Theory of $\Delta$ conditional randomized truth degree in Gödel $n$-valued propositional logic system of adding $\Delta$ operator

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ABSTRACT: In this paper, $\Delta$ conditional randomized truth degree of propositional formula is put forward in Gödel $n$-valued propositional logic system. It adds $\Delta$ operator compared with conditional randomized truth degree. On this basis, some inference rules such as MP, HS, intersection inference, union inference and their related properties are studied. At last, the concepts of $\Delta$ conditional randomized similarity degree, $\Delta$ conditional randomized pseudo-metric between propositional formulas are given, and their related good properties are discussed.

KEYWORDS: $\Delta$ conditional randomized truth degree, $\Delta$ conditional randomized similarity degree, $\Delta$ conditional randomized logic metric space

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INTRODUCTION

As we all know, mathematical logic is a formal theory characterized by symbolization. It focuses on formal reasoning rather than numerical calculation. However, numerical calculation pays more attention to solving problems and rarely uses formal reasoning methods. There are great differences, but the quantitative logic is initiated by Wang [1–4] is a new branch of research that tries to connect the two, which is the product of the combination of mathematical logic and probability calculation.

The idea of introducing probability methods into mathematical logic has gradually emerged since the 1970s, and a monograph on “probabilistic logic” has been published [5]. Later, many experts studied on this basis and some results are obtained. It is worth noting that Hui et al. [6–8] combined quantitative logic with probability logic, the concepts of randomized truth degree for binary and ternary logic systems are put forward, the randomized logic metric spaces are established.

Among the logical systems that have received widespread attention at present, related research has been hindered due to the strong negation in the Gödel system and the Goguen system. In order to solve this problem, the basic connectives $\sim$ and $\Delta$ are introduced in [9–12]. The quantification of $\Delta$ fuzzy logic system $\text{SBL}_\Delta$ is realized by Hui in [13], the theory of $t$-randomized truth degree on Gödel $n$-valued propositional logic system of adding two operators is proposed by Zhu in [14]. It is a very meaningful subject to combine the conditional probability part of probabilistic logic with the truth degree through an appropriate way [5].

Following the research results of the theory of $t$-randomized truth degree in Gödel $n$-valued propositional logic system. In this paper, $\Delta$ conditional randomized truth degree of propositional formula is put forward in Gödel $n$-valued propositional logic system. It adds $\Delta$ operator compared with conditional randomized truth degree. On this basis, some inference rules such as MP, HS, intersection inference, union inference and their related properties are studied. At last, the concepts of $\Delta$ conditional randomized similarity degree, $\Delta$ conditional randomized pseudo-metric between propositional formulas are given, and their related good properties are discussed.

PRELIMINARIES

Definition 1 ([11]) The axiom system of $\text{BL}_\Delta$ is as follows:

(Bl) the axiom system of BL

$(\Delta 1)$ $\Delta A \lor \sim \Delta A;
$(\Delta 2)$ $\Delta (A \lor B) \rightarrow (\Delta A \lor \Delta B);$
$(\Delta 3)$ $\Delta A \rightarrow B;$
$(\Delta 4)$ $\Delta A \rightarrow \Delta \Delta A;$
$(\Delta 5)$ $\Delta (A \rightarrow B) \rightarrow (\Delta A \rightarrow \Delta B).$

The inference rules in $\text{BL}_\Delta$ are MP rule and $\Delta$ rule, the MP rule is from $A, A \rightarrow B$, inferred $B$, the $\Delta$ rule is from $A$ inferred $\Delta A$.

Theorem 1 ([15], $\Delta$ deduction theorem) Let $L$ be an axiomatic extension of $\text{BL}_\Delta$, then for any theory $\Gamma$, the formulas $A$ and $B$, we have $\Gamma, A \vdash B$ if and only if $\Gamma \vdash \Delta A \rightarrow B$.

Definition 2 ([14]) Let $S = \{p_1, p_2, \ldots\}$ be a countable set, $\Delta$ is unary operation on $S$, $\lor$, $\land$, $\rightarrow$ are binary operations on $S$, respectively, $F(S)$ is a free algebra of type $\langle 1,2,2,2\rangle$ generated by $S$. Then the elements...
in $F(S)$ are called propositional formulas or formulas, and the elements in $S$ are called atomic formulas.

