\frac{a_{i,1} + a_{i,n}}{2} = 4 > 0.9 = c_{i, \frac{n^2-n}{2}}$$

for $i \in [2]$, $\frac{n^2-n}{2} = 1$ being odd and

$$2^{\frac{q}{2}-1} = 2 > 1 = \frac{n^2 - n + 2}{4},$$

we know that $\tilde{l}_i$ and $\tilde{u}_i$ can be computed by ② of Method 1 (c) in [12, p. 983] and hence

$$\begin{aligned}\tilde{l}_i &= L_6^i = a_{i,1} - \frac{1}{4}\left(a_{i,1} + a_{i,2}\right) + \frac{1}{2} c_{i,1} - r_i(\mathscr{A}) \\ &= \frac{1}{2}(d_1 + 3d_2) - 24|d_3| = 0.05, \\ \tilde{u}_i &= U_7^i = a_{i,2} + r_i(\mathscr{A}) = 6.4, \ i \in [2].\end{aligned}$$

From $\tilde{l}_i > 0$ for $i \in [2]$, it can be seen that the criterion in Theorem 4 is satisfied, which implies that $\mathscr{A}$ and $f(\mathrm{x}, \mathrm{y}) = \mathscr{A}\mathrm{x}^2\mathrm{y}^4$ are positive definite.

Moreover, Theorem 3 shows that all $l^{p,q/2}$-singular values of $\mathscr{A}$ lie in the interval

$$\begin{aligned}\Gamma(\mathscr{A}) &= ([\tilde{l}_1, \tilde{u}_1] \cup [\tilde{l}_2, \tilde{u}_2]) \cap ([\hat{l}_1, \hat{u}_1] \cup [\hat{l}_2, \hat{u}_2]) \\ &= [0.05, 6.4],\end{aligned}$$

which implies that all $l^{p,q/2}$-singular values of $\mathscr{A}$ are positive. This also shows that $f(\mathrm{x}, \mathrm{y}) = \mathscr{A}\mathrm{x}^2\mathrm{y}^4$ is positive definite.

Finally, all $l^{p,q/2}$-singular triplets of $\mathscr{A}$ are computed by the direct method in [12, Theorem 3.1] and listed in Table 3 as follows.

Table 3 All $l^{p,q/2}$-singular triplets $(\lambda, \mathrm{x}, \mathrm{y})$ of $\mathscr{A}$ in Case II.

No.	λ	x	y
1	4.1	$\pm(0.7071, -0.7071)^\top$	$\pm(1, 0)^\top$
2	4.1	$\pm(-0.7071, 0.7071)^\top$	$\pm(0, 1)^\top$
3	3.9242	$\pm(0.7071, 0.7071)^\top$	$\pm(-0.0607, 0.9982)^\top$
4	3.9242	$\pm(0.7071, 0.7071)^\top$	$\pm(0.9981, -0.0607)^\top$
5	2.65	$\pm(\sqrt{1-w^2}, w)^\top, w \in \mathbb{R}$,	$\pm(0.7071, -0.7071)^\top$
6	2.65	$\pm(0.7071, 0.7071)^\top$	$\pm(0.7071, -0.7071)^\top$
7	2.65	$\pm(0.7071, -0.7071)^\top$	$\pm(0.7071, 0.7071)^\top$
8	1.85	$\pm(0.7071, 0.7071)^\top$	$\pm(0.7071, 0.7071)^\top$

Table 3 verifies the correctness of Theorems 3, 4, 5, and 6. In summary, we have demonstrated that for this case, Theorems 5 and 6 cannot be used to determine the positive definiteness of $\mathscr{A}$ and $\mathscr{A}\mathrm{x}^p\mathrm{y}^q$, while Theorems 3 and 4 can. This fact is clearly demonstrated by Table 4, where $i \in [2]$ and the abbreviations are defined as follows: "CS" for "Criterion Satisfied", "DR" for "Determination Result" and "PD" for "Positive Definite".

Table 4 Comparisons of Theorems 3, 4, 5, and 6 for $\mathscr{A}$ in Case II.

Method	$\tilde{l}_i$	$\hat{l}_i$	$\bar{l}_i$	Interval	CS	DR
Theorem 3	0.05	−1.5		[0.05,6.4]		PD
Theorem 4	0.05	−1.5			Yes	PD
Theorem 5		−1.5	−0.6	$[-0.6, 6.4]$		Inconclusive
Theorem 6		−1.5	−0.6		No	Inconclusive

Finally, from Cases I and II, it can be seen that neither of the two intervals in Theorems 3 and 5 contains the other, and the criteria in Theorems 4 and 6 each have their own advantages.

