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Multiplicity and concentration of solutions for the fractional Schrödinger equation involving sign-changing potential

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ABSTRACT: In this paper, a class of fractional Schrödinger equations is investigated. We extend the general non-negative continuous potential functions to the sign-changing ones, i.e., $a_\lambda(x)$. Under certain conditions, the equation possesses two nontrivial solutions. Furthermore, based on the sign-changing $a_\lambda(x)$, some concentration phenomena are obtained.

KEYWORDS: variational methods, fractional Schrödinger equation, multiple solutions, concentration behavior

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INTRODUCTION

The aim of this paper is to study the following equation

$$(-\Delta)^s u + a_\lambda(x)u = K(x)f(u) + \mu W(x)|u|^{p-2}u, \quad (1)$$

for $x \in \mathbb{R}^N$, where $N > 2s$, $s \in (0,1)$, $a_\lambda(x) = \lambda a^+(x) - a^-(x)$ with $\lambda > 0$, $a^\pm(x) = \max\{\pm a(x), 0\}$, $a \in L^{N/2s}(\mathbb{R}^N, \mathbb{R})$ and $\mu > 0$. Here $(-\Delta)^s$ denotes the fractional Laplacian, which is defined as

$$(-\Delta)^s u(x) = \mathrm{P.V.}\int_{\mathbb{R}^N} \frac{\mathrm{u(x)-u(y)}}{\mathrm{|x-y|^{N+2s}}}\,\mathrm{dy},$$

for $x \in \mathbb{R}^N$, where P.V. denotes the principal value of the integral. We establish the existence and concentration behavior of nontrivial solutions of Eq. (1) under appropriate assumptions.

Variational methods applied to PDEs is an interesting field, as evidenced by numerous works such as [1–4]. Recently, the nonlocal problems have attracted widespread attention, such as the fractional problems (e.g., [5–7]), time-fractional models (e.g., [8]), p-Laplacian boundary value problems (e.g., [9]), the fractional GinzburgLandau equations (e.g., [10]) and so on. The fractional Laplacian equations have been widely studied using variational methods (e.g., [11–13]) after the paper [12] established the framework for the solvability of fractional Laplacian problems. Eq. (1) has also been widely studied, particularly when we focus on standing wave solutions $\Psi(x,t) = e^{-iwt}u(x)$ for the non-local problem

$$i\hbar\partial_t\Psi = (-\Delta)^s\Psi + a_\lambda(x)\Psi - g(x,\Psi),\ x \in \mathbb{R}^N,\ t \in \mathbb{R},$$

where w is a constant, $\hbar$ is the Planck's constant and g is a general nonlinear function.

There have been numerous works on problem (1) when $\mu = 0$ and $s = 1$, as seen in [14–17] and related references. For example, in [15], Ambrosetti et al investigated the zero mass case for problem

$$-\varepsilon^2\Delta u + a(x)u = K(x)|u|^\sigma (1 < \sigma < 2^* - 1), \quad (2)$$

where a and K are smooth functions from $\mathbb{R}^N$ to $\mathbb{R}$, $2^* = \frac{2N}{N-2s}$, and there exist positive constants b_1, b_2, b_3, B_1 and B_2 such that

$$\frac{b_3}{1+|x|^{b_1}} \leqslant a(x) \leqslant B_1 \text{ and } 0 < K(x) \leqslant \frac{B_2}{1+|x|^{b_2}},$$

$\forall x \in \mathbb{R}^N$. It is worth noting that this class of potentials $a(x)$ does not satisfy (A_0), which requires the existence of a positive constant such that $\{x : a(x) < b\} \neq \varnothing$ and has finite measure. Wang and Zhou also considered the similar type of equations as (2) in [18], where the vanishing potential $a \in C(\mathbb{R}^N, \mathbb{R})$ changes sign and is negative for some bounded set.

Furthermore, when the nonlinearities in Eq. (1) are more general mixed nonlinearities, some interesting results have been obtained, see [19–22]. However, there are few results for Eq. (1) with mixed nonlinearities and vanishing potential $a(x)$. For example, Yang and Zhao [22] studied a non-local problem with a sublinear perturbation in the nonlinearity involves, given by

$$(-\Delta)^s u + a(x)u = K(x)f(u) + \mu W(x)|u|^{p-2}u, \quad (3)$$

for $x \in \mathbb{R}^N, N > 2s$, where $W \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$ $(1 < p < 2)$, $\mu > 0$, and $a \in C(\mathbb{R}^N, \mathbb{R}^+)$ with $\lim_{x\to\infty} a(x) = 0$. They established the existence of two nontrivial solutions to (3).

It is worth noting that [19] and [20] considered the function $a_\lambda(x) = \lambda a^+(x) - a^-(x)$, where $a(x)$ satisfies certain conditions such as:

$(\bar{A}1)$ $a^+(x)$ is continuous and $a^- \in L^{\frac{N}{2}}(\mathbb{R}^N, \mathbb{R})$;

$(\bar{A}2)$ there exists a positive constant b such that

$$\Omega_1 := \{x \in \mathbb{R}^N : a^+(x) < b\} \neq \varnothing$$

and has finite measure.

In [19], they investigated the concave-convex nonlinearities (see [23]) under the above conditions for $a(x)$ and obtained results on the multiplicity and behavior of non-trivial solutions using the Nehari manifold. Similarly, Peng and Xia [20] obtained some analogous results for the corresponding fractional elliptic equations under similar conditions as in [19]. It is worth noting that the results in the papers [19, 20] rely heavily on the Nehari manifold.

Motivated by these works, the present paper aims to investigate Eq. (1), establish the existence of two nontrivial solutions and explore concentration phenomena. The main argument of this paper is to investigate a class of fractional Schrödinger equation. Based on the sign-changing potential $a_\lambda(x)$ and some certain conditions, the equation possesses two nontrivial solutions and some concentration phenomena are obtained. We specifically highlight the novelty of this article. In our approach, the Nehari manifold is not required, and there are no positive and continuity hypotheses on potential $a_\lambda(x)$. Additionally, the potential $a_\lambda(x)$ does not satisfy condition $(\bar{A}_2)$ of the papers [19, 20]. Compared with Yang and Zhao [22], we discussed Eq. (1) under a more general potential function $a_\lambda(x)$.