**Definition 3** ([14]) Let $L = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$. It is stipulated in $L$: $\forall x, y, x \in L, \Delta x = \{\frac{1}{n-1}, \frac{n-1}{n-1}, 1\}, x \vee y = \max\{x, y\}, x \wedge y = \min\{x, y\}, x \rightarrow y = $, then it is called type $(1, 2, 2, 2)$ algebra, which is called the expansion of Gödel $n$-valued propositional logic system. It is abbreviated as $G_\Delta$, if there is no special description, it is expanded in $G_\Delta$.

**Definition 4** ([14]) Let $A = (p_1, p_2, \ldots, p_n) \in F(S)$, Then $A$ corresponds to an $n$-valued $m$-element function $A$, in $G_\Delta, \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}^m \rightarrow [0, 1]$. Here $A(p_1, p_2, \ldots, p_n)$ is formed by the operation symbol $\Delta, \vee, \wedge, \rightarrow$ connecting the atomic formula $p_1, p_2, \ldots, p_n$ by the conjunction $\Delta, \vee, \wedge$. Let $\Delta$ is called the function induced by the formula $A$.

**Definition 5** ([16]) Let $N = (1, 2, \ldots), D = (p_1, p_2, p_3), 0 < p_0 < 1 (n = 1, 2, \ldots)$. Then $D$ is called a random number sequence in $(0, 1)$.

**Definition 6** ([16]) Let $D_0 = (p_{01}, p_{02}, \ldots), D_{\frac{1}{n-1}} = (p_{\frac{1}{n-1}1}, p_{\frac{1}{n-1}2}, \ldots), D_1 = (p_{11}, p_{12}, \ldots)$ be a series of random numbers in $(0, 1)$, and $p_{0k} + p_{\frac{1}{n-1}k} + \cdots + p_{1k} = 1 (k = 1, 2, \ldots)$. Then $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ $(n \geq 2)$ is said to be a $n$-valued random-number sequence in $(0, 1)$.

**Definition 7** ([16]) Let $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be a series of $n$-valued random numbers in $(0, 1), \forall a = (x_1, x_2, \ldots, x_m) \in \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}^m$. Let $\varphi(a) = Q_1 \times \cdots \times Q_m$. Here, when $x_0 = 0, x_0 = d_{0k}$; when $x_0 = 1, Q_k = d_{1k}$ $(k = 1, 2, \ldots, m)$, then a mapping $\varphi : \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}^m \rightarrow [0, 1]$. Then $\varphi$ is called the D-randomization map of $(0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1)^m$.

**Definition 8** ([16]) Let be a D-randomization map of $(0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1)^m$, then

$$\sum_{a \in \{0, \frac{n}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}^m} \varphi(a) = 1.$$  

**Definition 9** ([14]) Let $A = (p_1, p_2, \ldots, p_n) \in F(S)$, $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an $n$-valued random-number sequence in $(0, 1)$, and

$$[\Delta A]_{\frac{i}{n-1}} = \Delta A \left(\frac{i}{n-1}\right)$$

$$\mu([\Delta A]_{\frac{i}{n-1}}) = \sum_{\alpha \in \Delta A \left(\frac{i}{n-1}\right)} \varphi(a) : \alpha \in \Delta A \left(\frac{i}{n-1}\right)$$

$$\mu[A] = \sum_{i=1}^{n-1} \mu([\Delta A]_{\frac{i}{n-1}}), i = 1, 2, \ldots, n-1.$$

Denote $\mu[A]$ as $\tau_d(A)$, then $\tau_d(A)$ is called the randomized truth degree of $\Delta$ of the propositional formula $A$.

**Theorem 2** ([14]) Let $A, B \in F(S), D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an $n$-valued random-number sequence in $(0, 1)$. Then,

(i) $A$ is tautology if and only if $\tau_d(A) = 1$;

(ii) if $A \equiv B$, then $\tau_d(A)$ is $\tau_d(B)$;

(iii) $\tau_d(\Delta A \land \Delta B) = \tau_d(A) + \tau_d(B) - \tau_d(\Delta A \land \Delta B)$.

