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On entire solutions of certain types of nonlinear differential equations

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ABSTRACT: In this paper, we mainly investigate entire solutions of certain types of nonlinear differential equations that are related to trigonometric identities, and obtain some interesting results. Besides, we give the growth of entire solutions of nonlinear monomial differential-difference equations.

KEYWORDS: entire function, nonlinear differential equation, Nevanlinna theory

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INTRODUCTION

We use the standard notations of Nevanlinna theory [1, 2], i.e., $m(r, f)$, $N(r, f)$ and $T(r, f)$ denote the proximity function, the counting function and the characteristic function of f, respectively. For the growth order of f, we use the notation $\rho(f)$. It is an interesting and quite difficult question to find all the meromorphic solutions of a nonlinear differential equation or prove that a nonlinear differential equation has no meromorphic solution. There have been some results obtained lately that relate to the existence and the growth of meromorphic solutions of various types of differential equation, see [3–7] and references therein.

In 1939, by using Laguerre's theorem on the interlacing property of real zeros and critical points, Titchmarsh [8] showed that the following differential equation

$$ff'' = -\sin^2 z$$

has no real finite-order entire solutions other than $f(z) = \pm\sin z$. In 2019, Li et al [9] considered the more general equation

$$ff'' = p(z)\sin^2 z, \tag{1}$$

where $p(z) \not\equiv 0$ is a polynomial with real coefficients and real zeros. They obtained the following theorem:

Theorem A ([9]) *Suppose that f is an entire function satisfying (1). Then $p(z)$ must be a nonzero constant, and $f(z) = a\sin z$, where a is a constant satisfying $a^2 = -p$.*

Yang and Li [10] examined all the entire solutions of the nonlinear differential equation

$$4(f(z))^3 + 3f''(z) = -\sin 3z. \tag{2}$$

Theorem B ([10]) *Equation (2) admits exactly three entire solutions, namely $f_1(z) = \sin z$, $f_2(z) = \frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z$ and $f_3(z) = -\frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z$.*

Equation (2) considered by Yang [11], Yang and Li [10] comes from the formula: $\sin 3z = 3\sin z - 4\sin^3 z$. Later, Zhang and Yi [12] considered the more basic trigonometric formula: $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$. They gave all entire solutions of the following equation

$$f(z_1 + z_2) = f(z_1)f'(z_2) + f(z_2)f'(z_1),$$

where $f(z)$ is a meromorphic function (see [12, Theorem 1.2]). Particularly, the only entire solutions of the differential equation

$$2f(z)f'(z) = \sin 2z$$

are the four solutions $f_{1,2}(z) = \pm i\cos z$, $f_{3,4}(z) = \pm\sin z$ (see [12, Corollary 1.7]).

Recently, Gundersen et al [13] proved the following result, which is related to the trigonometric identity $(\cos z)^2 - (\sin z)^2 = \cos 2z$.

Theorem C ([13]) *The only entire solutions of the differential equation*

$$(f(z))^2 - (f'(z))^2 = \cos 2z$$

are the four solutions $f(z) = \pm\cos z, \pm i\sin z$.

Being enlightened by Theorems A, B and C, we will prove the next results, which are connected to the trigonometric identity $\cos 3z = \cos^3 z - 3\sin^2 z\cos z$.

Theorem 1 *The only entire solutions of the differential equation*

$$(f(z))^3 - 3f(z)(f'(z))^2 = \cos 3z \tag{3}$$

have the form $f(z) = c_1 e^{iz} + c_2 e^{-iz}$, *where* c_1, c_2 *are constants satisfying* $c_1^3 = \frac{1}{8}$, $c_2^3 = \frac{1}{8}$.

Theorem 2 *The only entire solutions of the differential equation*

$$(f(z))^3 + 3(f'(z))^2 f''(z) = \cos 3z \tag{4}$$

have the form $f(z) = c_1 e^{iz} + c_2 e^{-iz}$, *where* c_1, c_2 *are constants satisfying* $c_1^3 = \frac{1}{8}$, $c_2^3 = \frac{1}{8}$.

In the following, we continue to study the more general equation of the form

$$p(z)(f(z))^3 + q(z)f(z)(f'(z))^2 = \cos(\alpha(z)), \tag{5}$$

where $p(z)$, $q(z)$ are polynomials with $p(z)q(z) \not\equiv 0$, and $\alpha(z) = az + b$, where $a \in \mathbb{C}\backslash\{0\}$, $b \in \mathbb{C}$.

Theorem 3 *If* (5) *admits an entire solution* f, *then* $p(z) \equiv p$, $q(z) \equiv q$ *are nonzero constants, and* a *satisfying* $a \in \left\{3\sqrt{\frac{-3p}{q}}, -3\sqrt{\frac{-3p}{q}}, \sqrt{\frac{p}{q}}, -\sqrt{\frac{p}{q}}\right\}$. *Moreover,* f *has the form*

$$f(z) = C_1 e^{\lambda_1 z} + C_2 e^{\lambda_2 z},$$

where C_1, C_2 *are nonzero constants satisfying* $C_1C_2 = 1/4p^{\frac{2}{3}}$. *One of the following assertions holds:*

(i) $a = \sqrt{\frac{p}{q}}$, *then* $\lambda_1 = i\sqrt{\frac{p}{q}}$, $\lambda_2 = -i\sqrt{\frac{p}{q}}$, $8pC_1^2C_2 = e^{ib}$ *and* $8pC_1C_2^2 = e^{-ib}$;

(ii) $a = -\sqrt{\frac{p}{q}}$, *then* $\lambda_1 = i\sqrt{\frac{p}{q}}$, $\lambda_2 = -i\sqrt{\frac{p}{q}}$, $8pC_1^2C_2 = e^{-ib}$ *and* $8pC_1C_2^2 = e^{ib}$;

(iii) $a = 3\sqrt{\frac{-3p}{q}}$, *then* $\lambda_1 = \sqrt{\frac{3p}{q}}$, $\lambda_2 = -\sqrt{\frac{3p}{q}}$, $8pC_3^3 = e^{-ib}$ *and* $8pC_4^3 = e^{ib}$;

(iv) $a = -3\sqrt{\frac{-3p}{q}}$, *then* $\lambda_1 = \sqrt{\frac{3p}{q}}$, $\lambda_2 = -\sqrt{\frac{3p}{q}}$, $8pC_3^3 = e^{ib}$ *and* $8pC_4^3 = e^{-ib}$.

