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A note on the congruences with sums of powers of trinomial coefficients

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ABSTRACT: Trinomial coefficients $\binom{n}{k}_2$ are defined by

$$(1+x+x^2)^n=\sum_{k=0}^{2n}\binom{n}{k}_2 x^k.$$

Let $p>3$ be a prime number and n,m be positive integers, we obtained the congruences modulo p^2 with partial sums of powers of trinomial coefficients

$$\sum_{0\leqslant 3k+i\leqslant p-1}\binom{np-1}{3k+i}_2^m \text{ and } \sum_{0\leqslant 3k+i\leqslant \frac{p-1}{2}}\binom{np-1}{3k+i}_2^m (0\leqslant i\leqslant 2).$$

We also studied the congruences modulo p^2 with sums of powers of trinomial coefficients

$$\sum_{k=0}^{p-1}\binom{np-1}{k}_2^m \text{ and } \sum_{k=0}^{\frac{p-1}{2}}\binom{np-1}{k}_2^m.$$

KEYWORDS: trinomial coefficients, sum of powers, congruences

MSC2020: 11A07 11B65

INTRODUCTION

In 1819, Babbage [1] showed the congruence

$$\binom{2p-1}{p-1}\equiv 1 \pmod{p^2}$$

for any odd prime number p. In 1862, Wolstenholme [15] proved the above congruence about modulo p^3

$$\binom{2p-1}{p-1}\equiv 1 \pmod{p^3}$$

and in 1900, Glaisher [9] extended the congruence

$$\binom{np-1}{p-1}\equiv 1 \pmod{p^3}$$

for any prime number $p>3$ and positive integer n. In 1895, Morley [12] showed that for any prime $p\geqslant 5$,

$$(-1)^{\frac{p-1}{2}}\binom{p-1}{\frac{p-1}{2}}\equiv 4^{p-1} \pmod{p^3}.$$

And in 1953, Carlitz [5, 6] extended Morley's congruence and showed that, for any prime number $p\geqslant 5$,

$$(-1)^{\frac{p-1}{2}}\binom{p-1}{\frac{p-1}{2}}\equiv 4^{p-1}+\frac{p^3}{12} \pmod{p^4}.$$

In 2002, Cai and Granville [2] showed several arithmetic properties on the residues of binomial coefficients and their products modulo primes powers, e.g.,

$$\binom{pq-1}{\frac{pq-1}{2}}\equiv\binom{p-1}{\frac{p-1}{2}}\binom{q-1}{\frac{q-1}{2}} \pmod{pq},$$

for any distinct odd primes p and q. They also proved that if $p\geqslant 5$ is a prime and m is an integer, then

$$\sum_{s=0}^{p-1}\binom{p-1}{s}^m\equiv\begin{cases}2^{m(p-1)} \pmod{p^3}, \text{ if } 2\nmid m,\\ \binom{mp-2}{p-1} \pmod{p^4}, \text{ if } 2\mid m,\end{cases}$$

and

$$\sum_{s=0}^{p-1}(-1)^s\binom{p-1}{s}^m\equiv\begin{cases}2^{m(p-1)} \pmod{p^3}, \text{ if } 2\mid m,\\ \binom{mp-2}{p-1} \pmod{p^4}, \text{ if } 2\nmid m.\end{cases}$$

In 2018, for any prime $p\geqslant 7$, integer $l\geqslant 0$ and positive integers k, m , the first author and Cai [13] proved that

$$\sum_{s=lp}^{(l+1)p-1}\binom{kp-1}{s}^m \equiv\begin{cases}\binom{k-1}{l}^m 2^{km(p-1)} \pmod{p^3}, \text{ if } 2\nmid m,\\ \binom{k-1}{l}^m\binom{kmp-2}{p-1} \pmod{p^4}, \text{ if } 2\mid m,\end{cases}$$

and

$$\sum_{s=lp}^{(l+1)p-1}(-1)^s\binom{kp-1}{s}^m \equiv \begin{cases}\binom{k-1}{l}^m 2^{km(p-1)} \pmod{p^3}, \text{ if } 2\mid m,\\ \binom{k-1}{l}^m\binom{kmp-2}{p-1} \pmod{p^4}, \text{ if } 2\nmid m.\end{cases}$$