**CONDITIONAL RANDOMIZED TRUTH DEGREE**

**Definition 10** Let $A = (p_1, p_2, \ldots, p_n) \in F(S), D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an $n$-valued random-number sequence in $(0, 1), \forall \in F(S), \tau_d(\Delta A) > 0$, and

$$\tau_d(\Delta A | \Delta A) = \frac{\tau_d(\Delta A \land \Delta B)}{\tau_d(\Delta A)}.$$

Then $\tau_d(\Delta A | \Delta A)$ is called the conditional randomized truth degree of formula $A$ under condition $\Lambda$.

**Remark 1** Definition 9 gives the $\Delta$ randomized truth degree of the propositional formula, after converting Definition 9, the following Proposition 1 is obtained.

**Proposition 1** Let $A = (p_1, p_2, \ldots, p_n) \in F(S), D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an $n$-valued random-number sequence in $(0, 1), \forall \in F(S), \tau_d(\Delta A) > 0$, and

$$\tau_d(A) = \sum_{a \in \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}} [\Delta A]_a \varphi(a) : \alpha \in \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}^m.$$

**Proof:**

$$\tau_d(A) = \sum_{i=1}^{n-1} \{\sum_{a \in \Delta A \left(\frac{i}{n-1}\right)} \varphi(a) : \alpha \in \Delta A \left(\frac{i}{n-1}\right)\}$$

$$= \sum_{i=1}^{n-1} \{\sum_{a \in \Delta A \left(\frac{i}{n-1}\right)} \varphi(a) : \alpha \in \Delta A \left(\frac{i}{n-1}\right)\}$$

$$= \sum_{i=1}^{n-1} \{\sum_{a \in \Delta A \left(\frac{i}{n-1}\right)} \varphi(a) : \alpha \in \Delta A \left(\frac{i}{n-1}\right)\}$$

$$= \sum_{i=1}^{n-1} \{\sum_{a \in \Delta A \left(\frac{i}{n-1}\right)} \varphi(a) : \alpha \in \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}\}.$$

**Definition 11** Let $A = (p_1, p_2, \ldots, p_n) \in F(S), D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an $n$-valued random-number sequence in $(0, 1), \forall \in F(S),$ and $\tau_d(\Delta A) > 0$.

(i) If there is $\Delta A \left(\frac{i}{n-1}\right) \subseteq \Delta A \left(\frac{i}{n-1}\right)$ when $i = 1, 2, \ldots, n-1$, then $A$ is called tautology under the condition $\Lambda$, denoted as $A=_{\Lambda} 1$.
(ii) if there is \((\Delta A \land \Delta A)^{-1}(\frac{i}{n}) = \phi\) when \(i = 1,2,\ldots,n-1\), then \(A\) is called contradiction under the condition \(\Lambda\), denoted as \(A = \alpha \Lambda\);

(iii) if there is \((\Delta A \land \Delta A)^{-1}(\frac{i}{n}) = (\Delta B \land \Delta A)^{-1}(\frac{i}{n})\) when \(i = 1,2,\ldots,n-1\), then \(A\) and \(B\) are said to be logically equivalent under the condition \(\Lambda\), denoted as \(A \equiv \Lambda B\).

**Theorem 3** Let \(A, B \in F(S)\), \(D_0, D_1, \ldots, D_{n-1}, D_1\) \((n \geq 2)\) be an \(n\)-valued randomized numbers sequence in \((0,1)\), \(\Lambda \in F(S)\), and \(\tau_D(\Delta A) > 0\).

(i) If \(A = \omega_\Lambda\), then \(\tau_D(\Delta A \land \Delta A) = 1\);

(ii) if \(A = \alpha\), then \(\tau_D(\Delta A \land \Delta A) = 0\);

(iii) if \(A = \alpha\), then \(\tau_D(\Delta A \land \Delta A) = \tau_D(\Delta B \land \Delta A)\);

(iv) \(\tau_D(\Delta A \lor \Delta B) = \tau_D(\Delta A \land \Delta A) + \tau_D(\Delta B \land \Delta A)\).