Clearly, Theorem 3 is an extension of Theorem 1. For possible future investigations, we hope to reduce the condition $\alpha(z) = az + b$ to $\alpha(z)$ is an non-constant polynomial in (5). In addition, we also would like to consider (5) with the condition $\alpha(z)$ being a non-constant entire function.

Theorems A and C lead one to consider differential equation of the form $ff^{(k)} = H(z)$, where $k \geqslant 1$ and H is an entire function with $H(z) \not\equiv 0$. Gundersen et al [13] considered entire solutions of nonlinear monomial differential equations of the more general form

$$f^{n_0}(f')^{n_1}(f'')^{n_2} \cdots (f^{(k)})^{n_k} = H(z), \tag{6}$$

where $H(z)$ is an entire function with $H(z) \not\equiv 0$, $k \geqslant 1$, $n_0 \geqslant 1$ and $n_k \geqslant 1$. They showed a double inequality for the growth of entire solutions of (6) and $\rho(f) = \rho(H)$, where f is an entire solution of (6).

Closely related to differential expressions are difference expressions where the usual shift $f'(z)$ of a meromorphic function will be replaced by the shift $f(z+c)$. We consider the following nonlinear monomial differential-difference equations

$$f^{n_0}(f_c')^{n_1}(f_c'')^{n_2} \cdots (f_c^{(k)})^{n_k} = H(z), \tag{7}$$

where $H(z)$ is an entire function with $H(z) \not\equiv 0$, c is a nonzero constant, $f_c = f(z+c)$, $k \geqslant 1$, $n_0 \geqslant 1$ and $n_k \geqslant 1$.

Theorem 4 *If* f *is an entire solution with finite order of a monomial differential-difference equation* (7), *then* $\rho(f) = \rho(H)$.

LEMMAS

Lemma 1 ([15]) *Let* η_1, η_2 *be two complex numbers such that* $\eta_1 \neq \eta_2$ *and let* f *be a finite order meromorphic function. Let* ρ *be the order of* f, *then for each* $\epsilon > 0$, *we have*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O(r^{\rho-1+\epsilon}).$$

Lemma 2 ([15]) *Let* f *be a meromorphic function with order* $\rho = \rho(f)$, $\rho < \infty$, *and let* η *be a fixed nonzero complex number, then for each* $\epsilon > 0$, *we have*

$$T(r, f(z+\eta)) = T(r, f) + O(r^{\rho-1+\epsilon}) + O(\log r).$$

Lemma 3 ([14]) *If* $f_j(z)$, $g_j(z)$, $(1 \leqslant j \leqslant n, n \geqslant 2)$, *are entire functions satisfying*

(i) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv 0$;

(ii) *The orders of* f_j *are less than that of* $e^{g_h - g_k}$ *for* $1 \leqslant j \leqslant n$, $1 \leqslant h < k \leqslant n$.

Then $f_j(z) \equiv 0$ *for* $1 \leqslant j \leqslant n$.

Lemma 4 (Proposition 5.1, [2]) *All non-trivial solutions* f *of*

$$f'' + P(z)f = 0,$$

where $P(z) = a_n z^n + \cdots + a_0$, $a_n \neq 0$, *have the order of growth* $\rho(f) = \frac{n+2}{2}$.

PROOF OF Theorem 1

Suppose that f is an entire solution of (3). Differentiation of (3) gives

$$3f^2 f' - 3(f')^3 - 6ff'f'' = -3\sin 3z.$$

Thus, we have

$$-f^2 f' + (f')^3 + 2ff'f'' = \sin 3z. \tag{8}$$

Differentiating (8) yields

$$-2f(f')^2 - f^2 f'' + 5(f')^2 f'' + 2f(f'')^2 + 2ff'f''' = 3\cos 3z. \tag{9}$$

Combining (3) with (9), we have

$$-2f(f')^2 - f^2 f'' + 5(f')^2 f'' + 2f(f'')^2 + 2ff'f''' = 3f^3 - 9f(f')^2,$$

which gives that

$$f[-3f^2 + 7(f')^2 - ff'' + 2(f'')^2 + 2f'f'''] = -5(f')^2 f''. \quad (10)$$

Next, we will prove that f''/f is an entire function. Suppose that z_0 is a zero of f, i.e., $f(z_0) = 0$. From (10), we have either $f'(z_0) = 0$ or $f''(z_0) = 0$. If $f'(z_0) = 0$, then $f^3 - 3f(f')^2$ has a multiple zero at z_0, which contradicts (3) because $\cos 3z$ has only simple zeros. Thus, we have $f'(z_0) \neq 0$ and $f''(z_0) = 0$. Therefore, any zero of f must be both a simple zero of f and a zero of f'', which implies that f''/f must be an entire function.

Taking Nevanlinna characteristic function of both sides of (3), by using the lemma on the logarithmic derivative, we get that

$$\begin{aligned} T(r, \cos 3z) = T(r, f^3 - 3f(f')^2) &= m(r, f^3 - 3f(f')^2) \\ &\leqslant 3m(r,f) + 2m\left(r, \tfrac{f'}{f}\right) + O(1) \\ &\leqslant 3T(r,f) + S(r,f). \end{aligned}$$

This gives that $\rho(f) \geqslant 1$. Similarly,

$$\begin{aligned} 3T(r,f) &= T\left(r, \frac{\cos 3z}{1 - 3(f'/f)^2}\right) \\ &\leqslant T(r, \cos 3z) + 2T\left(r, \tfrac{f'}{f}\right) + O(1) \\ &\leqslant T(r, \cos 3z) + 2N\left(r, \tfrac{f'}{f}\right) + S(r,f) \\ &\leqslant T(r, \cos 3z) + 2N(r,f) + 2N\left(r, \tfrac{1}{f}\right) + S(r,f) \\ &\leqslant T(r, \cos 3z) + 2T(r,f) + S(r,f), \end{aligned}$$

which implies that $\rho(f) \leqslant 1$. Hence, $\rho(f) = 1$.