In 2014, Sun [14] gave some properties and congruences involving the trinomial coefficients $\binom{n}{k}_2$ defined by

$$(1+x+x^2)^n=\sum_{k=0}^{2n}\binom{n}{k}_2 x^k,$$

also see [3, 4]. Recently, for any prime number $p>3$ and positive integer n, Elkhiri and Mihoubi [10] proved following congruences involving trinomial coefficients

$$\binom{np-1}{p-1}_2 \equiv \begin{cases}1+npq_3 \pmod{p^2}, \text{ if } p\equiv 1 \pmod 3,\\ -1-npq_3 \pmod{p^2}, \text{ if } p\equiv 2 \pmod 3,\end{cases}$$

$$\binom{np-1}{\frac{p-1}{2}}_2 \equiv \begin{cases}1+np\left(2q_2+\frac{1}{2}q_3\right) \pmod{p^2}, \text{ if } p\equiv 1 (\mathrm{mod}\, 6),\\ -\frac{np}{2}q_3 \pmod{p^2}, \text{ if } p\equiv 5 \pmod 6,\end{cases}$$

$$\sum_{k=0}^{p-1}\binom{np-1}{k}_2 \equiv \begin{cases}1+npq_3 \pmod{p^2}, \text{ if } p\equiv 1 \pmod 3,\\ 0 \pmod{p^2}, \text{ if } p\equiv 2 \pmod 3,\end{cases}$$

and

$$\sum_{k=0}^{\frac{p-1}{2}}\binom{np-1}{k}_2 \equiv \begin{cases}1+np\left(\frac{4}{3}q_2+q_3\right) \pmod{p^2}, \text{ if } p\equiv 1 (\mathrm{mod}\, 6),\\ -\frac{2np}{3}q_2 \pmod{p^2}, \text{ if } p\equiv 5 \pmod 6,\end{cases}$$

where q_a is the Fermat quotient defined for a given prime number $p>3$ by $q_a=\frac{a^{p-1}-1}{p}$, $\gcd(a,p)=1$.

The idea of this work is inspired from [10] and [13], generalizing the congruences involving trinomial coefficients $\sum_{k=0}^{p-1}\binom{np-1}{k}_2$ and $\sum_{k=0}^{\frac{p-1}{2}}\binom{np-1}{k}_2$ in [10] to the congruences involving higher powers of trinomial coefficients $\sum_{k=0}^{p-1}\binom{np-1}{k}_2^m$ and $\sum_{k=0}^{\frac{p-1}{2}}\binom{np-1}{k}_2^m$, where m is a positive integer. Meanwhile, we will also extend the congruences related to binomial coefficients to the congruences related to trinomial coefficients. The paper also considered the congruences with partial sums of powers of trinomial coefficients $\sum_{k=0}^{[\frac{p}{3}]}\binom{np-1}{3k}_2^m$, $\sum_{k=0}^{[\frac{p-2}{3}]}\binom{np-1}{3k+1}_2^m$, $\sum_{k=0}^{[\frac{p}{3}]-1}\binom{np-1}{3k+2}_2^m$ and obtained the following theorems by changing the order of summation and classical congruence calculation methods.

Theorem 1 *Let $p>3$ be a prime number and n,m be positive integers. We have*

$$\sum_{k=0}^{[\frac{p}{3}]}\binom{np-1}{3k}_2^m \equiv \begin{cases}\frac{p+2}{3}+mnp\left(\frac{2}{3}q_3-\frac{1}{3}\right) \pmod{p^2}, \text{ if } p\equiv 1 \pmod 3,\\ \frac{p+1}{3}+mnp\left(\frac{1}{3}q_3-\frac{2}{3}\right) \pmod{p^2}, \text{ if } p\equiv 2 \pmod 3,\end{cases}$$

where $[x]$ is the greatest integer not greater than x.

Theorem 2 *Let $p>3$ be a prime number and n,m be positive integers. We have*

$$\sum_{k=0}^{[\frac{p-2}{3}]}\binom{np-1}{3k+1}_2^m \equiv \begin{cases}(-1)^m\left(\frac{p-1}{3}-mnp\left(\frac{1}{3}q_3+\frac{1}{3}\right)\right) \pmod{p^2}, \text{ if } p\equiv 1 \pmod 3,\\ (-1)^m\left(\frac{p+1}{3}+mnp\left(\frac{1}{3}q_3-\frac{1}{3}\right)\right) \pmod{p^2}, \text{ if } p\equiv 2 \pmod 3.\end{cases}$$

Theorem 3 *Let $p>3$ be a prime number and n,m be positive integers. We have*

$$\sum_{k=0}^{[\frac{p}{3}]-1}\binom{np-1}{3k+2}_2^m \equiv \begin{cases}0 \pmod{p^2}, \text{ if } m>1,\\ 0 \pmod{p^2}, \text{ if } p\equiv 1 \pmod 3 \text{ and } m=1,\\ \frac{np}{3} \pmod{p^2}, \text{ if } p\equiv 2 \pmod 3 \text{ and } m=1.\end{cases}$$

Theorem 4 *Let $p>3$ be a prime number and n,m be positive integers. We have*

$$\sum_{k=0}^{p-1}\binom{np-1}{k}_2 \equiv \begin{cases}1+npq_3 \pmod{p^2}, \text{ if } p\equiv 1 \pmod 3,\\ 0 \pmod{p^2}, \text{ if } p\equiv 2 \pmod 3.\end{cases}$$