Before presenting our main results, we define $a^+(x) = \max\{a(x), 0\}$ and $a^-(x) = \max\{-a(x), 0\}$ and assume that the functions f and K are continuous on $\mathbb{R}^N$. We also impose that the following conditions hold.

(A_0) $a_\lambda(x) := \lambda a^+(x) - a^-(x)$ with $\lambda > 0$ and $a \in L^{N/2s}(\mathbb{R}^N, \mathbb{R})$.

(A_1) There exists $R_0 > 0$ such that

$$a(x) > 0, \text{ for a.e. } |x| \geqslant R_0,$$

and set

$$\Omega^* = \{x \in \mathbb{R}^N | a(x) < 0, \ |x| \geqslant R_0\}.$$

(A_2) There exists $\Omega_2 \subset \mathbb{R}^N$, and the Lebesgue measure $m(\Omega_2) > 0$, where

$$\Omega_2 = \{x \in \mathbb{R}^N | a(x) < 0\},$$

and we set $\Omega_3 = \Omega_2 \backslash \Omega^*$.

(A_3) There exists a positive constant $\eta_0 > 1$ such that

$$\eta_1 := \inf_{u \in H^s(\mathbb{R}^N)\backslash\{0\}} \frac{\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} \lambda a^+(x) u^2 \,\mathrm{d}x}{\int_{\mathbb{R}^N} a^-(x) u^2 \,\mathrm{d}x} \geqslant \eta_0,$$

for all $\lambda > 0$.

(A_4) The set

$$\Omega_4 = \operatorname{int}\{x \in \mathbb{R}^N : a^+(x) = 0\}$$

is nonempty and has a smooth boundary with $\bar{\Omega}_4$.

(K_1) $K(x) > 0, \ \forall x \in \mathbb{R}^N$ and $K \in L^\infty(\mathbb{R}^N)$.

(K_2) If $\{A_n\} \subset \mathbb{R}^N$ is a sequence of Borel sets such that $|A_n| \leqslant R$ for all n and some $R > 0$, then

$$\lim_{r\to\infty} \int_{A_n \cap B_r^c(0)} K(x) dx = 0, \text{ uniformly in } n \in \mathbb{N}.$$

(K_3) There exists $q \in (2, 2_s^*)$ such that

$$\frac{K(x)}{[a(x)]^{\frac{2_s^*-q}{2_s^*-2}}} \to 0 \text{ as } |x| \to \infty,$$

where $2_s^* = \frac{2N}{N-2s}$.

(f_1) $\lim_{t\to 0} \frac{f(t)}{|t|^{q-1}} < \infty$.

(f_2) f has a quasicritical growth, that is,

$$\lim_{t\to\infty} \frac{f(t)}{|t|^{2_s^*-1}} = 0.$$

(f_3) There exists $\theta > 2$ such that $0 \leqslant \theta F(t) \leqslant t f(t)$, for all $t \in \mathbb{R}$, where $F(t) = \int_0^t f(\tau)\,\mathrm{d}\tau$.

We will now provide some explanations for conditions listed above.

Remark 1 (i) Obviously, (A_1) implies that $m(\Omega^*) = 0$, and the set Ω_3 defined in (A_2) is finite measure. Since $m(\Omega_2) > 0$ and $m(\Omega^*) = 0$, $m(\Omega_3) \neq 0$. It follows that $a_\lambda(x)$ is sign-changing.

(ii) By (A_3) and a simple calculation, for each $\lambda \geqslant 0$,

$$[v]^2 + \int_{\mathbb{R}^N} \lambda a^+ |v|^2 \,\mathrm{d}x \geqslant [v]^2 + \int_{\mathbb{R}^N} a_\lambda |v|^2 \,\mathrm{d}x \geqslant \frac{\eta_0 - 1}{\eta_0}\Big([v]^2 + \int_{\mathbb{R}^N} \lambda a^+ |v|^2 \,\mathrm{d}x\Big), \quad (4)$$

where $[v]^2 = \int_{\mathbb{R}^{2N}} \frac{(v(x)-v(y))^2}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y$. More details about (A_3) can be found in [19].

(iii) The hypotheses on functions $K(x)$ were firstly introduced in [14]. As stated in [14], it is important to observe that (K_2) is weaker than any of the below conditions:

(a) There are $r \geqslant 1$ and $\rho \geqslant 0$ such that $K \in L^r(\mathbb{R}^N \backslash B_\rho(0))$;

(b) $K(x) \to 0$ as $|x| \to \infty$;

(c) $K = H_1 + H_2$, with H_1 and H_2 verifying (a) and (b).

Based on the above conditions, we can state our main results, which establish the existence of two nontrivial solutions to Eq. (1) and explore concentration phenomena.

Theorem 1 *Assume* (A_0)–(A_3), (K_1)–(K_3), (f_1)–(f_3) *hold and* $W \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$, *where* $1 < p < 2$. *Then there exists* $\mu_0 > 0$, *for each* $\mu (< \mu_0)$ *and* $\lambda \geqslant 0$, *Eq. (1) has two nontrivial weak solutions.*

Moreover, to obtain the following theorem, we recall condition (A_4) defined above. By observing that $m(\Omega_2) > 0$ and the definition of $a^+(x)$, we can see that (A_4) is well defined. Furthermore, by using conditions (A_1) and (A_4), we can conclude that Ω_4 is bounded and $0 < m(\Omega_4) < \infty$.