Set $\alpha = f''/f$. Since α is an entire solution and $\rho(f) = 1$, then we have, as $r \to \infty$,

$$T(r, \alpha) = m(r, \alpha) = m\left(r, \tfrac{f''}{f}\right) = S(r,f) = O(\log r).$$

Thus, α is a polynomial.

Rewrite $\alpha = f''/f$ as

$$f'' - \alpha f = 0. \quad (11)$$

By Lemma 4, we have $1 = \rho(f) = \frac{n+2}{2}$, where n denotes the degree of α. Therefore, we obtain $n = 0$, which implies that α is a constant. From (10) and (11), it follows that

$$(2\alpha^2 - \alpha - 3)f^2 + (7\alpha + 7)(f')^2 + 2f'(f''' - \alpha f') = 0.$$

(11) gives that $f''' - \alpha f' = 0$. Substituting this expression into the above equation yields

$$(7\alpha + 7)(f')^2 + (2\alpha^2 - \alpha - 3)f^2 = 0. \quad (12)$$

Clearly, $f \not\equiv 0$ and $f' \not\equiv 0$. We consider the following two cases.

If $7\alpha + 7 = 0$ and $2\alpha^2 - \alpha - 3 = 0$, then we have $\alpha = -1$. By (11), we get that

$$f(z) = c_1 e^{iz} + c_2 e^{-iz},$$

where c_1, c_2 are constants. Substituting this expression into (3), we have

$$4c_1^3 e^{3iz} + 4c_2^3 e^{-3iz} = \cos 3z = \frac{e^{3iz} + e^{-3iz}}{2}.$$

By Lemma 3, we have $4c_1^3 - \frac{1}{2} = 0$, $4c_2^3 - \frac{1}{2} = 0$. Hence, we obtain $c_1^3 = \frac{1}{8}$, $c_2^3 = \frac{1}{8}$.

If $7\alpha + 7 \neq 0$ and $2\alpha^2 - \alpha - 3 = 0$, then from (12) we have $f' \equiv 0$. This implies that f is a constant, a contradiction.

If $7\alpha + 7 \neq 0$ and $2\alpha^2 - \alpha - 3 \neq 0$, then $f'/f = \beta$, where $\beta = \sqrt{\frac{3-2\alpha}{7}}$ is a nonzero constant. Then, we have $f = c_3 e^{\beta z}$, where $c_3 \in \mathbb{C}\backslash\{0\}$. Substituting this expression into (3), we get that

$$c_3^3(1 - 3\beta^2)e^{3\beta z} - \frac{e^{3iz}}{2} - \frac{e^{-3iz}}{2} = 0.$$

By Lemma 3, we get a contradiction.

The proof of Theorem 1 is complete.

PROOF OF Theorem 2

Suppose that f is an entire solution of (4). Differentiation of (4) gives

$$3f^2 f' + 6f'(f'')^2 + 3(f')^2 f''' = -3\sin 3z.$$

Thus, we have

$$f^2 f' + 2f'(f'')^2 + (f')^2 f''' = -\sin 3z. \quad (13)$$

Differentiation of (13) gives

$$2f(f')^2 + f^2 f'' + 2(f'')^3 + 6f'f''f''' + (f')^2 f^{(4)} = -3\cos 3z. \quad (14)$$

Differentiating (14) yields

$$2(f')^3 + 6ff'f'' + f^2 f''' + 8f'f''f^{(4)} + 6f'(f''')^2 + 12(f'')^2 f''' + (f')^2 f^{(5)} = 9\sin 3z. \quad (15)$$

From (13) and (15), it follows that

$$2(f')^3 + 6ff'f'' + f^2 f''' + 8f'f''f^{(4)} + 6f'(f''')^2 + 12(f'')^2 f''' + (f')^2 f^{(5)} = -9f^2 f' - 18f'(f'')^2 - 9(f')^2 f'''.$$

Rewrite the above equation as

$$f'(9f^2+2(f')^2+6ff''+18(f'')^2+8f''f^{(4)}+f'f^{(5)} + 6(f''')^2+9f'f''') = -f'''(f^2+12(f'')^2). \quad (16)$$

Next, we will prove that f'''/f' is an entire function. Suppose that z_0 is a zero of f', i.e., $f'(z_0)=0$. From (16), we have either $f'''(z_0)=0$ or $f^2(z_0)+12(f''(z_0))^2=0$.

If $f'''(z_0)\neq 0$, then $f^2(z_0)+12(f''(z_0))^2=0$. It follows from (14) that

$$10(f''(z_0))^3 = 3\cos 3z_0.$$

From (4), we have

$$f^3(z_0)=\cos 3z_0.$$

If $\cos 3z_0 \neq 0$, then we have $\left(\frac{f(z_0)}{f''(z_0)}\right)^3 = \frac{10}{3}$, which contradicts with $f^2(z_0)+12(f''(z_0))^2=0$. Thus, we have $f(z_0)=f''(z_0)=\cos 3z_0=0$. Recall that $f'(z_0)=0$. We may observe that $f^3+3(f')^2f''$ has a multiple zero at z_0, which contradicts (4) because $\cos 3z$ has only simple zeros.

Hence, we have $f'''(z_0)=0$. Since $\sin 3z$ has only simple zeros, $f'(z_0)=0$ and $f'''(z_0)=0$, it follows from (13) and (14) that $f''(z_0)\neq 0$. Therefore, any zero of f' must be both a simple zero of f' and a zero of f''', which implies that f'''/f' must be an entire function.

By a similar method to that in the proof of Theorem 1, we have $\rho(f)=1$.

Set $\kappa = f'''/f'$ is an entire solution. Clearly, as $r\to\infty$,

$$T(r,\kappa)=m(r,\kappa)=m\left(r,\frac{f'''}{f'}\right)=S(r,f)=O(\log r).$$

Thus, we have that κ is a polynomial.

Rewrite $\kappa = f'''/f'$ in the form

$$f'''-\kappa f' = 0. \quad (17)$$

Using Lemma 4 into (17), we have $1=\rho(f)=\rho(f')=\frac{n+2}{2}$, where n denotes the degree of κ. Thus, we have $n=0$, which implies that κ is a constant.