When even $m>1$, we have

$$\sum_{k=0}^{p-1}\binom{np-1}{k}_2^m \equiv \begin{cases}\frac{2p+1}{3}+mnp\left(\frac{1}{3}q_3-\frac{2}{3}\right) \pmod{p^2}, \text{ if } p\equiv 1 \pmod 3,\\ \frac{2p+2}{3}+mnp\left(\frac{2}{3}q_3-1\right) \pmod{p^2}, \text{ if } p\equiv 2 \pmod 3.\end{cases}$$

When odd $m>1$, we have

$$\sum_{k=0}^{p-1}\binom{np-1}{k}_2^m \equiv \begin{cases}1+mnpq_3 \pmod{p^2}, \text{ if } p\equiv 1 \pmod 3,\\ -\frac{mnp}{3} \pmod{p^2}, \text{ if } p\equiv 2 \pmod 3.\end{cases}$$

Theorem 5 *Let $p>3$ be a prime number and n,m be positive integers. We have*

$$\sum_{k=0}^{\frac{p-1}{2}}\binom{np-1}{k}_2 \equiv \begin{cases}1+np\left(\frac{4}{3}q_2+q_3\right) \pmod{p^2}, \text{ if } p\equiv 1 (mod 3),\\ -\frac{2np}{3}q_2 \pmod{p^2}, \text{ if } p\equiv 2 \pmod 3.\end{cases}$$

When even $m>1$, we have

$$\sum_{k=0}^{\frac{p-1}{2}}\binom{np-1}{k}_2^m \equiv \begin{cases}\frac{p+2}{3}+mnp\left(\frac{4}{3}q_2+\frac{5}{12}q_3-\frac{1}{3}\right) \pmod{p^2}, \text{ if } p\equiv 1 \pmod 3,\\ \frac{p+1}{3}+mnp\left(\frac{2}{3}q_2+\frac{1}{12}q_3-1\right) \pmod{p^2}, \text{ if } p\equiv 2 \pmod 3.\end{cases}$$

When odd $m>1$, we have

$$\sum_{k=0}^{\frac{p-1}{2}}\binom{np-1}{k}_2^m \equiv \begin{cases}1+mnp\left(\frac{14}{9}q_2+\frac{3}{4}q_3\right) \pmod{p^2}, \text{ if } p\equiv 1 \pmod 3,\\ mnp\left(-\frac{4}{9}q_2+\frac{1}{4}q_3\right) \pmod{p^2}, \text{ if } p\equiv 2 \pmod 3.\end{cases}$$

AUXILIARY RESULTS

Let H_n be the n-th harmonic number defined by

$$H_0=0, \quad H_n=\sum_{j=1}^{n}\frac{1}{j}.$$

Lemma 1 ([7, 8, 11]) *Let $p>3$ be a prime number, we have*

$$H_{[\frac{p}{3}]}\equiv -\frac{3}{2}q_3 \pmod p,$$
$$H_{[\frac{p}{6}]}\equiv -2q_2-\frac{3}{2}q_3 \pmod p.$$

Lemma 2 ([10]) *Let $p>3$ be a prime number, we have*

$$\sum_{j=0}^{[\frac{p}{3}]-1}\frac{1}{3j+2}\equiv \begin{cases}0 \pmod p, \text{ if } p\equiv 1 \pmod 3,\\ \frac{1}{2}q_3 \pmod p, \text{ if } p\equiv 2 \pmod 3.\end{cases}$$

$$\sum_{j=0}^{[\frac{p}{3}]-1}\frac{1}{3j+1}\equiv \begin{cases}\frac{1}{2}q_3 \pmod p, \text{ if } p\equiv 1 \pmod 3,\\ 1 \pmod p, \text{ if } p\equiv 2 \pmod 3.\end{cases}$$

Lemma 3 ([10]) *Let $p>3$ be a prime number, we have*

$$\sum_{j=0}^{[\frac{p}{6}]}\frac{1}{3j+2}\equiv \begin{cases}-\frac{2}{3}q_2+\frac{1}{2}q_3+\frac{2}{3} \pmod p, \text{ if } p\equiv 1 \pmod 6,\\ -\frac{2}{3}q_2 \pmod p, \text{ if } p\equiv 5 \pmod 6.\end{cases}$$

$$\sum_{j=0}^{[\frac{p}{6}]}\frac{1}{3j+1}\equiv \begin{cases}-\frac{2}{3}q_2+2 \pmod p, \text{ if } p\equiv 1 \pmod 6,\\ -\frac{2}{3}q_2+\frac{1}{2}q_3 \pmod p, \text{ if } p\equiv 5 \pmod 6.\end{cases}$$

Lemma 4 ([10]) *Let $p>3$ be a prime number and $n>0$, k be integers. We have*

$$\binom{np-1}{3k}_2\equiv 1-np\left(\frac{2}{3}H_k+\sum_{j=0}^{k-1}\frac{1}{3j+2}\right) \pmod{p^2}, \quad 1\leqslant 3k\leqslant p-1.$$

$$\binom{np-1}{3k+1}_2\equiv -1+np\left(\frac{2}{3}H_k+\sum_{j=0}^{k}\frac{1}{3j+1}\right) \pmod{p^2}, \quad 1\leqslant 3k+1\leqslant p-1.$$

$$\binom{np-1}{3k+2}_2\equiv np\left(\sum_{j=0}^{k}\frac{1}{3j+2}-\sum_{j=0}^{k}\frac{1}{3j+1}\right) \pmod{p^2}, \quad 1\leqslant 3k+2\leqslant p-1.$$