Theorem 2 *Under the conditions of Theorem 1 and assuming that* (A_4) *holds, we have that for every* $r \in (2, 2_s^*)$,

$$u_\lambda^{(2)} \to u^{(2)} \text{ strongly in } L^r(\mathbb{R}^N) \text{ as } \lambda \to \infty,$$

where $u_\lambda^{(2)}$ *be one of the nontrivial solutions obtained by Theorem 1 with the negative energy functional. Furthermore,* $u^{(2)} \in X_0$ *is a nontrivial solution of the following equation*

$$\begin{cases} (-\Delta)^s u - a^-(x)u \\ \quad = K(x)f(u) + \mu W(x)|u|^{p-2}u, \ x \in \Omega_4; \\ u = 0, \ x \in \mathbb{R}^N \backslash \Omega_4. \end{cases} \tag{5}$$

The key contributions of our work lies in elucidating the intricate interplay between the fractional Laplacian and the sign-changing potential, thereby offering valuable insights into the multiplicity and concentration phenomena within the framework of the fractional Schrödinger equation.

PRELIMINARY RESULTS

Firstly, recall some basic results of fractional Sobolev space. For any $0 < s < 1$, the fractional Sobolev space $H^s(\mathbb{R}^N)$ (see [12, 13]) is well-defined as follows:

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{N+2s}{2}}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\}.$$

$D^s(\mathbb{R}^N)$ denotes the completion space $C_0^\infty(\mathbb{R}^N)$ with the norm

$$[u]^2 = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \,\mathrm{d}x \,\mathrm{d}y.$$

Recall the subspace

$$E = \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} a^+(x)|u(x)|^2 \,\mathrm{d}x < \infty \right\}.$$

E is a separable Hilbert space with the inner product

$$\begin{aligned} \langle u, v \rangle = & \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \,\mathrm{d}x \,\mathrm{d}y \\ & + \int_{\mathbb{R}^N} a^+(x) u(x) v(x) \,\mathrm{d}x \end{aligned}$$

and the norm

$$\|u\|^2 = \langle u, u \rangle = [u]^2 + \int_{\mathbb{R}^N} a^+(x) u^2 \,\mathrm{d}x.$$

For each $\lambda > 0$, set

$$E_\lambda = \left\{ u \in H^s(\mathbb{R}^N) : \lambda \int_{\mathbb{R}^N} a^+(x)|u|^2 \,\mathrm{d}x < \infty \right\}$$

equipped with the following inner product

$$\begin{aligned} \langle u, v \rangle_\lambda = & \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \,\mathrm{d}x \,\mathrm{d}y \\ & + \lambda \int_{\mathbb{R}^N} a^+(x) u(x) v(x) \,\mathrm{d}x \end{aligned}$$

and the corresponding norm

$$\|u\|_\lambda^2 = \langle u, u \rangle_\lambda = [u]^2 + \lambda \int_{\mathbb{R}^N} a^+(x) u^2 \,\mathrm{d}x.$$

For $\lambda \geqslant 1$, $\|u\|_\lambda^2 \geqslant \|u\|^2$ and the embedding $H^s(\mathbb{R}^N) \subset L^r(\mathbb{R}^N)$ is continuous for any $r \in [2, 2_s^*]$ and is locally compact whenever $r \in [2, 2_s^*)$. And set $\|u\|_{H^s(\mathbb{R}^N)}^2 = [u]^2 + \int_{\mathbb{R}^N} u^2 \,\mathrm{d}x$. Moreover, there exist $C_1(\lambda)$ and $C_2(\lambda)$, $C_1(\lambda) \|u\|_{H^s(\mathbb{R}^N)}^2 \leqslant \|u\|_\lambda^2 \leqslant C_2(\lambda) \|u\|_{H^s(\mathbb{R}^N)}^2$, $\forall u \in E_\lambda$, see [24]. Let L_K^r be the weighted Lebesgue space of measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ satisfying

$$|u|_{K,r} = \left[\int_{\mathbb{R}^N} K(x)|u(x)|^r \,\mathrm{d}x \right]^{1/r} < \infty,$$

where $1 \leqslant r < \infty$.

Define the energy functional $\mathscr{I}_\lambda : E_\lambda \to \mathbb{R}$,

$$\begin{aligned} \mathscr{I}_\lambda(u) = & \frac{1}{2} \|u\|_\lambda^2 - \frac{1}{2} \int_{\mathbb{R}^N} a^-(x)|u|^2 \,\mathrm{d}x \\ & - \int_{\mathbb{R}^N} K(x) F(u) \,\mathrm{d}x - \frac{\mu}{p} \int_{\mathbb{R}^N} W(x)|u|^p \,\mathrm{d}x. \end{aligned}$$

Obviously, $\mathscr{I}_\lambda(u) \in C^1(E_\lambda, \mathbb{R})$ and its differential is given by

$$\begin{aligned}\langle \mathscr{J}'_\lambda(u), v\rangle = &\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\\ &+\int_{\mathbb{R}^N} \lambda a^+(x)u(x)v(x)\,\mathrm{d}x\\ &-\int_{\mathbb{R}^N} a^-(x)u(x)v(x)\,\mathrm{d}x\\ &-\int_{\mathbb{R}^N} K(x)f(u(x))v(x)\,\mathrm{d}x\\ &-\int_{\mathbb{R}^N} \mu W(x)|u(x)|^{p-1}v(x)\,\mathrm{d}x,\end{aligned}$$

for all $v\in E_\lambda$. We say that if $u\in E_\lambda$ is a critical point of $\mathscr{J}_\lambda$, then u is a weak solution of Eq. (1).

Throughout this paper, $B_r(x)$ or $B(x,r)$ devotes the open ball with center at x and radius r. And C devote some positive constants may change from line to line. From this now, we give some lemmas. The following lemmas have a slight differences from the existing conclusions. For example, in a standard way (see [25]), one can check that the functional $\mathscr{J}_\lambda$ satisfies the mountain pass geometry. To clarify the proof, we provide the proof process.