Combining (16) with (17), we have

$$(9+\kappa)f^2+6ff''+(18+12\kappa)(f'')^2+8f''f^{(4)} + f'f^{(5)}+(6\kappa^2+9\kappa+2)(f')^2=0.$$

From (17), we have $f^{(4)}-\kappa f''=0$ and $f^{(5)}-\kappa f'''=0$. By substituting those expressions into the above equation, we have

$$(9+\kappa)f^2+6ff''+(18+20\kappa)(f'')^2 + (7\kappa^2+9\kappa+2)(f')^2=0. \quad (18)$$

Differentiating (18) yields

$$(18+2\kappa)ff'+6f'f''+6ff'''+(36+40\kappa)f''f''' + (14\kappa^2+18\kappa+4)f'f''=0.$$

Combining the above equation with (17), we have

$$(9+4\kappa)f+(27\kappa^2+27\kappa+5)f''=0. \quad (19)$$

It follows from (17) and the differentiation of (19) that

$$(9+4\kappa)f'+(27\kappa^3+27\kappa^2+5\kappa)f'=0.$$

Since f is a transcendental function, we get that

$$27\kappa^3+27\kappa^2+5\kappa+9+4\kappa = 9(3\kappa^2+1)(\kappa+1)=0.$$

Hence, we have $\kappa=-1$ or $\kappa=\pm\frac{\sqrt{3}i}{3}$.

If $\kappa=-1$, then (19) gives that $f+f''=0$. Its general solution is $f(z)=c_1\,e^{iz}+c_2\,e^{-iz}$, where c_1,c_2 are constants. Substituting $f(z)=c_1\,e^{iz}+c_2\,e^{-iz}$ into (4), we have

$$(8c_1^3-1)\,e^{3iz}+(8c_2^3-1)\,e^{-3iz}=0.$$

Applying Lemma 3 to the above equation, we have $8c_1^3-1=0$ and $8c_2^3-1=0$. Hence, we obtain $c_1^3=\frac{1}{8}$, $c_2^3=\frac{1}{8}$.

If $\kappa=\frac{\sqrt{3}i}{3}$, then (19) gives that $f+\sqrt{3}if''=0$. Its general solution is $f(z)=c_3\,e^{\lambda z}+c_4\,e^{-\lambda z}$, where λ is a nonzero constant satisfying $\lambda^2=\frac{\sqrt{3}i}{3}$, and c_3, c_4 are constants. Substituting $f(z)=c_3\,e^{\lambda z}+c_4\,e^{-\lambda z}$ into (4), we have

$$\tfrac{4}{3}c_3^3\,e^{3\lambda z}+\tfrac{4}{3}c_4^3\,e^{-3\lambda z}+8c_3^2c_4\,e^{\lambda z}+8c_3c_4^2\,e^{-\lambda z}=e^{3iz}+e^{-3iz}.$$

Since $\lambda^2=\frac{\sqrt{3}i}{3}$, we have $\lambda\neq\pm 3i$. By Lemma 3, we get a contradiction.

If $\kappa=-\frac{\sqrt{3}i}{3}$, by the same arguments, we also get a contradiction.

The proof of Theorem 2 is complete.

PROOF OF Theorem 3

Suppose that f is an entire solution of (5). Rewrite (5) as

$$pf^3+qf(f')^2=\cos\alpha \quad (20)$$

for simplicity. Differentiating (20) yields

$$3pf^2f'+p'f^3+q(f')^3+2qff'f''+q'f(f')^2 = -a\sin\alpha. \quad (21)$$

Differentiation of (21) gives

$$(6p+q'')f(f')^2+3pf^2f''+6p'f^2f'+p''f^3 + 5q(f')^2f''+2q'(f')^3+2qf(f'')^2+2qff'f''' + 4q'ff'f''=-a^2\cos\alpha. \quad (22)$$

From (20) and (22), it follows that

$$(6p+q''+a^2q)f(f')^2+3pf^2f''+6p'f^2f' \\ +(p''+a^2p)f^3+5q(f')^2f''+2q'(f')^3 \\ +2qf(f'')^2+2qff'f'''+4q'ff'f''=0.$$

Then, we have

$$f\{(6p+q''+a^2q)(f')^2+3pff''+6p'ff' \\ +(p''+a^2p)f^2+2q(f'')^2+2qf'f'''+4q'f'f''\} \\ =(f')^2(-5qf''-2q'f'). \quad (23)$$

Denote

$$\beta := \frac{-5qf''-2q'f'}{f}. \quad (24)$$

If z_0 is a zero of f, i.e., $f(z_0)=0$, then from (23) we have either z_0 is a zero of f' or a zero of $-5qf''-2q'f'$. From (20), we may observe that z_0 cannot be zero of f' because $\cos\alpha$ has only simple zeros. Therefore, any zero of f must be a simple zero of f and a zero of $-5qf''-2q'f'$. It follows from (24) that β is an entire function.

By the lemma on logarithmic derivative, we get that

$$T(r,\beta)=m(r,\beta)=m\left(r,\frac{-5qf''-2q'f'}{f}\right)=S(r,f).$$

Rewrite (24) in the form

$$5qf''+2q'f'+\beta f=0. \quad (25)$$

Differentiating (25) yields

$$5qf'''+7q'f''+(2q''+\beta)f'+\beta'f=0. \quad (26)$$

Elimination of f''' from (26) and (23), we get that

$$f\left\{\left(6p+q''+a^2q-\frac{2(2q''+\beta)}{5}\right)(f')^2+3pff'' \\ +\left(6p'-\frac{2\beta'}{5}\right)ff'+(p''+a^2p)f^2+2q(f'')^2 \\ +\left(4q'-\frac{14q'}{5}\right)f'f''\right\}=(f')^2[-5qf''-2q'f']. \quad (27)$$

Similarly, elimination of f'' from (27) and (25), we have

$$f\left\{\left(6p+q''+a^2q-\frac{2(2q''+\beta)}{5}-\frac{4(q')^2}{25q}\right)(f')^2 \\ +\left(6p'-\frac{2\beta'}{5}-\frac{6pq'}{5q}+\frac{2q'\beta}{25q}\right)ff' \\ +\left(p''+a^2p-\frac{3p\beta}{5q}+\frac{2\beta^2}{25q}\right)f^2\right\}=\beta f(f')^2.$$

Then, we obtain

$$\mu(f')^2+\nu ff'+\iota f^2=0, \quad (28)$$

where $\mu=6p+a^2q+\frac{q''}{5}-\frac{4(q')^2}{25q}-\frac{7\beta}{5}$, $\nu=6p'-\frac{2\beta'}{5}-\frac{6pq'}{5q}+\frac{2q'\beta}{25q}$ and $\iota=p''+a^2p-\frac{3p\beta}{5q}+\frac{2\beta^2}{25q}$ are small functions of f.