PROOFS

Proof of Theorem 1

Proof: By Lemma 4, we have

$$\begin{aligned}\sum_{k=0}^{[\frac{p}{3}]}\binom{np-1}{3k}_2^m &= 1+\sum_{k=1}^{[\frac{p}{3}]}\binom{np-1}{3k}_2^m\\ &\equiv 1+\sum_{k=1}^{[\frac{p}{3}]}\left(1-mnp\left(\frac{2}{3}H_k+\sum_{j=0}^{k-1}\frac{1}{3j+2}\right)\right)\\ &\equiv 1+\left[\frac{p}{3}\right]-mnp\sum_{k=1}^{[\frac{p}{3}]}\left(\frac{2}{3}H_k+\sum_{j=0}^{k-1}\frac{1}{3j+2}\right) \pmod{p^2}. \quad (1)\end{aligned}$$

Changing the sum order of j and k in (1), we get

$$\begin{aligned}\sum_{k=0}^{[\frac{p}{3}]}\binom{np-1}{3k}_2^m &\equiv 1+\left[\frac{p}{3}\right]-mnp\left(\frac{2}{3}\sum_{j=1}^{[\frac{p}{3}]}\frac{1}{j}\sum_{k=j}^{[\frac{p}{3}]}1+\sum_{j=0}^{[\frac{p}{3}]-1}\frac{1}{3j+2}\sum_{k=j+1}^{[\frac{p}{3}]}1\right)\\ &\equiv 1+\left[\frac{p}{3}\right]-mnp\left(\frac{2}{3}\sum_{j=1}^{[\frac{p}{3}]}\frac{[\frac{p}{3}]-j+1}{j}\right.\\ &\qquad\left.+\sum_{j=0}^{[\frac{p}{3}]-1}\frac{[\frac{p}{3}]-j}{3j+2}\right) \pmod{p^2}. \quad (2)\end{aligned}$$

For $p \equiv 1 \pmod 3$ in (2), we have $\left[\frac{p}{3}\right] = \frac{p-1}{3}$, then

$$\begin{aligned}
&\sum_{k=0}^{\left[\frac{p}{3}\right]} \binom{np-1}{3k}_2^m \\
&\equiv 1 + \frac{p-1}{3} - mnp\left(\frac{2}{3}\sum_{j=1}^{\left[\frac{p}{3}\right]} \frac{\frac{p-1}{3}-j+1}{j} + \sum_{j=0}^{\left[\frac{p}{3}\right]-1} \frac{\frac{p-1}{3}-j}{3j+2}\right) \\
&\equiv \frac{p+2}{3} - mnp\left(\frac{4}{9}H_{\left[\frac{p}{3}\right]} - \frac{2}{3}\left[\frac{p}{3}\right] - \frac{1}{3}\sum_{j=0}^{\left[\frac{p}{3}\right]-1} \frac{(3j+2)-1}{3j+2}\right) \\
&\equiv \frac{p+2}{3} - mnp\left(\frac{4}{9}H_{\left[\frac{p}{3}\right]} - \left[\frac{p}{3}\right] + \frac{1}{3}\sum_{j=0}^{\left[\frac{p}{3}\right]-1} \frac{1}{3j+2}\right) \\
&\equiv \frac{p+2}{3} - mnp\left(\frac{4}{9}H_{\left[\frac{p}{3}\right]} + \frac{1}{3} + \frac{1}{3}\sum_{j=0}^{\left[\frac{p}{3}\right]-1} \frac{1}{3j+2}\right) \pmod{p^2}. \qquad (3)
\end{aligned}$$

By Lemma 1 and Lemma 2, for $p \equiv 1 \pmod 3$, (3) is congruent to

$$\sum_{k=0}^{\left[\frac{p}{3}\right]} \binom{np-1}{3k}_2^m \equiv \frac{p+2}{3} + mnp\left(\frac{2}{3}q_3 - \frac{1}{3}\right) \pmod{p^2}. \qquad (4)$$