Lemma 1 *Assume* (A_0)*–*(A_3)*,* (K_1)*–*(K_3)*,* (f_1)*–*(f_3) *and* $W\in L^{\frac{2}{2-p}}(\mathbb{R}^N,\mathbb{R}^+)$*, where* $1<p<2$*. Then, there exists* $\mu_0>0$*, for each* $\lambda>0$ *and* $0<\mu<\mu_0$*,* $\mathscr{J}_\lambda$ *satisfies the Mountain Pass geometry:*

(i) *there exist* $\beta>0$ *and* $\rho>0$ *such that* $\mathscr{J}_\lambda\geqslant\beta$ *for* $\|u\|_\lambda=\rho$ *and* $\mu\in(0,\mu_0)$*;*

(ii) *there exists* $e\in E_\lambda$ *satisfying* $\|e\|_\lambda>\rho$ *such that* $\mathscr{J}_\lambda(e)<0$*.*

Proof: By (4), $\|u\|_\lambda^2\geqslant[u]^2+\int_{\mathbb{R}^N}a_\lambda(x)|u|^2dx\geqslant\frac{\eta_0-1}{\eta_0}\|u\|_\lambda^2$. For any $\varepsilon>0$, it follows from (f_1) and (f_2) that there exists $C_\varepsilon>0$ such that

$$K(x)F(x,u)\leqslant\varepsilon C\|u\|_\lambda^2+C_\varepsilon\|u\|_\lambda^{2_s^*}.$$

Consequently, for $1<p<2$,

$$\begin{aligned}\mathscr{J}_\lambda(u)=&\frac{1}{2}\|u\|_\lambda^2-\frac{1}{2}\int_{\mathbb{R}^N}a^-(x)|u|^2\,\mathrm{d}x\\ &-\int_{\mathbb{R}^N}K(x)F(u)\,\mathrm{d}x-\frac{\mu}{p}\int_{\mathbb{R}^N}W(x)|u|^p\,\mathrm{d}x\\ \geqslant&\Big(\frac{\eta_0-1}{\eta_0}-\varepsilon C\Big)\|u\|_\lambda^2-\frac{\mu C}{p}|W|_{\frac{2}{2-p}}\|u\|_\lambda^p-C_\varepsilon\|u\|_\lambda^{2_s^*}\\ =&\|u\|_\lambda^p\Big[\Big(\frac{\eta_0-1}{\eta_0}-\varepsilon C\Big)\|u\|_\lambda^{2-p}-C_\varepsilon\|u\|_\lambda^{2^*-p}-\frac{\mu C}{p}|W|_{\frac{2}{2-p}}\Big].\end{aligned}$$

Fixed ε such that $\frac{\eta_0-1}{\eta_0}-\varepsilon C>0$ and set $H(t)=At^{2-p}-Bt^{2^*-p}-\frac{\mu C}{p}|W|_{\frac{2}{2-p}}$. Hence, it is not difficult to see, there is $\mu_0>0$, for $\mu\in(0,\mu_0)$ and small $t(>0)$, $H(t)>0$. Then by a standard way, we can derive the lemma. □

Let

$$c_\lambda=\inf_{\gamma\in\Gamma_\lambda}\max_{0<t\leqslant1}\mathscr{J}_\lambda(\gamma(t)),$$

where $\Gamma_\lambda=\{\gamma\in(C[0,1],E_\lambda):\gamma(0)=0,\gamma(1)=e\}$. By Mountain Pass theorem (see [25]), there exists a (PS) sequence $\{u_n\}$, that is,

$$\mathscr{J}_\lambda(u_n)\to c_\lambda \text{ and } \mathscr{J}'_\lambda(u_n)\to0. \tag{6}$$

Lemma 2 (A_0)*–*(A_3)*,* (K_1)*–*(K_3)*,* (f_1)*–*(f_3) *and* $W\in L^{\frac{2}{2-p}}(\mathbb{R}^N,\mathbb{R}^+)$*, where* $1<p<2$*. For each* $\lambda>0$ *and* $\mu>0$*, (PS) sequence* $\{u_n\}$ *is bounded in* E_λ*.*

Proof: As in [22] and (f_3), we derive that

$$\begin{aligned}1+c+\|u_n\|_\lambda\geqslant&\mathscr{J}_\lambda(u_n)-\frac{1}{\theta}\langle\mathscr{J}'_\lambda(u_n),u_n\rangle\\ =&\Big(\frac{1}{2}-\frac{1}{\theta}\Big)\Big([u_n]^2+\int_{\mathbb{R}^N}a_\lambda(x)|u_n|^2\,\mathrm{d}x\Big)\\ &+\int_{\mathbb{R}^N}K(x)\Big(\frac{1}{\theta}f(u_n)u_n-F(u_n)\Big)\,\mathrm{d}x\\ &+\Big(\frac{1}{\theta}-\frac{1}{p}\Big)\mu\int_{\mathbb{R}^N}W(x)|u_n|^p\,\mathrm{d}x\\ \geqslant&\Big(\frac{1}{2}-\frac{1}{\theta}\Big)\frac{\eta_0-1}{\eta_0}\|u_n\|_\lambda^2\\ &+\int_{\mathbb{R}^N}K(x)\Big(\frac{1}{\theta}f(u_n)u_n-F(u_n)\Big)\,\mathrm{d}x\\ &+\Big(\frac{1}{\theta}-\frac{1}{p}\Big)\mu\int_{\mathbb{R}^N}W(x)|u_n|^p\,\mathrm{d}x\\ \geqslant&\Big(\frac{1}{2}-\frac{1}{\theta}\Big)\frac{\eta_0-1}{\eta_0}\|u_n\|_\lambda^2\\ &+\Big(\frac{1}{\theta}-\frac{1}{p}\Big)\mu\int_{\mathbb{R}^N}W(x)|u_n|^p\,\mathrm{d}x.\end{aligned}$$

It implies that

$$\begin{aligned}1+c+\|u_n\|_\lambda+\Big(\frac{1}{p}-\frac{1}{\theta}\Big)\mu\int_{\mathbb{R}^N}W(x)|u_n|^p\,\mathrm{d}x&\\ \geqslant\Big(\frac{1}{2}-\frac{1}{\theta}\Big)\frac{\eta_0-1}{\eta_0}\|u\|_\lambda^2.&\end{aligned}\tag{7}$$

Consequently, for $1<p<2$, by Hölder and Sobolev inequalities, $\{u_n\}$ in bounded in E_λ. □