Now, we show that $\mu\equiv0$, $\nu\equiv0$ and $\iota\equiv0$. To this end, suppose that $\mu\not\equiv0$. If f has finitely many zeros, then we have $f=\widetilde{p}\,e^{\widetilde{q}}$, where $\widetilde{p}$ is a nonzero polynomial and $\widetilde{q}$ is a non-constant polynomial. Substituting $f=\widetilde{p}\,e^{\widetilde{q}}$ into (20) yields

$$[p(\widetilde{p})^3+q(\widetilde{p})^3(\widetilde{q}')^2]\,e^{3\widetilde{q}}=\cos\alpha.$$

The left-hand side of the above equation has at most finitely many zeros, but the right-hand side of the above equation has infinitely many zeros. This is impossible. Hence, f has infinitely many zeros. Recall that any zero of f must be a simple zero of f. Therefore, f has infinitely many simple zeros. We can choose a simple zero z_0 of f, but not the zero and the pole of the coefficients of (28). From (28), it follows that $f'(z_0)=0$, which contradicts with z_0 being a simple zero of f. Hence, $\mu\equiv0$. Moreover, (28) reduces to

$$\nu f'+\iota f=0. \quad (29)$$

By the same method as above, we can obtain $\nu\equiv0$, and then $\iota\equiv0$. Therefore,

$$6p+a^2q+\frac{q''}{5}-\frac{4(q')^2}{25q}-\frac{7\beta}{5}\equiv0, \\ 6p'-\frac{2\beta'}{5}-\frac{6pq'}{5q}+\frac{2q'\beta}{25q}\equiv0, \\ p''+a^2p-\frac{3p\beta}{5q}+\frac{2\beta^2}{25q}\equiv0.$$

Rewrite the above three equations as

$$150pq+25a^2q^2+5qq''-4(q')^2-35q\beta\equiv0, \quad (30)$$

$$150p'q-10q\beta'-30pq'+2q'\beta\equiv0, \quad (31)$$

$$25p''q+25a^2pq-15p\beta+2\beta^2\equiv0. \quad (32)$$

Eliminating β from (30) and (31) yields

$$150p'q^2-30pqq'+\frac{12q'A}{35}-\frac{2qA'}{7}\equiv0, \quad (33)$$

where $A=150pq+25a^2q^2+5qq''-4(q')^2$.

Similarly, from (30) and (32), it follows that

$$25p''q^3+25a^2pq^3-\frac{3pqA}{7}+\frac{2A^2}{35^2}\equiv0. \quad (34)$$

Let $\deg p=m$ and $\deg q=n$, where m,n are non-negative integers. Set

$$p(z):=\mu_m z^m+\mu_{m-1}z^{m-1}+\cdots+\mu_0 \quad (35)$$

and

$$q(z) := \nu_n z^n + \nu_{n-1} z^{n-1} + \cdots + \nu_0, \qquad (36)$$

where $\mu_m, \nu_n \in \mathbb{C}\backslash\{0\}$ and $\mu_i\,(i = 0, 1, \ldots, m-1)$, $\nu_j\,(j = 0, 1, \ldots, n-1) \in \mathbb{C}$.

Note that $\deg A = \max\{m+n, 2n\}$, $\deg(a^2pq^3) = m+3n$, $\deg(pqA) = m+n+\deg A$ and $\deg(A^2) = 2\deg A$.

Case 1. If $m < n$, then $\deg A = \max\{m+n, 2n\} = 2n$. Thus, we have

$$\deg(p''q^3) < \deg(a^2pq^3) = \deg(pqA) = m+3n < \deg(A^2) = 4n.$$

From (34), we observe that the coefficient of A^2 is a nonzero constant. This gives a contradiction.

Case 2. If $m > n$, then by (35) and (36), we have

$$A = 150pq + 25(a)^2q^2 + 5qq'' - 4(q')^2 = 150\mu_m\nu_n z^{m+n} + t_{m+n-1}z^{m+n-1} + \cdots + t_0, \qquad (37)$$

where $t_0, \ldots, t_{m+n-1} \in \mathbb{C}$.

Substituting (35), (36) and (37) into (33), we get that

$$150p'q^2 - 30pqq' + \frac{12q'A}{35} - \frac{2qA'}{7} = \left(\frac{750m\mu_m\nu_n^2}{7} - \frac{150n\mu_m\nu_n^2}{7}\right)z^{m+2n-1} + k_{m+2n-2}z^{m+2n-2} + \cdots + k_0 \equiv 0,$$

where $k_{m+2n-2}, \ldots, k_0 \in \mathbb{C}$. Thus, we have

$$\frac{750m\mu_m\nu_n^2}{7} - \frac{150n\mu_m\nu_n^2}{7} = 0.$$

Then, we have $5m = n$. Since $m > n$ and m, n are non-negative integers, this is impossible. We get a contradiction.

Case 3. If $m = n$, then $\deg A = \max\{m+n, 2n\} = 2m$.