For $p \equiv 2 \pmod 3$ in (2), we have $\left[\frac{p}{3}\right] = \frac{p-2}{3}$, then

$$\begin{aligned}
&\sum_{k=0}^{\left[\frac{p}{3}\right]} \binom{np-1}{3k}_2^m \\
&\equiv 1 + \frac{p-2}{3} - mnp\left(\frac{2}{3}\sum_{j=1}^{\left[\frac{p}{3}\right]} \frac{\frac{p-2}{3}-j+1}{j} + \sum_{j=0}^{\left[\frac{p}{3}\right]-1} \frac{\frac{p-2}{3}-j}{3j+2}\right) \\
&\equiv \frac{p+1}{3} - mnp\left(\frac{2}{9}H_{\left[\frac{p}{3}\right]} - \frac{2}{3}\left[\frac{p}{3}\right] - \frac{1}{3}\sum_{j=0}^{\left[\frac{p}{3}\right]-1} \frac{3j+2}{3j+2}\right) \\
&\equiv \frac{p+1}{3} - mnp\left(\frac{2}{9}H_{\left[\frac{p}{3}\right]} - \frac{2}{3}\left[\frac{p}{3}\right] - \frac{1}{3}\left[\frac{p}{3}\right]\right) \\
&\equiv \frac{p+1}{3} - mnp\left(\frac{2}{9}H_{\left[\frac{p}{3}\right]} + \frac{2}{3}\right) \pmod{p^2}. \qquad (5)
\end{aligned}$$

By Lemma 1, for $p \equiv 2 \pmod 3$, (5) is congruent to

$$\sum_{k=0}^{\left[\frac{p}{3}\right]} \binom{np-1}{3k}_2^m \equiv \frac{p+1}{3} + mnp\left(\frac{1}{3}q_3 - \frac{2}{3}\right) \pmod{p^2}. \qquad (6)$$

Combining (4) and (6), we obtain Theorem 1. □

Proof of Theorem 2

Proof: By Lemma 4, we have

$$\begin{aligned}
\sum_{k=0}^{\left[\frac{p-2}{3}\right]} \binom{np-1}{3k+1}_2^m \equiv (-1)^m \sum_{k=0}^{\left[\frac{p-2}{3}\right]} \Bigg(1 \\
- mnp\left(\frac{2}{3}H_k + \sum_{j=0}^{k} \frac{1}{3j+1}\right)\Bigg) \pmod{p^2}. \qquad (7)
\end{aligned}$$

Changing the sum order of j and k in (7), we get

$$\begin{aligned}
&\sum_{k=0}^{\left[\frac{p-2}{3}\right]} \binom{np-1}{3k+1}_2^m \equiv (-1)^m\Bigg(\left[\frac{p-2}{3}\right] + 1 \\
&\quad - mnp\left(\frac{2}{3}\sum_{j=1}^{\left[\frac{p-2}{3}\right]} \frac{1}{j}\sum_{k=j}^{\left[\frac{p-2}{3}\right]} 1 + \sum_{j=0}^{\left[\frac{p-2}{3}\right]} \frac{1}{3j+1}\sum_{k=j}^{\left[\frac{p-2}{3}\right]} 1\right)\Bigg) \\
&\equiv (-1)^m\Bigg(\left[\frac{p-2}{3}\right] + 1 - mnp\Bigg(\frac{2}{3}\sum_{j=1}^{\left[\frac{p-2}{3}\right]} \frac{\left[\frac{p-2}{3}\right]-j+1}{j} \\
&\quad + \sum_{j=0}^{\left[\frac{p-2}{3}\right]} \frac{\left[\frac{p-2}{3}\right]-j+1}{3j+1}\Bigg)\Bigg) \pmod{p^2}. \qquad (8)
\end{aligned}$$

For $p \equiv 1 \pmod 3$ in (8), we have $\left[\frac{p-2}{3}\right] = \frac{p-4}{3}$, then

$$\begin{aligned}
&\sum_{k=0}^{\left[\frac{p-2}{3}\right]} \binom{np-1}{3k+1}_2^m \\
&\equiv (-1)^m\left(\frac{p-1}{3} - mnp\left(\frac{2}{3}\sum_{j=1}^{\left[\frac{p-2}{3}\right]} \frac{\frac{p-1}{3}-j}{j} + \sum_{j=0}^{\left[\frac{p-2}{3}\right]} \frac{\frac{p-1}{3}-j}{3j+1}\right)\right) \\
&\equiv (-1)^m\left(\frac{p-1}{3} - mnp\left(-\frac{2}{9}H_{\left[\frac{p-2}{3}\right]} - \frac{2}{3}\left[\frac{p-2}{3}\right] - \frac{1}{3}\sum_{j=0}^{\left[\frac{p-2}{3}\right]} \frac{3j+1}{3j+1}\right)\right) \\
&\equiv (-1)^m\Bigg(\frac{p-1}{3} - mnp\Bigg(-\frac{2}{9}\left(H_{\left[\frac{p}{3}\right]} - \frac{3}{p-1}\right) \\
&\quad - \frac{2}{3}\left[\frac{p-2}{3}\right] - \frac{1}{3}\left(\left[\frac{p-2}{3}\right] + 1\right)\Bigg)\Bigg) \\
&\equiv (-1)^m\left(\frac{p-1}{3} - mnp\left(-\frac{2}{9}H_{\left[\frac{p}{3}\right]} + \frac{1}{3}\right)\right) \pmod{p^2}. \qquad (9)
\end{aligned}$$