Lemma 3 *Let* $a(x)\in L^{N/2s}(\mathbb{R}^N,\mathbb{R})$ *and* $W(x)\in L^{\frac{2}{2-p}}(\mathbb{R}^N,\mathbb{R}^+)$*, where* $N>2s$ *and* $1<p<2$*. Suppose that* $u_n\rightharpoonup u$ *in* E_λ*, then*

$$\int_{\mathbb{R}^N}a(x)|u_n|^2\,\mathrm{d}x\to\int_{\mathbb{R}^N}a(x)|u|^2\,\mathrm{d}x \tag{8}$$

and

$$\int_{\mathbb{R}^N} W(x)|u_n|^p \,\mathrm{d}x \to \int_{\mathbb{R}^N} W(x)|u|^p \,\mathrm{d}x. \tag{9}$$

Proof: The proof is standard, for the sake of completeness, and we provide the details. We know $a(x) \in L^{\frac{N}{2s}}(\mathbb{R}^N, \mathbb{R})$. $\forall\ \varepsilon > 0$, there exists $R_\varepsilon > 0$,

$$\Big(\int_{\mathbb{R}^N \setminus B_{R_\varepsilon}} |a(x)|^{\frac{N}{2s}} \,\mathrm{d}x\Big)^{\frac{2s}{N}} < \varepsilon.$$

By Hölder inequality and the boundedness of $\{u_n\}$, it is not difficult to obtian that

$$\int_{\mathbb{R}^N \setminus B_{R_\varepsilon}} |a(x)|(u_n^2 - u^2) \,\mathrm{d}x \leqslant \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}} |a(x)|u_n^2 \,\mathrm{d}x \leqslant \varepsilon C.$$

So, if we get the following result, we shall obtain (8),

$$\int_{B_{R_\varepsilon}} |a(x)|(u_n^2 - u^2) \,\mathrm{d}x \leqslant \varepsilon, \tag{10}$$

which is not obvious. It remains to prove (10). Because of the absolute continuity of integral, $\exists \delta$, whatever $\bar{\Omega} \subset B_{R_\varepsilon}$ with $|\bar{\Omega}| < \delta$,

$$\int_{\bar{\Omega}} |a(x)|^{\frac{N}{2s}} \,\mathrm{d}x < \varepsilon,$$

for all n. Then,

$$\int_{\bar{\Omega}} a(x)|u_n(x)|^2 \,\mathrm{d}x \leqslant |a|_{\frac{N}{2s}(\bar{\Omega})} |u_n(x)|^2_{2_s^*(\bar{\Omega})} < M\varepsilon.$$

On the other hand, by the boundenss of $\{u_n\}$ in E_λ, $\{u_n\}$ is bounded in $L^{2_s^*}(B_{R_\varepsilon})$. So, we get

$$\begin{cases} u_n \rightharpoonup u \text{ in } L^{2_s^*}(B_{R_\varepsilon}), \\ u_n \to u \text{ in } L^2(B_{R_\varepsilon}), \\ u_n(x) \to u(x) \text{ a.e. in } B_{R_\varepsilon}, \\ a(x)|u_n(x)|^2 \to a(x)|u(x)|^2 \text{ a.e. in } B_{R_\varepsilon}. \end{cases}$$

Then we get it, by Vitaly theorem.

For (9), we notice the following facts. Since $W \in L^{\frac{2}{2-p}} (1 < p < 2)$, we can choose $\bar{R}_\varepsilon > 0$ such that

$$\Big(\int_{\mathbb{R}^N \setminus B_{\bar{R}_\varepsilon}} |W(x)|^{\frac{2}{2-p}} \,\mathrm{d}x\Big)^{\frac{2-p}{2}} < \varepsilon.$$

As $u_n \rightharpoonup u$ in E_λ, it implies $u_n \to u$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$. Then by a standard way, we can derive the lemma. This completes the proof. □

Lemma 4 (Hardy-type inequality) *Suppose that (A_0)–(A_2) and (K_1)–(K_3) hold. For each $\lambda > 0$, E_λ is compactly embedded in $L^q_K(\mathbb{R}^N)$, $q \in (2, 2_s^*)$.*

Proof: By (A_1) and for each $\lambda > 0$, we obtain $a_\lambda(x) > 0$, $x \in \hat{\Omega} := \{x : |x| \geqslant R_0\} \setminus (\Omega^* \cup \{x | a(x) = 0, |x| \geqslant R_0\})$. For each $x \in \hat{\Omega}$ fixed, set

$$H(s) = a_\lambda(x) s^{2-q} + s^{2_s^*-q}, \qquad \forall s > 0.$$

By [14], we derive that $H(s)$ has

$$C_q a_\lambda(x)^{\frac{2_s^*-q}{2_s^*-2}} := \Big(\frac{2_s^*-2}{2_s^*-q}\Big)\Big(\frac{q-2}{2_s^*-q}\Big)^{\frac{2-q}{2_s^*-2}} a_\lambda(x)^{\frac{2_s^*-q}{2_s^*-2}}$$

as its minimum value. Hence

$$C_q a_\lambda(x)^{\frac{2_s^*-q}{2_s^*-2}} \leqslant a_\lambda(x) s^{2-q} + s^{2_s^*-q}, \quad \forall x \in \hat{\Omega} \text{ and } s > 0.$$

By (K_3), given $\varepsilon \in (0, C_q)$, there is $r > 0$ large enough such that

$$K(x) \leqslant \varepsilon [a_\lambda(x)]^{\frac{2_s^*-q}{2_s^*-2}}, \ \forall x \in \hat{\Omega} \cap \{x : |x| > r\}.$$