Substituting (35) and (36) into $A = 150pq + 25a^2q^2 + 5qq'' - 4(q')^2$, we get that

$$A = (150\mu_m\nu_n + 25a^2\nu_n^2)z^{2m} + l_{2m-1}z^{2m-1} + \cdots + l_0,$$

where $l_{2m-1}, \ldots, l_0 \in \mathbb{C}$. Substituting this expression into (33) and (34), we get that

$$150p'q^2 - 30pqq' + \frac{12q'A}{35} - \frac{2qA'}{7} = \left(\frac{450m\mu_m\nu_n^2}{7} + \frac{150n\mu_m\nu_n^2}{7} + \frac{60na^2\nu_n^3}{7} - \frac{100ma^2\nu_n^3}{7}\right)z^{3m-1} + \cdots \equiv 0$$

and

$$25p''q^3 + 25a^2pq^3 - \frac{3pqA}{7} + \frac{2A^2}{35^2} = \left(25a^2\mu_m\nu_n^3 - \frac{3}{7}\mu_m\nu_n(150\mu_m\nu_n + 25a^2\nu_n^2) + \frac{2}{35^2}(150\mu_m\nu_n + 25a^2\nu_n^2)^2\right)z^{4m} + \cdots \equiv 0,$$

respectively. Then, we have

$$\frac{450m\mu_m\nu_n^2}{7} + \frac{150n\mu_m\nu_n^2}{7} + \frac{60na^2\nu_n^3}{7} - \frac{100ma^2\nu_n^3}{7} = 0,$$

$$25a^2\mu_m\nu_n^3 - \frac{3}{7}\mu_m\nu_n(150\mu_m\nu_n + 25a^2\nu_n^2) + \frac{2}{35^2}(150\mu_m\nu_n + 25a^2\nu_n^2)^2 = 0$$

Thus,

$$(45m + 15n)\mu_m + (6n - 10m)a^2\nu_n = 0,$$
$$(a^2\nu_n + 27\mu_m)(a^2\nu_n - \mu_m) = 0.$$

From the above equation, we have $15m - 7n = 0$ or $5m + 3n = 0$. Since m, n are non-negative integers and $m = n$, then we have $m = n = 0$. Hence, $p(z) \equiv p$, $q(z) \equiv q$ are nonzero constants.

Then, (34) becomes

$$25a^2pq^3 - \frac{3pqA}{7} + \frac{2A^2}{35^2} \equiv 0, \qquad (38)$$

and $A = 150pq + 25a^2q^2 + 5qq'' - 4(q')^2 = 150pq + 25a^2q^2$ is a constant. Substituting this expression into (38), we have

$$a^4q^2 + 26a^2pq - 27p^2 = [a^2q + 27p][a^2q - p] \equiv 0,$$

which means that $a^2q + 27p = 0$ or $a^2q - p = 0$. Therefore, we have $a = \pm 3\sqrt{\frac{-3p}{q}}$ or $a = \pm\sqrt{\frac{p}{q}}$. Thus, $a \in \left\{3\sqrt{\frac{-3p}{q}}, -3\sqrt{\frac{-3p}{q}}, \sqrt{\frac{p}{q}}, -\sqrt{\frac{p}{q}}\right\}$.

Next, we will show the precise expression of f. Recall that p, q are nonzero constants. (30), (31) and (32) reduces to

$$150pq + 25a^2q^2 - 35q\beta \equiv 0, \qquad (39)$$
$$-10q\beta' \equiv 0, \qquad (40)$$
$$25a^2pq - 15p\beta + 2\beta^2 \equiv 0. \qquad (41)$$

It follows from (39) and (41) that

$$20p\beta - 150p^2 + 2\beta^2 = (2\beta - 10p)(\beta + 15p) \equiv 0.$$

This gives that $\beta = 5p$ or $\beta = -15p$. Since q is a nonzero constant, then (24) reduces to

$$5qf'' + \beta f = 0. \qquad (42)$$

Case 3.1. If $\beta = 5p$, by substituting $\beta = 5p$ into (42), we have

$$qf'' + pf = 0.$$

Solving the above equation, we obtain

$$f(z) = C_1 e^{\lambda_1 z} + C_2 e^{\lambda_2 z}, \qquad (43)$$

where C_1, C_2, $\lambda_1 = i\sqrt{\frac{p}{q}} \neq 0$, $\lambda_2 = -i\sqrt{\frac{p}{q}} \neq 0$ are constants. Clearly, $\lambda_1 + \lambda_2 = 0$.

Substituting $\alpha(z) = az+b$ and (43) into (20) yields

$$\begin{aligned}(pC_1^3 + qC_1^3\lambda_1^2)\,\mathrm{e}^{3\lambda_1 z} + (pC_2^3 + qC_2^3\lambda_2^2)\,\mathrm{e}^{3\lambda_2 z}\\ + (3pC_1^2C_2 + 2qC_1^2C_2\lambda_1\lambda_2 + qC_1^2C_2\lambda_1^2)\,\mathrm{e}^{\lambda_1 z}\\ + (3pC_1C_2^2 + 2qC_1C_2^2\lambda_1\lambda_2 + qC_1C_2^2\lambda_2^2)\,\mathrm{e}^{\lambda_2 z}\\ = \frac{\mathrm{e}^{iaz}\,\mathrm{e}^{ib}}{2} + \frac{\mathrm{e}^{-iaz}\,\mathrm{e}^{-ib}}{2}. \quad (44)\end{aligned}$$

Note that $a = \pm 3\sqrt{\frac{-3p}{q}}$ or $a = \pm\sqrt{\frac{p}{q}}$.

If $a = \pm 3\sqrt{\frac{-3p}{q}}$, then $a = 3\sqrt{3}\lambda_1$ or $a = 3\sqrt{3}\lambda_2$. Substituting $a = 3\sqrt{3}\lambda_1$ into (44), we have

$$\begin{aligned}(pC_1^3 + qC_1^3\lambda_1^2)\,\mathrm{e}^{3\lambda_1 z} + (pC_2^3 + qC_2^3\lambda_2^2)\,\mathrm{e}^{3\lambda_2 z}\\ + (3pC_1^2C_2 + 2qC_1^2C_2\lambda_1\lambda_2 + qC_1^2C_2\lambda_1^2)\,\mathrm{e}^{\lambda_1 z}\\ + (3pC_1C_2^2 + 2qC_1C_2^2\lambda_1\lambda_2 + qC_1C_2^2\lambda_2^2)\,\mathrm{e}^{\lambda_2 z}\\ = \frac{\mathrm{e}^{3\sqrt{3}i\lambda_1 z}\,\mathrm{e}^{ib}}{2} + \frac{\mathrm{e}^{-3\sqrt{3}i\lambda_1 z}\,\mathrm{e}^{-ib}}{2}.\end{aligned}$$

We can observe that $3\lambda_1$, $3\lambda_2$, λ_1, λ_2, $3\sqrt{3}i\lambda_1$, $-3\sqrt{3}i\lambda_1$ are distinct from each other. By Lemma 3, we have $\mathrm{e}^{ib}/2 \equiv 0$. This gives a contradiction. Similarly, by substituting $a = 3\sqrt{3}i\lambda_2$ into (44), we also get a contradiction.