By Lemma 1, for $p \equiv 1 \pmod 3$, (9) is congruent to

$$\begin{aligned}
\sum_{k=0}^{\left[\frac{p-2}{3}\right]} \binom{np-1}{3k+1}_2^m \equiv (-1)^m\Bigg(\frac{p-1}{3} \\
- mnp\left(\frac{1}{3}q_3 + \frac{1}{3}\right)\Bigg) \pmod{p^2}. \qquad (10)
\end{aligned}$$

For $p \equiv 2 \pmod 3$ in (8), we have $\left[\frac{p-2}{3}\right] = \frac{p-2}{3}$, then

$$\begin{aligned}
&\sum_{k=0}^{\left[\frac{p-2}{3}\right]} \binom{np-1}{3k+1}_2^m \\
&\equiv (-1)^m \left(\frac{p+1}{3} - mnp \left(\frac{2}{3} \sum_{j=1}^{\left[\frac{p-2}{3}\right]} \frac{\frac{p+1}{3}-j}{j} + \sum_{j=0}^{\left[\frac{p-2}{3}\right]} \frac{\frac{p+1}{3}-j}{3j+1} \right) \right) \\
&\equiv (-1)^m \left(\frac{p+1}{3} - mnp \left(\frac{2}{9} H_{\left[\frac{p-2}{3}\right]} - \frac{2}{3} \left[\frac{p-2}{3} \right] \right.\right. \\
&\left.\left. - \frac{1}{3} \sum_{j=0}^{\left[\frac{p-2}{3}\right]} \frac{(3j+1)-2}{3j+1} \right) \right) \\
&\equiv (-1)^m \left(\frac{p+1}{3} - mnp \left(\frac{2}{9} H_{\left[\frac{p}{3}\right]} - \frac{2}{3} \left[\frac{p-2}{3} \right] \right.\right. \\
&\left.\left. - \frac{1}{3} (\left[\frac{p-2}{3} \right] + 1) + \frac{2}{3} \sum_{j=0}^{\left[\frac{p-2}{3}\right]} \frac{1}{3j+1} \right) \right) \\
&\equiv (-1)^m \left(\frac{p+1}{3} - mnp \left(\frac{2}{9} H_{\left[\frac{p}{3}\right]} + \frac{1}{3} \right.\right. \\
&\left.\left. + \frac{2}{3} \sum_{j=0}^{\left[\frac{p}{3}\right]-1} \frac{1}{3j+1} - \frac{2}{3} \right) \right) \pmod{p^2}. \quad (11)
\end{aligned}$$

By Lemma 1 and Lemma 2, for $p \equiv 2 \pmod 3$, (11) is congruent to

$$\sum_{k=0}^{\left[\frac{p-2}{3}\right]} \binom{np-1}{3k+1}_2^m \equiv (-1)^m \left(\frac{p+1}{3} + mnp \left(\frac{1}{3} q_3 - \frac{1}{3} \right) \right) \pmod{p^2}. \quad (12)$$

Combining (10) and (12), we obtain Theorem 2. □

Proof of Theorem 3

Proof: By Lemma 4, if $m > 1$, we obtain

$$\sum_{k=0}^{\left[\frac{p}{3}\right]-1} \binom{np-1}{3k+2}_2^m \equiv 0 \pmod{p^2}. \quad (13)$$

By Lemma 4, if $m = 1$, changing the sum order of j and k, we obtain

$$\begin{aligned}
\sum_{k=0}^{\left[\frac{p}{3}\right]-1} \binom{np-1}{3k+2}_2 &\equiv \sum_{k=0}^{\left[\frac{p}{3}\right]-1} np \left(\sum_{j=0}^{k} \frac{1}{3j+2} - \sum_{j=0}^{k} \frac{1}{3j+1} \right) \\
&\equiv np \left(\sum_{j=0}^{\left[\frac{p}{3}\right]-1} \frac{1}{3j+2} \sum_{k=j}^{\left[\frac{p}{3}\right]-1} 1 - \sum_{j=0}^{\left[\frac{p}{3}\right]-1} \frac{1}{3j+1} \sum_{k=j}^{\left[\frac{p}{3}\right]-1} 1 \right) \\
&\equiv np \left(\sum_{j=0}^{\left[\frac{p}{3}\right]-1} \frac{\left[\frac{p}{3}\right]-j}{3j+2} - \sum_{j=0}^{\left[\frac{p}{3}\right]-1} \frac{\left[\frac{p}{3}\right]-j}{3j+1} \right) \pmod{p^2}. \quad (14)
\end{aligned}$$