Then, for each $x \in \hat{\Omega} \cap \{x : |x| > r\}$ fixed and $s > 0$,

$$\begin{aligned} K(x) C_q &\leqslant \varepsilon C_q [a_\lambda(x)]^{\frac{2_s^*-q}{2_s^*-2}} \leqslant \varepsilon s^{-q}(a_\lambda(x) s^2 + s^{2_s^*}), \\ K(x) s^q &\leqslant \varepsilon (a_\lambda(x) s^2 + s^{2_s^*}) C_q^{-1}, \end{aligned}$$

leading to

$$K(x) s^q \leqslant \varepsilon (a_\lambda(x) s^2 + s^{2_s^*}) C_q^{-1},$$

$\forall x \in \hat{\Omega} \cap \{x : |x| > r\}$ and $s > 0$. By (A_1) and (A_2), we obtain $m(\Omega^* \cup \{x | a(x) = 0, |x| \geqslant R_0\}) = 0$. Therefore, for $r_0 := \max\{r, R_0\}$, we derive that

$$\int_{B^c_{r_0}(0)} K(x)|u|^q \,\mathrm{d}x \leqslant \varepsilon C_q^{-1} \int_{B^c_{r_0}(0)} \Big(a_\lambda(x)|u|^2 + |u|^{2_s^*}\Big) \mathrm{d}x,$$

$\forall u \in E_\lambda$. If $u_n \rightharpoonup u$ in E_λ as $n \to \infty$, then, repeating the progress of Proposition 2.1 in [14],

$$\int_{B^c_{r_0}(0)} K(x)|u_n|^q \,\mathrm{d}x \leqslant \varepsilon C C_q^{-1}, \quad \forall n \in N. \tag{11}$$

Furthermore, for $q \in (2, 2_s^*)$, it follows form Sobolev embeddings

$$\int_{B_{r_0}(0)} K(x)|u_n|^q \,\mathrm{d}x \to \int_{B_{r_0}(0)} K(x)|u|^q \,\mathrm{d}x. \tag{12}$$

Therefore, from (11) and (12), we obtain

$$\int_{\mathbb{R}^N} K(x)|u_n|^q \,\mathrm{d}x \to \int_{\mathbb{R}^N} K(x)|u|^q \,\mathrm{d}x.$$

This completes the proof. □

Lemma 5 *Under the conditions of Theorem 1, any (PS) sequence $\{u_n\}$ of $\mathscr{J}_\lambda$ has a convergent subsequence.*

Proof: By Lemma 2, $\{u_n\}$ is bounded in E_λ. Passing a subsequence, we may assume that $u_n \rightharpoonup u$ in E_λ. Repeating the same argument explored in Lemma 2.2 of [14] and Lemma 4, we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)f(u_n)u_n\,\mathrm{d}x=\int_{\mathbb{R}^N} K(x)f(u)u\,\mathrm{d}x \tag{13}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)f(u_n)u\,\mathrm{d}x=\int_{\mathbb{R}^N} K(x)f(u)u\,\mathrm{d}x. \tag{14}$$

Furthermore, easily to know

$$\begin{aligned}\|u_n-u\|_\lambda^2=\langle \mathscr{I}_\lambda'(u_n)-\mathscr{I}_\lambda'(u),u_n-u\rangle&\\ +\int_{\mathbb{R}^N}\mu W(x)(|u_n|^{p-1}-|u|^{p-1})(u_n-u)\,\mathrm{d}x&\\ +\int_{\mathbb{R}^N}K(x)(f(u_n)-f(u))(u_n-u)\,\mathrm{d}x&\\ +\int_{\mathbb{R}^N}a^-|u_n-u|^2\,\mathrm{d}x.&\end{aligned}$$

By Lemma 2, Lemma 4, (13) and (14), we complete the proof. □

PROOFS OF THE THEOREMS

In this section, we present the proof of our main results. Let $Q=\mathbb{R}^{2N}\backslash(\Omega^c\times\Omega^c)$, where Ω is an open bounded set in $\mathbb{R}^N$, $\Omega^c=\mathbb{R}^N\backslash\Omega$. And define the function space X_0 as follows

$$X_0=\left\{u\in L^2(\Omega)\,\middle|\,\int_Q\frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y<\infty,\ u(x)=0 \text{ if } x\in\Omega^c\right\}.$$

For more details about X_0, refer to [12].

Proof of Theorem 1

By combining Lemmas 1–5 and the Mountain Pass theorem, we can conclude that for each $\lambda>0$ and $0<\mu<\mu_0$, there is a $u_\lambda^{(1)}\in E_\lambda$ such that $\mathscr{I}_\lambda(u_\lambda^{(1)})\to c_\lambda$ and $\mathscr{I}_\lambda'(u_\lambda^{(1)})\to 0$. Furthermore, $\mathscr{I}_\lambda(u_\lambda^{(1)})=c_\lambda>0$, and $u_\lambda^{(1)}$ is a nontrivial weak solution to Eq. (1).

The second solution will be constructed using the local minimization. We aim to show the existence of $\phi\in E_\lambda$ such that $\mathscr{I}_\lambda(l\phi)<0$ for all $l>0$ sufficiently small. Consider $W\in L^{\frac{2}{2-p}}(\mathbb{R}^N,\mathbb{R}^+)$ and choose $\phi\in E_\lambda$ such that $\int_{\mathbb{R}^N}W(x)|\phi|^p dx>0$. One follows

$$\begin{aligned}\mathscr{I}_\lambda(l\phi)&=\frac12\|l\phi\|_\lambda^2-\frac12\int_{\mathbb{R}^N}a^-(x)|l\phi|^2\,\mathrm{d}x\\&\quad-\int_{\mathbb{R}^N}KF(l\phi)\,\mathrm{d}x-\frac{\mu}{p}\int_{\mathbb{R}^N}W|l\phi|^p\,\mathrm{d}x\\&\leqslant\frac12\|l\phi\|_\lambda^2-\frac{\mu}{p}\int_{\mathbb{R}^N}W|l\phi|^p\,\mathrm{d}x<0,\end{aligned} \tag{15}$$

for $l>0$ small enough, which implies $\vartheta:=\inf\{\mathscr{I}_\lambda(u):u\in\bar{B_\rho}\}<0$ and ρ is given in Lemma 1. Moreover, (15) implies that there exist $l_0>0$ and $\kappa<0$ being independent of λ such that $\mathscr{I}_\lambda(l_0\phi)=\kappa$ and $\|l_0\phi\|<\rho$. By Ekeland's variational principle (see [26]), there exists a minimizing sequence $\{u_n\}\subset\bar{B_\rho}$ such that $\mathscr{I}_\lambda(u_n)\leqslant\kappa<0$ and $\mathscr{I}'{}_\lambda(u_n)\to0$ as $n\to\infty$. Therefore, by Lemma 5, for each $\lambda>0$ and $0<\mu<\mu_0$, we obtain that Eq. (1) possesses a second solution $u_\lambda^{(2)}$ satisfying $\mathscr{I}_\lambda(u_\lambda^2)\leqslant\kappa<0$ and $\left\|u_\lambda^2\right\|_\lambda\leqslant\rho$.