**Proof:** (i): Let \(A\) and \(B\) contain the same atomic formulas \(q_1,\ldots,q_m\).

\[
\forall \alpha \in (\Delta A \land \Delta A)^{-1}(\frac{i}{n}) = \phi \text{ that } (\Delta A \land \Delta A)(\alpha) = (\alpha^{-1}), \\
\text{It follows from the homorphism of the induced function that } (\Delta A \land \Delta A)(\alpha) = (\alpha^{-1}), \\
\text{min}(\Delta A \land \Delta A) = (\alpha^{-1}), \\
\text{It follows from } \Lambda = \Lambda_1 \\
\text{that } (\Delta A \land \Delta A)(\alpha) = (\alpha^{-1}), \\
\text{min}(\Delta A \land \Delta A) = (\alpha^{-1}). \\
\text{Then it follows from } \min(\Delta A \land \Delta A) = (\alpha^{-1}) \\
\text{Conversely, } \forall \alpha \in (\Delta A \land \Delta A)^{-1}(\frac{i}{n}) = \phi \text{ that } (\Delta A \land \Delta A)(\alpha) = (\alpha^{-1}), \\
\text{min}(\Delta A \land \Delta A) = (\alpha^{-1}).
\]

(ii): Because \(\tau_D(\Delta A \land \Delta A) = \sum_{i=0}^{n-1} \frac{i}{n} \sum_{k=0}^{n-1} \langle \phi(a) : a \in (\Delta A \land \Delta A)^{-1}(\frac{i}{n}) \rangle, \)

\[
\text{it follows from } (\Delta A \land \Delta A)^{-1}(\frac{i}{n}) = \phi \text{ that } \tau_D(\Delta A \land \Delta A) = 0. \\
\text{Hence, } \tau_D(\Delta A \land \Delta A) = 0.
\]

(iii): It follows from \(A \equiv \Lambda B\) that \(\overline{(\Delta A \land \Delta A)^{-1}(\frac{i}{n})} = (\Delta B \land \Delta A)^{-1}(\frac{i}{n}).\)

\[
\tau_D(\Delta A \land \Delta A) = \sum_{i=0}^{n-1} \frac{i}{n} \sum_{k=0}^{n-1} \langle \phi(a) : a \in (\Delta B \land \Delta A)^{-1}(\frac{i}{n}) \rangle
\]

\[
= \tau_D(\Delta B \land \Delta A).
\]

Hence, \(\tau_D(\Delta A \land \Delta A) = \tau_D(\Delta B \land \Delta A).
\]

(iv): It follows from **Theorem 2**(ii) that \(\tau_D(\Delta A \land \Delta A) = \tau_D(\Delta A \land \Delta A) \lor \Delta A) = \tau_D(\Delta A \land \Delta A) + \tau_D(\Delta B \land \Delta A) - \tau_D(\Delta A \land \Delta A)\). Dividing both sides by \(\tau_D(\Delta A)\) to get \(\tau_D(\Delta A \land \Delta A) = \tau_D(\Delta A \land \Delta A) + \tau_D(\Delta B \land \Delta A) - \tau_D(\Delta A \land \Delta A)\).
That is, \( \tau_p((\Delta A \to (\Delta B \land \Delta C)) \mid \Delta A) = \tau_p((\Delta A \to \Delta B) \land (\Delta A \to \Delta C)) \mid \Delta A) \). It follows from Theorem 3(iv) that \( \tau_p((\Delta A \to \Delta B) \land (\Delta A \to \Delta C)) \mid \Delta A) = \tau_p((\Delta A \to \Delta B) \mid \Delta A) + \tau_p((\Delta A \to \Delta C) \mid \Delta A) \). Hence, \( \tau_p((\Delta A \to (\Delta B \land \Delta C)) \mid \Delta A) = \tau_p((\Delta A \to \Delta B) \mid \Delta A) + \tau_p((\Delta A \to \Delta C) \mid \Delta A) - \tau_p(((\Delta A \to \Delta B) \lor (\Delta A \to \Delta C)) \mid \Delta A) \).