For $p \equiv 1 \pmod 3$ in (14), we have $\left[\frac{p}{3}\right] = \frac{p-1}{3}$, then

$$\begin{aligned}
\sum_{k=0}^{\left[\frac{p}{3}\right]-1} \binom{np-1}{3k+2}_2 &\equiv np \left(\sum_{j=0}^{\left[\frac{p}{3}\right]-1} \frac{\frac{p-1}{3}-j}{3j+2} - \sum_{j=0}^{\left[\frac{p}{3}\right]-1} \frac{\frac{p-1}{3}-j}{3j+1} \right) \\
&\equiv np \left(\frac{1}{3} \sum_{j=0}^{\left[\frac{p}{3}\right]-1} \frac{1+3j}{3j+1} - \frac{1}{3} \sum_{j=0}^{\left[\frac{p}{3}\right]-1} \frac{(2+3j)-1}{3j+2} \right) \\
&\equiv \frac{np}{3} \sum_{j=0}^{\left[\frac{p}{3}\right]-1} \frac{1}{3j+2} \pmod{p^2}. \quad (15)
\end{aligned}$$

By Lemma 2, for $p \equiv 1 \pmod 3$, (15) is congruent to

$$\sum_{k=0}^{\left[\frac{p}{3}\right]-1} \binom{np-1}{3k+2}_2 \equiv 0 \pmod{p^2}. \quad (16)$$

For $p \equiv 2 \pmod 3$ in (14), we have $\left[\frac{p}{3}\right] = \frac{p-2}{3}$, then

$$\begin{aligned}
\sum_{k=0}^{\left[\frac{p}{3}\right]-1} \binom{np-1}{3k+2}_2 &\equiv np \left(\sum_{j=0}^{\left[\frac{p}{3}\right]-1} \frac{\frac{p-2}{3}-j}{3j+2} - \sum_{j=0}^{\left[\frac{p}{3}\right]-1} \frac{\frac{p-2}{3}-j}{3j+1} \right) \\
&\equiv np \left(\frac{1}{3} \sum_{j=0}^{\left[\frac{p}{3}\right]-1} \frac{1+3j+1}{3j+1} - \frac{1}{3} \sum_{j=0}^{\left[\frac{p}{3}\right]-1} \frac{2+3j}{3j+2} \right) \\
&\equiv \frac{np}{3} \sum_{j=0}^{\left[\frac{p}{3}\right]-1} \frac{1}{3j+1} \pmod{p^2}. \quad (17)
\end{aligned}$$

By Lemma 2, for $p \equiv 2 \pmod 3$, (17) is congruent to

$$\sum_{k=0}^{\left[\frac{p}{3}\right]-1} \binom{np-1}{3k+2}_2 \equiv \frac{np}{3} \pmod{p^2}. \quad (18)$$

Combining (16) and (18), we obtain Theorem 3. □

Proof of Theorem 4

Proof: By Theorem 1–Theorem 3, when $m = 1$ and $p \equiv 1 \pmod 3$, we obtain

$$\begin{aligned}
&\sum_{k=0}^{p-1} \binom{np-1}{k}_2 \\
&= \sum_{k=0}^{\left[\frac{p}{3}\right]} \binom{np-1}{3k}_2 + \sum_{k=0}^{\left[\frac{p-2}{3}\right]} \binom{np-1}{3k+1}_2 + \sum_{k=0}^{\left[\frac{p}{3}\right]-1} \binom{np-1}{3k+2}_2 \\
&\equiv \frac{p+2}{3} + np \left(\frac{2}{3} q_3 - \frac{1}{3} \right) - \frac{p-1}{3} + np \left(\frac{1}{3} q_3 + \frac{1}{3} \right) \\
&\equiv 1 + npq_3 \pmod{p^2}. \quad (19)
\end{aligned}$$

By Theorem 1–Theorem 3, when $m=1$ and $p\equiv 2 \pmod 3$, we obtain

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{np-1}{k}_2\\&=\sum_{k=0}^{[\frac{p}{3}]}\binom{np-1}{3k}_2+\sum_{k=0}^{[\frac{p-2}{3}]}\binom{np-1}{3k+1}_2+\sum_{k=0}^{[\frac{p}{3}]-1}\binom{np-1}{3k+2}_2\\&\equiv\frac{p+1}{3}+np\left(\frac{1}{3}q_3-\frac{2}{3}\right)-\frac{p+1}{3}-np\left(\frac{1}{3}q_3-\frac{1}{3}\right)+\frac{np}{3}\\&\equiv 0 \pmod{p^2}. \quad (20)\end{aligned}$$

The (19) and (20) has been proved in [10]. By Theorem 1–Theorem 3, when $m>1$ and $p\equiv 1 \pmod 3$, we obtain

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{np-1}{k}_2^m\\&=\sum_{k=0}^{[\frac{p}{3}]}\binom{np-1}{3k}_2^m+\sum_{k=0}^{[\frac{p-2}{3}]}\binom{np-1}{3k+1}_2^m+\sum_{k=0}^{[\frac{p}{3}]-1}\binom{np-1}{3k+2}_2^m\\&\equiv\frac{p+2}{3}+mnp\left(\frac{2}{3}q_3-\frac{1}{3}\right)+(-1)^m\left(\frac{p-1}{3}-mnp\left(\frac{1}{3}q_3+\frac{1}{3}\right)\right)\\&\equiv\frac{((-1)^m+1)p+2-(-1)^m}{3}\\&\quad+mnp\left(\frac{2-(-1)^m}{3}q_3-\frac{1+(-1)^m}{3}\right) \pmod{p^2}. \quad (21)\end{aligned}$$