If $a = \pm\sqrt{\frac{p}{q}}$, then $a = -i\lambda_1$ or $a = -i\lambda_2$. Substituting $a = -i\lambda_1$ into (44), we have

$$\begin{aligned}(pC_1^3 + qC_1^3\lambda_1^2)\,\mathrm{e}^{3\lambda_1 z} + (pC_2^3 + qC_2^3\lambda_2^2)\,\mathrm{e}^{3\lambda_2 z}\\ + \left(3pC_1^2C_2 + 2qC_1^2C_2\lambda_1\lambda_2 + qC_1^2C_2\lambda_1^2 - \tfrac{\mathrm{e}^{ib}}{2}\right)\mathrm{e}^{\lambda_1 z}\\ + \left(3pC_1C_2^2 + 2qC_1C_2^2\lambda_1\lambda_2 + qC_1C_2^2\lambda_2^2 - \tfrac{\mathrm{e}^{-ib}}{2}\right)\mathrm{e}^{\lambda_2 z} = 0.\end{aligned}$$

Applying Lemma 3 to the above equation, we have

$$\begin{aligned}pC_1^3 + qC_1^3\lambda_1^2 &= 0,\\ pC_2^3 + qC_2^3\lambda_2^2 &= 0,\\ 3pC_1^2C_2 + 2qC_1^2C_2\lambda_1\lambda_2 + qC_1^2C_2\lambda_1^2 - \tfrac{\mathrm{e}^{ib}}{2} &= 0,\\ 3pC_1C_2^2 + 2qC_1C_2^2\lambda_1\lambda_2 + qC_1C_2^2\lambda_2^2 - \tfrac{\mathrm{e}^{-ib}}{2} &= 0.\end{aligned}$$

Solving the above system, we have $8pC_1^2C_2 = \mathrm{e}^{ib}$ and $8pC_1C_2^2 = \mathrm{e}^{-ib}$. Then, we obtain $C_1C_2 = 1/4p^{\frac{2}{3}}$.

Substituting $a = -i\lambda_2$ into (44), we get that

$$\begin{aligned}(pC_1^3 + qC_1^3\lambda_1^2)\,\mathrm{e}^{3\lambda_1 z} + (pC_2^3 + qC_2^3\lambda_2^2)\,\mathrm{e}^{3\lambda_2 z}\\ + (3pC_1^2C_2 + 2qC_1^2C_2\lambda_1\lambda_2 + qC_1^2C_2\lambda_1^2 - \tfrac{\mathrm{e}^{-ib}}{2})\,\mathrm{e}^{\lambda_1 z}\\ + (3pC_1C_2^2 + 2qC_1C_2^2\lambda_1\lambda_2 + qC_1C_2^2\lambda_2^2 - \tfrac{\mathrm{e}^{ib}}{2})\,\mathrm{e}^{\lambda_2 z} = 0.\end{aligned}$$

By the same argument as above, we also obtain $8pC_1^2C_2 = \mathrm{e}^{-ib}$ and $8pC_1C_2^2 = \mathrm{e}^{ib}$. Moreover, $C_1C_2 = 1/4p^{\frac{2}{3}}$, which implies that C_1, C_2 are nonzero constants.

Case 3.2. If $\beta = -15p$, by substituting $\beta = -15p$ into (42), we have

$$qf'' - 3pf = 0.$$

Solving the above equation, we obtain

$$f(z) = C_3\,\mathrm{e}^{\lambda_3 z} + C_4\,\mathrm{e}^{\lambda_4 z}, \quad (45)$$

where C_3, C_4, $\lambda_3 = \sqrt{\frac{3p}{q}} \neq 0$, $\lambda_4 = -\sqrt{\frac{3p}{q}} \neq 0$ are constants. Clearly, $\lambda_3 + \lambda_4 = 0$.

Substituting $\alpha(z) = az+b$ and (45) into (20) yields

$$\begin{aligned}(3pC_3^2C_4 + qC_3^2C_4\lambda_3^2 + 2qC_3^2C_4\lambda_3\lambda_4)\,\mathrm{e}^{\lambda_3 z}\\ + (3pC_3C_4^2 + qC_3C_4^2\lambda_4^2 + 2qC_3C_4^2\lambda_3\lambda_4)\,\mathrm{e}^{\lambda_4 z}\\ + (pC_3^3 + qC_3^3\lambda_3^2)\,\mathrm{e}^{3\lambda_3 z} + (pC_4^3 + qC_4^3\lambda_4^2)\,\mathrm{e}^{3\lambda_4 z}\\ = \frac{\mathrm{e}^{iaz}\,\mathrm{e}^{ib}}{2} + \frac{\mathrm{e}^{-iaz}\,\mathrm{e}^{-ib}}{2}. \quad (46)\end{aligned}$$

If $a = \pm\sqrt{\frac{p}{q}}$, then $a = \frac{\sqrt{3}}{3}\lambda_3$ or $a = \frac{\sqrt{3}}{3}\lambda_4$. Substituting $a = \frac{\sqrt{3}}{3}\lambda_3$ into (46), we have

$$\begin{aligned}(3pC_3^2C_4 + qC_3^2C_4\lambda_3^2 + 2qC_3^2C_4\lambda_3\lambda_4)\,\mathrm{e}^{\lambda_3 z}\\ + (3pC_3C_4^2 + qC_3C_4^2\lambda_4^2 + 2qC_3C_4^2\lambda_3\lambda_4)\,\mathrm{e}^{\lambda_4 z}\\ + (pC_3^3 + qC_3^3\lambda_3^2)\,\mathrm{e}^{3\lambda_3 z} + (pC_4^3 + qC_4^3\lambda_4^2)\,\mathrm{e}^{3\lambda_4 z}\\ = \frac{\mathrm{e}^{ib}}{2}\,\mathrm{e}^{i\frac{\sqrt{3}}{3}\lambda_3 z} + \frac{\mathrm{e}^{-ib}}{2}\,\mathrm{e}^{i\frac{\sqrt{3}}{3}\lambda_4 z}.\end{aligned}$$

We may observe that λ_3, λ_4, $3\lambda_3$, $3\lambda_4$, $i\frac{\sqrt{3}}{3}\lambda_3$, $i\frac{\sqrt{3}}{3}\lambda_4$ are distinct from each other. By Lemma 3, we get $\mathrm{e}^{ib}/2 = 0$. This gives a contradiction. Similarly, if $a = \frac{\sqrt{3}}{3}\lambda_4$, by the same method as above, we get a contradiction.