**Corollary 1 (\( \Delta \) conditional randomised truth degree intersection inference rule)** Let \( A, B, C \in F(S), D_0, D_1, D_2, \ldots, D_{n-1}, D_n \) be a \( n \)-valued randomised numbers sequence in \((0, 1), \Lambda \in F(S), \) and \( \tau_p(\Delta A) > 0 \). If \( \tau_p((\Delta A \to \Delta B) \mid \Delta A) \geq a, \tau_p((\Delta A \to \Delta C) \mid \Delta A) \geq b, \) then \( \tau_p((\Delta A \to (\Delta B \land \Delta C)) \mid \Delta A) \geq a + b - 1 \).

**Theorem 7** Let \( A, B, C \in F(S), D_0, D_1, D_2, \ldots, D_{n-1}, D_n \) be an \( n \)-valued randomised numbers sequence in \((0, 1), \Lambda \in F(S), \) and \( \tau_p(\Delta A) > 0 \). Then \( \tau_p((\Delta A \lor \Delta B) \to \Delta C) \mid \Delta A) = \tau_p((\Delta A \to \Delta C) \mid \Delta A) + \tau_p((\Delta B \to \Delta C) \mid \Delta A) \).

**Proof:** Let \( A, B, C \) and \( \Lambda \) contain the same atomic formulas \( q_1, \ldots, q_m \). For all \( a, b, c \in \mathbb{D}_A \) we have \( (\Delta A \to \Delta B) \lor (\Delta B \to \Delta C) \). That is, \( \forall a \in \{0, \frac{1}{n-1}, \ldots, \frac{n-1}{n-1}, 1\} \) there is \( (\Delta A \to \Delta B) \lor (\Delta B \to \Delta C) \). Hence, \( \sum((\Delta A \to \Delta B) \lor (\Delta B \to \Delta C)) = (\Delta A \to \Delta B) + (\Delta B \to \Delta C) \). Therefore, \( \sum((\Delta A \to \Delta B) \lor (\Delta B \to \Delta C)) = (\Delta A \to \Delta B) + (\Delta B \to \Delta C) \).

**Δ CONDITIONAL RANDOMIZED SIMILARITY DEGREE AND Δ CONDITIONAL RANDOMIZED PSEUDO-DISTANCE**

**Definition 12** Let \( A, B \in F(S), D_0, D_1, D_2, \ldots, D_{n-1}, D_n \) be an \( n \)-valued randomised numbers sequence in \((0, 1), \Lambda \in F(S), \) and \( \tau_p(\Delta A) > 0 \). Let

\[ \xi_p((\Delta A \lor \Delta B) \mid \Delta A) = \tau_p((\Delta A \to \Delta B) \lor (\Delta B \to \Delta A)) \mid \Delta A) \]

then, \( \xi_p((\Delta A \lor \Delta B) \mid \Delta A) \) is called the \( \Delta \) conditional similarity degree between Proposition 1 formulas \( A \) and \( B \).

**Theorem 9** Let \( A, B \in F(S), D_0, D_1, D_2, \ldots, D_{n-1}, D_n \) be a \( n \)-valued randomised numbers sequence in \((0, 1), \Lambda \in F(S), \) and \( \tau_p(\Delta A) > 0 \). (i) If \( A \equiv \Lambda B \), then \( \xi_p((\Delta A \lor \Delta B) \mid \Delta A) = 1 \); (ii) \( \xi_p((\Delta A \lor \Delta B) \mid \Delta A) = \xi_p((\Delta B \lor \Delta A) \mid \Delta A) \); (iii) \( \xi_p((\Delta A \lor \Delta B) \mid \Delta A) = \xi_p((\Delta B \lor \Delta A) \mid \Delta A) \); (iv) \( \xi_p((\Delta A \lor \Delta B) \mid \Delta A) = \xi_p((\Delta B \lor \Delta A) \mid \Delta A) \).