Moreover, we can conclude that

$$\mathscr{I}_\lambda(u_\lambda^{(2)})\leqslant\kappa<0<c_\lambda=\mathscr{I}_\lambda(u_\lambda^{(1)}).$$

Thus, we have completed the proof.

Proof of Theorem 2

We follows the argument in [16] (or see [19]). Consider any sequence $\lambda_n\to\infty$, and let $u_n^{(2)}:=u_{\lambda_n}^{(2)}$ denote the critical points of $\mathscr{I}_{\lambda_n}$ obtained in Theorem 1.

Step 1. We claim that $u^{(2)}=0$ a.e. in $\mathbb{R}^N\backslash\Omega_4$. Fixed $\mu\in(0,\mu_0)$. Since we have $\mathscr{I}_{\lambda_n}(u_n^{(2)})\leqslant\kappa<0$, and referring to (7), it follows that

$$\begin{aligned}1+\kappa+\|u_n\|_{\lambda_n}+\Big(\frac1p-\frac1\theta\Big)\mu_0\int_{\mathbb{R}^N}W(x)|u_n|^p\,\mathrm{d}x&\\ \geqslant\Big(\frac12-\frac1\theta\Big)\frac{\eta_0-1}{\eta_0}\|u\|_{\lambda_n}^2.&\end{aligned}$$

Since $\eta>2$, $\left\|u_n^{(2)}\right\|_{\lambda_n}$ is bounded, and we can assume that $u_n^{(2)}\rightharpoonup u^{(2)}$ in E. Moreover, we have the following convergence result

$$\begin{cases}u_n^{(2)}\to u^{(2)} \text{ in } L^r_{\mathrm{loc}}(\mathbb{R}^N), \text{ for } 2\leqslant r<2_s^*,\\ u_n^{(2)}(x)\to u^{(2)}(x) \text{ a.e. in } \mathbb{R}^N.\end{cases}$$

According to the Fatou lemma, we derive that

$$\begin{aligned}0&\leqslant\int_{\mathbb{R}^N}a^+(x)|u^{(2)}|^2dx\\&\leqslant\liminf_{n\to\infty}\int_{\mathbb{R}^N}a^+(x)|u_n^{(2)}|^2dx\\&\leqslant\liminf_{n\to\infty}\frac{\left\|u_n^{(2)}\right\|_{\lambda_n}}{\lambda_n}=0,\end{aligned}$$

which implies that $u^{(2)} = 0$ a.e. in $\mathbb{R}^N \backslash \Omega_4$, and $u^{(2)} \in X_0$.

Step 2. We will establish that $u^{(2)}$ is the solution to problem (5). For any $\phi \in C_0^\infty(\Omega_4)$, using the fact that $\langle \mathscr{I}'(u_n^{(2)}), \phi \rangle = 0$, we have

$$\begin{aligned}\int_Q \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y \\ -\int_{\Omega_4} a^- u^{(2)}(x)\phi \,\mathrm{d}x = \int_{\Omega_4} K(x) f(u^{(2)})\phi \,\mathrm{d}x \\ +\int_{\Omega_4} \mu W(x)|u^{(2)}|^{p-1}\phi \,\mathrm{d}x.\end{aligned}$$

By the density of $C_0^\infty(\Omega_4)$ in X_0, we conclude that $u^{(2)}$ is the weak solution of (5).

Step 3. Next, we will show that $u_n^{(2)} \to u^{(2)}$ in $L^r(\mathbb{R}^N)$ for $2 < r < 2_s^*$. Assume, to the contrary, that there exist $\delta > 0$, $R > 0$ and $x_n \in \mathbb{R}^N$ satisfying

$$\int_{B(x_n,R)} |u_n^{(2)} - u^{(2)}|^2 \,\mathrm{d}x \geqslant \delta > 0. \tag{16}$$

Moreover, we have $|x_n| \to \infty$. By (A_1) and (16), we obtain that

$$\int_{B(x_n,R)} a^+(x)|u_n^{(2)} - u^{(2)}|^2 \,\mathrm{d}x \neq 0. \tag{17}$$

Since $m(\Omega_4) < \infty$, we have $m(B(x_n,R) \cap \Omega_4) \to 0$. Thereby, we derive that

$$\int_{B(x_n,R)\cap\Omega_4} a^+(x)|u_n^{(2)} - u^{(2)}|^2 \,\mathrm{d}x \to 0. \tag{18}$$

By combining (17), (18) and the fact that $u^{(2)} = 0$ a.e. in $\mathbb{R}^N \backslash \Omega_4$, one follows that

$$\begin{aligned}\left\|u_n^{(2)}\right\|_{\lambda_n}^2 &= [u_n^{(2)}]^2 + \int_{\mathbb{R}^N} \lambda_n a^+(x)|u_n^{(2)}|^2 \,\mathrm{d}x \\ &\geqslant \int_{B(x_n,R)\cap\Omega_4^c} \lambda_n a^+(x)|u_n^{(2)}|^2 \,\mathrm{d}x \\ &= \int_{B(x_n,R)\cap\Omega_4^c} \lambda_n a^+(x)|u_n^{(2)} - u^{(2)}|^2 \,\mathrm{d}x \\ &= \lambda_n \bigg(\int_{B(x_n,R)} a^+(x)|u_n^{(2)} - u^{(2)}|^2 \,\mathrm{d}x \\ &\quad - \int_{B(x_n,R)\cap\Omega_4} a^+|u_n^{(2)} - u^{(2)}|^2 \,\mathrm{d}x \bigg) \to \infty,\end{aligned}$$

as $n \to \infty$. However, this contradicts the boundedness of $\left\|u_n^{(2)}\right\|_{\lambda_n}^2$. Thus, we conclude that $u_n^{(2)} \to u^{(2)}$ in $L^r(\mathbb{R}^N)$ for $2 < r < 2_s^*$.