If $a = \pm 3\sqrt{\frac{-3p}{q}}$, then $a = 3i\lambda_3$ or $a = 3i\lambda_4$. Substituting $a = 3i\lambda_3$ into (46), we have

$$\begin{aligned}(3pC_3^2C_4 + qC_3^2C_4\lambda_3^2 + 2qC_3^2C_4\lambda_3\lambda_4)\,\mathrm{e}^{\lambda_3 z}\\ + (3pC_3C_4^2 + qC_3C_4^2\lambda_4^2 + 2qC_3C_4^2\lambda_3\lambda_4)\,\mathrm{e}^{\lambda_4 z}\\ + (pC_3^3 + qC_3^3\lambda_3^2 - \tfrac{\mathrm{e}^{-ib}}{2})\,\mathrm{e}^{3\lambda_3 z} + (pC_4^3 + qC_4^3\lambda_4^2 - \tfrac{\mathrm{e}^{ib}}{2})\,\mathrm{e}^{3\lambda_4 z} = 0.\end{aligned}$$

By Lemma 3, we have

$$\begin{aligned}3pC_3^2C_4 + qC_3^2C_4\lambda_3^2 + 2qC_3^2C_4\lambda_3\lambda_4 &= 0,\\ 3pC_3C_4^2 + qC_3C_4^2\lambda_4^2 + 2qC_3C_4^2\lambda_3\lambda_4 &= 0,\\ pC_3^3 + qC_3^3\lambda_3^2 - \tfrac{\mathrm{e}^{-ib}}{2} &= 0,\\ pC_4^3 + qC_4^3\lambda_4^2 - \tfrac{\mathrm{e}^{ib}}{2} &= 0.\end{aligned}$$

Solving the above system, we have $8pC_3^3 = \mathrm{e}^{-ib}$ and $8pC_4^3 = \mathrm{e}^{ib}$. Hence, we obtain $C_3C_4 = 1/4p^{\frac{2}{3}}$.

Similarly, if $a = 3i\lambda_4$, by the same arguments, we have $8pC_3^3 = \mathrm{e}^{ib}$ and $8pC_4^3 = \mathrm{e}^{-ib}$. Then, we also obtain $C_3C_4 = 1/4p^{\frac{2}{3}}$.

The proof of Theorem 3 is complete.

PROOF OF Theorem 4

Let f be an entire solution with finite order of differential-difference equation (7). By Nevanlinna's fundamental estimate, it is well know that $T(r,f^{(k)}) \leqslant T(r,f)+S(r,f)$.

From (7), it follows that

$$\begin{aligned} T(r,H) &= T(r,f^{n_0}(f_c')^{n_1}(f_c'')^{n_2}\cdots(f_c^{(k)})^{n_k}) \\ &\leqslant n_0T(r,f)+(n_1+\cdots+n_k)T(r,f_c)+S(r,f_c). \end{aligned}$$

Using Lemma 2 to the above equation, we have

$$T(r,H) \leqslant qT(r,f)+S(r,f), \tag{47}$$

where $q=n_0+n_1+\cdots+n_k$.

Next, we prove an estimate in the other direction. Rewrite (7) as

$$f^q = \frac{f^{q-n_0}}{(f_c')^{n_1}(f_c'')^{n_2}\cdots(f_c^{(k)})^{n_k}}H(z). \tag{48}$$

Taking the Nevanlinna characteristic function of both sides of (48), by using Nevanlinna's first fundamental theorem and Lemma 1, we have

$$\begin{aligned} qT(r,f) &= qm(r,f)=m(r,f^q) \\ &\leqslant m\left(r,\frac{f^{q-n_0}}{(f_c')^{n_1}(f_c'')^{n_2}\cdots(f_c^{(k)})^{n_k}}\right)+m(r,H) \\ &\leqslant m\left(r,\frac{f_c^{q-n_0}}{(f_c')^{n_1}(f_c'')^{n_2}\cdots(f_c^{(k)})^{n_k}}\right) \\ &\quad +(q-n_0)m\left(r,\frac{f}{f_c}\right)+m(r,H) \\ &\leqslant T\left(r,\frac{(f_c')^{n_1}(f_c'')^{n_2}\cdots(f_c^{(k)})^{n_k}}{f_c^{q-n_0}}\right) \\ &\quad -N\left(r,\frac{f_c^{q-n_0}}{(f_c')^{n_1}(f_c'')^{n_2}\cdots(f_c^{(k)})^{n_k}}\right)+T(r,H)+S(r,f) \\ &\leqslant N\left(r,\frac{(f_c')^{n_1}(f_c'')^{n_2}\cdots(f_c^{(k)})^{n_k}}{f_c^{q-n_0}}\right) \\ &\quad -N\left(r,\frac{f_c^{q-n_0}}{(f_c')^{n_1}(f_c'')^{n_2}\cdots(f_c^{(k)})^{n_k}}\right) \\ &\quad +T(r,H)+S(r,f_c)+S(r,f) \\ &\leqslant N\left(r,\frac{1}{f^{q-n_0}}\right)-N\left(r,\frac{1}{(f_c')^{n_1}(f_c'')^{n_2}\cdots(f_c^{(k)})^{n_k}}\right) \\ &\quad +T(r,H)+S(r,f) \\ &\leqslant (q-n_0)N\left(r,\tfrac{1}{f}\right)+T(r,H)+S(r,f) \\ &\leqslant (q-n_0)T(r,f)+T(r,H)+S(r,f). \end{aligned}$$

Hence, we get that

$$n_0T(r,f) \leqslant T(r,H)+S(r,f). \tag{49}$$

It follows from (47) and (49) that $\rho(f)=\rho(H)$.

The proof of Theorem 4 is complete.

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