**Proof:** (i) It follows from \( A \equiv \Lambda B \) that both \( A \to B \) and \( B \to A \) are tautologies based on \( \Lambda \), then \( (A \to B) \mid (B \to A) = 1 \). It follows from Theorem 3(i) that \( \tau_p((\Delta A \lor \Delta B) \mid (\Delta A \to \Delta B) \mid (\Delta B \to \Delta A)) = 1 \). (ii) For all \( a, b, c \in \mathbb{D}_A \), we have \( (\Delta a \to \Delta b) \lor (\Delta b \to \Delta a) = (\Delta b \to \Delta a) \lor (\Delta a \to \Delta b) \). Hence, \( (\Delta a \to \Delta b) \lor (\Delta b \to \Delta a) \lor (\Delta c \to \Delta d) \lor (\Delta d \to \Delta c) \). That is, \( (\Delta A \to \Delta B) \lor (\Delta B \to \Delta A) \lor (\Delta A \to \Delta B) \lor (\Delta B \to \Delta A) \). It follows from Theorem 2(ii) that \( \xi_p((\Delta A \lor \Delta B) \mid (\Delta A \to \Delta B) \mid (\Delta B \to \Delta A)) = 1 \). Dividing both sides by \( \tau_p(\Delta A) \) to get \( \xi_p((\Delta A \lor \Delta B) \mid (\Delta A \to \Delta B) \mid (\Delta B \to \Delta A)) = 1 \). (iii) For all \( a, b, c \in \mathbb{D}_A \), we have \( (\Delta a \to \Delta b) \lor (\Delta b \to \Delta a) = (\Delta a \to \Delta b) \lor (\Delta b \to \Delta a) \lor (\Delta a \to \Delta b) \lor (\Delta b \to \Delta a) \).
Proof: For all \( a, b, c \in G_{\Delta} \), we have \( (\Delta a \land \Delta b) \rightarrow \Delta b = (\Delta a \rightarrow \Delta b) \lor (\Delta b \rightarrow \Delta b) = \Delta b \lor \Delta b, \) then \( \Delta b \rightarrow (\Delta a \land \Delta b) = (\Delta b \rightarrow \Delta a) \land (\Delta b \rightarrow \Delta b) = \Delta b \rightarrow \Delta a. \) Hence,

\[
\begin{align*}
\xi_\rho((\Delta A \land \Delta B, |\Delta \Lambda) = &\; \tau_\rho(((\Delta A \land \Delta B) \rightarrow \Delta B) \land (\Delta b \rightarrow (\Delta A \land \Delta B))) |\Delta \Lambda) \\
= &\; \tau_\rho(((\Delta A \land \Delta B) \rightarrow \Delta B) \land (\Delta b \rightarrow (\Delta A \land \Delta B))) |\Delta \Lambda) \\
= &\; \tau_\rho((\Delta A \rightarrow \Delta B) \land (\Delta b \rightarrow (\Delta A \land \Delta B))) |\Delta \Lambda) \\
= &\; \tau_\rho((\Delta A \land \Delta B, |\Delta \Lambda) \\
= &\; \tau_\rho((\Delta B \rightarrow \Delta A) \land (\Delta a \rightarrow \Delta A) \land (\Delta b \rightarrow (\Delta A \land \Delta B))) |\Delta \Lambda) \\
= &\; \tau_\rho((\Delta B \rightarrow \Delta A) \land (\Delta \Lambda).)
\end{align*}
\]

\( \Box \)

Theorem 10 Let \( A, B \in F(S), D_0, D_1, \ldots, D_{n-2}, D_1 \) \((n \geq 2)\) be an \( n \)-valued randomized numbers sequence in \((0, 1), L \in F(S), \) and \( \tau_\rho(\Delta \Lambda) > 0. \) Then, \( \xi_\rho((\Delta A, \Delta B) \land (\Delta A \rightarrow \Delta B)) |\Delta \Lambda) = \tau_\rho((\Delta A \rightarrow \Delta B) \land (\Delta B \rightarrow (\Delta A \land \Delta B))) |\Delta \Lambda) + \tau_\rho((\Delta B \rightarrow \Delta A) \land (\Delta B \rightarrow (\Delta A \land \Delta B))) |\Delta \Lambda) = 1. \)