Step 4. Finally, we will establish the convergence of $u_n^{(2)}$ to $u^{(2)}$ in the energy space E. For a sufficiently large $r > 0$, we can have the following result

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} W(x)|u_n^{(2)}|^p \,\mathrm{d}x - \int_{\mathbb{R}^N} W(x)|u_n^{(2)}|^{p-2}u_n^{(2)}u^{(2)} \,\mathrm{d}x\right| \\ &\quad= \left|\int_{\mathbb{R}^N} W(x)|u_n^{(2)}|^{p-2}(|u_n^{(2)}|^2 - u_n^{(2)}u^{(2)}) \,\mathrm{d}x\right| \\ &\quad\leqslant \int_{\mathbb{R}^N} |W(x)||u_n^{(2)}|^{p-1}|u_n^{(2)} - u^{(2)}| \,\mathrm{d}x \\ &\quad\leqslant |W|_{\frac{2}{2-p}(B_r(0))}|u_n^{(2)}|_{2(B_r(0))}^{p-1}|u_n^{(2)} - u^{(2)}|_{2(B_r(0))}^2 \\ &\qquad + |W|_{\frac{2}{2-p}(B_r^c(0))}|u_n^{(2)}|_{2(B_r(0))}^{p-1}|u_n^{(2)} - u^{(2)}|_{2(B_r(0))}^2.\end{aligned}$$

Taking into account that $W(x) \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$ and $u_n^{(2)} \to u^{(2)}$ in $L_{\mathrm{loc}}^2(\mathbb{R}^N)$, we can deduce that the following approximation

$$\begin{aligned}&\int_{\mathbb{R}^N} W(x)|u_n^{(2)}|^p \,\mathrm{d}x \\ &\quad= \int_{\mathbb{R}^N} W(x)|u_n^{(2)}|^{p-2}u_n^{(i)}u^{(2)} \,\mathrm{d}x + o(1).\end{aligned} \tag{19}$$

Furthermore, we have the following expressions

$$\begin{aligned}&\int_{\mathbb{R}^N} K(x)f(u_n^{(2)})u_n^{(2)} \,\mathrm{d}x \\ &\quad= \int_{\mathbb{R}^N} K(x)f(x, u_n^{(2)})u^{(2)} \,\mathrm{d}x + o(1).\end{aligned} \tag{20}$$

Consider the relation

$$\int_{\mathbb{R}^N} a^-(x)|u_n^{(2)}|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} a^-(x)u_n^{(2)}u^{(2)} \,\mathrm{d}x + o(1). \tag{21}$$

Noticing

$$\langle \mathscr{I}'_{\lambda_n}(u_n^{(2)}), u_n^{(2)} \rangle = \langle \mathscr{I}'_{\lambda_n}(u_n^{(2)}), u^{(2)} \rangle = o(1), \tag{22}$$

we can obtain the following equations

$$\begin{aligned}\left\|u_n^{(2)}\right\|_{\lambda_n}^2 &= \int_{\mathbb{R}^N} K(x)f(u_n^{(2)})u_n^{(2)} \,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} \mu W(x)|u_n^{(2)}|^p \,\mathrm{d}x + \int_{\mathbb{R}^N} a^-(x)|u_n^{(2)}|^2 \,\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\langle u_n^{(2)}, u^{(2)} \rangle_{\lambda_n} &= \int_{\mathbb{R}^N} K(x)f(u_n^{(2)})u^{(2)} \,\mathrm{d}x \\ &+ \int_{\mathbb{R}^N} \mu W(x)|u_n^{(2)}|^{p-2}u_n^{(2)}u^{(2)} \,\mathrm{d}x + \int_{\mathbb{R}^N} a^-(x)|u_n^{(2)}|u^{(2)} \,\mathrm{d}x.\end{aligned}$$

Hence, utilizing the previously established results (19)–(21) and (22), we have the following convergence

$$\begin{aligned}\lim_{n\to\infty} \left\|u_n^{(2)}\right\|_{\lambda_n}^2 &= \lim_{n\to\infty} \langle u_n^{(2)}, u^{(2)}\rangle_{\lambda_n} \\ &= \lim_{n\to\infty} \langle u_n^{(2)}, u^{(2)}\rangle = \left\|u^{(2)}\right\|^2.\end{aligned}$$

On the other hand, using the weakly lower semicontinuity of norm, we have

$$\begin{aligned}\left\|u^{(2)}\right\|^2 &\leqslant \liminf_{n\to\infty} \left\|u_n^{(2)}\right\|^2 \leqslant \limsup_{n\to\infty} \left\|u_n^{(2)}\right\|^2 \\ &\leqslant \limsup_{n\to\infty} \left\|u_n^{(2)}\right\|_{\lambda_n}^2 = \left\|u^{(2)}\right\|^2.\end{aligned}$$

Therefore, we conclude that $u_n^{(2)}$ converges to $u^{(2)}$ in the energy space E. Furthermore, by repeating the steps of Theorem 1, we have

$$\begin{aligned}\frac{1}{2}[u^{(2)}]^2 - \frac{1}{2}\int_{\Omega_4} a^-|u^{(2)}|^2\,\mathrm{d}x - \int_{\Omega_4} K(x)F(u^{(2)})\,\mathrm{d}x \\ - \frac{\mu}{p}\int_{\Omega_4} \mu W(x)|u^{(2)}|^p\,\mathrm{d}x \leqslant \kappa < 0,\end{aligned}$$

which implies $u^{(2)} \neq 0$. This completes the proof.

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