By Theorem 1–Theorem 3, when $m>1$ and $p\equiv 2 \pmod 3$, we obtain

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{np-1}{k}_2^m\\&=\sum_{k=0}^{[\frac{p}{3}]}\binom{np-1}{3k}_2^m+\sum_{k=0}^{[\frac{p-2}{3}]}\binom{np-1}{3k+1}_2^m+\sum_{k=0}^{[\frac{p}{3}]-1}\binom{np-1}{3k+2}_2^m\\&\equiv\frac{p+1}{3}+mnp\left(\frac{1}{3}q_3-\frac{2}{3}\right)+(-1)^m\left(\frac{p+1}{3}+mnp\left(\frac{1}{3}q_3-\frac{1}{3}\right)\right)\\&\equiv\frac{((-1)^m+1)(p+1)}{3}\\&\quad+mnp\left(\frac{1+(-1)^m}{3}q_3-\frac{2+(-1)^m}{3}\right) \pmod{p^2}. \quad (22)\end{aligned}$$

Combining (19)–(22), we obtain Theorem 4. □

Proof of Theorem 5

Proof:

$$\sum_{k=0}^{\frac{p-1}{2}}\binom{np-1}{k}_2^m=\sum_{k=0}^{[\frac{p}{6}]}\binom{np-1}{3k}_2^m+\sum_{k=0}^{[\frac{p-2}{6}]}\binom{np-1}{3k+1}_2^m+\sum_{k=0}^{[\frac{p-2}{6}]}\binom{np-1}{3k+2}_2^m \pmod{p^2}.$$

By Lemma 4, Lemma 1 and Lemma 3, similar to the proof of Theorem 1, we obtain that

$$\sum_{k=0}^{[\frac{p}{6}]}\binom{np-1}{3k}_2^m\equiv\begin{cases}\frac{p+5}{6}+mnp\left(\frac{13}{9}q_2+\frac{7}{12}q_3-\frac{1}{6}\right) \pmod{p^2},\\ \qquad \text{if } p\equiv 1 \pmod 3,\\ \frac{p+1}{6}+mnp\left(\frac{1}{9}q_2+\frac{1}{6}q_3-\frac{1}{2}\right) \pmod{p^2},\\ \qquad \text{if } p\equiv 2 \pmod 3.\end{cases}$$

By Lemma 4, Lemma 1 and Lemma 3, similar to the proof of Theorem 2, we obtain that

$$\sum_{k=0}^{[\frac{p-2}{6}]}\binom{np-1}{3k+1}_2^m\equiv\begin{cases}(-1)^m\left(\frac{p-1}{6}-mnp\left(\frac{1}{9}q_2+\frac{1}{6}q_3+\frac{1}{6}\right)\right) \pmod{p^2},\\ \qquad \text{if } p\equiv 1 \pmod 3,\\ (-1)^m\left(\frac{p+1}{6}+mnp\left(\frac{5}{9}q_2-\frac{1}{12}q_3-\frac{1}{2}\right)\right) \pmod{p^2},\\ \qquad \text{if } p\equiv 2 \pmod 3.\end{cases}$$

By Lemma 4, Lemma 1 and Lemma 3, similar to the proof of Theorem 3, we obtain that

$$\sum_{k=0}^{[\frac{p-2}{6}]}\binom{np-1}{3k+2}_2^m\equiv\begin{cases}0 \pmod{p^2}, \qquad \text{if } m>1,\\ -np\left(\frac{2}{9}q_2-\frac{1}{4}q_3\right) \pmod{p^2},\\ \quad \text{if } p\equiv 1 \pmod 3 \text{ and } m=1,\\ -np\left(\frac{2}{9}q_2+\frac{1}{4}q_3\right) \pmod{p^2},\\ \quad \text{if } p\equiv 2 \pmod 3 \text{ and } m=1.\end{cases}$$

Then, similar to the proof of Theorem 4, we obtain Theorem 5. □

CONCLUSION

As far as we know, changing the summation order can be considered in multiple summations when the result cannot be directly calculated through a formula or when changing the summation order can help simplify the calculation. In this paper, we have successfully established the congruences with partial sums of powers of trinomial coefficients by changing the order of summation and classical congruence calculation methods. We have extended the results of the congruences of binomial coefficients to the results of the congruences of trinomial coefficients, and the congruences of the partial summation of trinomial coefficients $\sum_{k=0}^{[\frac{p}{3}]}\binom{np-1}{3k}_2^m$, $\sum_{k=0}^{[\frac{p-2}{3}]}\binom{np-1}{3k+1}_2^m$ and $\sum_{k=0}^{[\frac{p}{3}]-1}\binom{np-1}{3k+2}_2^m$, also indicate that their distribution is not evenly distributed.

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