Proof: Let \( A, B, \) and \( \Lambda \) contain the same atomic formulas \( q_1, \ldots, q_m. \) It follows from Definition 12 and Theorem 3(iv) that \( \xi_\rho((\Delta A, \Delta B) \land (\Delta A \rightarrow \Delta B)) |\Delta \Lambda) = \tau_\rho((\Delta A \rightarrow \Delta B) \land (\Delta B \rightarrow (\Delta A \land \Delta B))) |\Delta \Lambda) + \tau_\rho((\Delta B \rightarrow \Delta A) \land (\Delta B \rightarrow (\Delta A \land \Delta B))) |\Delta \Lambda) + \tau_\rho((\Delta B \rightarrow \Delta A) \land (\Delta B \rightarrow (\Delta A \land \Delta B))) |\Delta \Lambda) = 1, \) hence, \( \tau_\rho((\Delta A \rightarrow \Delta B) \lor (\Delta B \rightarrow (\Delta A \land \Delta B))) |\Delta \Lambda) = 1, \) that is, \( \xi_\rho((\Delta A, \Delta B) \land (\Delta A \rightarrow \Delta B)) |\Delta \Lambda) = \tau_\rho((\Delta A \rightarrow \Delta B) \land (\Delta B \rightarrow (\Delta A \land \Delta B))) |\Delta \Lambda) + \tau_\rho((\Delta B \rightarrow \Delta A) \land (\Delta B \rightarrow (\Delta A \land \Delta B))) |\Delta \Lambda) = 1. \)

\( \Box \)

Theorem 11 Let \( A, B, C \in F(S), D_0, D_1, \ldots, D_{n-2}, D_1 \) \((n \geq 2)\) be an \( n \)-valued randomized numbers sequence in \((0, 1), L \in F(S), \) and \( \tau_\rho(\Delta \Lambda) > 0. \) Then, \( \xi_\rho((\Delta A, \Delta B, \Delta C) \land (\Delta A \rightarrow \Delta B) \land (\Delta B \rightarrow (\Delta A \land \Delta C))) |\Delta \Lambda) = \tau_\rho((\Delta A \rightarrow \Delta B) \lor (\Delta B \rightarrow (\Delta A \land \Delta C))) |\Delta \Lambda) = 1. \)

Proof: Let \( A, B, C, \) and \( \Lambda \) contain the same atomic formulas \( q_1, \ldots, q_m. \) It follows from Theorem 5 and Theorem 10 that \( \xi_\rho((\Delta A, \Delta B, \Delta C) \land (\Delta A \rightarrow \Delta B)) |\Delta \Lambda) = \tau_\rho((\Delta A \rightarrow \Delta B) \lor (\Delta B \rightarrow (\Delta A \land \Delta C))) |\Delta \Lambda) = 1 \) and \( \xi_\rho((\Delta A, \Delta B, \Delta C) \land (\Delta A \rightarrow \Delta B) \land (\Delta B \rightarrow (\Delta A \land \Delta C))) |\Delta \Lambda) = \tau_\rho((\Delta A \rightarrow \Delta B) \lor (\Delta B \rightarrow (\Delta A \land \Delta C))) |\Delta \Lambda) = 1. \)

\( \Box \)

CONCLUSION

In this paper, \( \Delta \) conditional randomized truth degree of propositional formulas is put forward in Gödel \( n \)-valued propositional logic system. It adds \( \Delta \) operator compared with conditional randomized truth degree. On this basis, some inference rules such as MP, HS, intersection inference, union inference and their related properties are studied. At last, the concepts of \( \Delta \) conditional randomized similarity degree, \( \Delta \) conditional randomized pseudo-metric between propositional formulas are given, and their related good properties are discussed. Thus, how to further develop the approximate reasoning and topological properties in the \( \Delta \) conditional randomized metric space of the Gödel \( n \)-valued propositional logic system with the addition of \( \Delta \) operator will be discussed in another paper.
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REFERENCES