CONCLUSION

Let $\mathscr{A} \in \mathbb{S}^{[p,q;m,n]}$ with p and $q/2$ positive even integers and $m, n \geqslant 2$. To determine the positive definiteness of $\mathscr{A}$, we first provided, in Theorem 5, an interval $\Omega(\mathscr{A})$ that captures all $l^{p,q/2}$-singular values of $\mathscr{A}$. Then, in Theorem 6, we presented a criterion for this determination. Finally, via Example 1, we showed that:

(a) the two intervals $\Gamma(\mathscr{A})$ in Theorem 3 and $\Omega(\mathscr{A})$ in Theorem 5 do not contain each other;

(b) the two criteria in Theorem 4 and Theorem 6 each have their own advantages.

However, as can be seen from Tables 1 and 2 (or, Tables 3 and 4), the intervals provided by Theorems 3 and 5, while capable of capturing all singular values, are nevertheless not precise. Hence, how to derive a more precise localization interval remains a question worthy of consideration.

Acknowledgements: This work is supported by Guizhou Provincial Science and Technology Projects, China (Grant No. QKHJC-ZK[2022]YB215)

REFERENCES

1. Knowles JK, Sternberg E (1975) On the ellipticity of the equations of nonlinear elastostatics for a special material. *J Elast* **5**, 341–361.
2. Wang Y, Aron M (1996) A reformulation of the strong ellipticity conditions for unconstrained hyperelastic media. *J Elast* **44**, 89–96.
3. Dahl G, Leinaas JM, Myrheim J, Ovrum E (2007) A tensor product matrix approximation problem in quantum physics. *Linear Algebra Appl* **420**, 711–725.
4. Einstein A, Podolsky B, Rosen N (1935) Can quantum-mechanical description of physical reality be considered complete? *Phys Rev* **47**, 777–780.
5. Xiong L, Qin Q, Liu J, Gong Z, Jiang Z, Sze NS (2023) Multipartite strongly symmetric states and applications to geometric entanglement and classicality. *Quantum Inf Process* **22**, 291.
6. Ling C, Qi L (2013) $l^{k,s}$-singular values and spectral radius of rectangular tensors. *Front Math China* **8**, 63–83.
7. Yao H, Long B, Bu C, Zhou J (2016) $l^{k,s}$-singular values and spectral radius of partially symmetric rectangular tensors. *Front Math China* **11**, 605–622.
8. Chang K, Qi L, Zhou G (2010) Singular values of a real rectangular tensor. *J Math Anal Appl* **370**, 284–294.
9. He J, Liu Y, Xu G, Liu G (2019) V-singular values of rectangular tensors and their applications. *J Inequal Appl* **2019**, 84.
10. Zhao J (2021) $l^{p,q}$-singular values of a partially symmetric rectangular tensor. *Pac J Optim* **17**, 151–173.
11. Zhao J (2023) $l^{p/2,q/2}$-singular values of a real partially symmetric rectangular tensor. *Japan J Indust Appl Math* **40**, 843–875.
12. Zhao J (2024) $l^{p,q/2}$-singular values of a real partially symmetric rectangular tensor. *J Ind Manage Optim* **20**, 967–993.
13. Yao H, Zhang C, Liu L, Zhou J, Bu C (2019) Singular value inclusion sets of rectangular tensors. *Linear Algebra Appl* **576**, 181–199.
14. Yao H, Ma L, Liu C, Bu C (2020) Brualdi-type inclusion sets of Z-eigenvalues and $l^{k,s}$-singular values for tensors. *Front Math China* **15**, 601–612.
15. Liu D, Li W, Vong SW, Li J, Chen Y (2025) Riemannian conjugate gradient algorithms for solving λ-approximation of stochastic tensors and applications. *Numer Linear Algebra Appl* **32**, e70010.
16. Zhao J, Li C (2018) Singular value inclusion sets for rectangular tensors. *Linear Multilinear Algebra* **66**, 1333–1350.
17. Zhao J (2019) Two new singular value inclusion sets for rectangular tensors. *Linear Multilinear Algebra* **67**, 2451–2470.
18. Sang C, Zhao J (2022) Criteria for the strong ellipticity condition of a partially symmetric tensor. *Acta Appl Math* **180**, 10.