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Formal deformations of associative and Lie H-pseudoalgebras

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ABSTRACT: The aim of this paper is to study the 1-parameter formal deformations of associative (resp. Lie) H-pseudoalgebras and describe them with the cohomology groups. As an application, we derive Poisson H-pseudoalgebra from the first-order term of a deformation of the commutative associative H-pseudoalgebra.

KEYWORDS: associative H-pseudoalgebra, Lie H-pseudoalgebra, deformation

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INTRODUCTION

The theory of pseudotensor category was introduced by Beilinson and Drinfeld [1] as a way of expressing notions such as Lie algebras and representations in purely categorical terms. A pseudotensor category is a category equipped with an extra pseudotensor structure formed by a collection of functors $P_n\{L_i\}_{1\leqslant i\leqslant n} \to M$ and a way to compose them. Let H be a cocommutative Hopf algebra, then $\mathscr{M}^*(H)$ is a pseudotensor category with the same objects as ${}_H\mathscr{M}$ (the category of left H-modules), and the pseudotensor structure is defined by

$$\mathrm{Lin}(\{L_i\}_{i\in I}, M) = \mathrm{Hom}_{H^{\otimes I}}(\boxtimes_{i\in I} L_i, H^{\otimes I} \otimes_H M),$$

where I is a finite non-empty set and $\boxtimes_{i\in I}$ is the tensor product functor ${}_H\mathscr{M}^I \to_{H^{\otimes I}} \mathscr{M}$. An algebra in this pseudotensor category is called an H-pseudoalgebra or simply a pseudoalgebra. When H is the polynomial algebra $\mathbb{C}[\partial]$, it is actually the conformal algebra [2–4], which is closely related to the vertex algebra. Moreover, conformal algebra provide an axiomatic description of the operator product expansion (OPE) of chiral fields in conformal field theory.

In [5], Bakalov, D'Andrea and Kac studied Lie (resp. associative) algebras in pseudotensor category $\mathscr{M}^*(H)$, called Lie (resp. associative) H-pseudoalgebras, which are closely related to the differential Lie algebras of Ritt and Hamiltonian formalism in the theory of nonliner evolution equation [6–8]. They established representation theory, cohomology theory, and simultaneously solved classification problems in a series of papers [5, 9–11]. Other algebraic structures in pseudotensor category $\mathscr{M}^*(H)$ have been introduced afterwards, such as Jordan pseudoalgebra [12], Leibniz H-pseudoalgebra [13], left symmetric pseudoalgebra [14] and so on. Moreover, Boyallian and Liberati [15] introduced the notions of Lie H-coalgebra and Lie H-pseudobialgebra from the point of pseudo-dual. Later, Liu and Wang et al. studied the infinitesimal H-pseudobialgebra and associative pseudo-Yang-Baxter equation [16]. Further research on pseudoalgebras can be found in [17–20].

The deformation is a tool to study a mathematical object by deforming it into a family of the same kind of objects depending on a certain parameter. The deformation theory of algebras was introduced by Gerstenhaber for rings and algebras in [21], and by Nijenhuis and Richardson for Lie algebras in [22]. They studied 1-parameter formal deformations and connected them with Hochschild cohomology and Chevally-Eilenberg cohomology, respectively. The deformation problem of other algebraic structures, such as Lie triple systems, Yamaguti algebras, Novikov algebras, Hom-type algebras, etc., has been intensively studied in the literature recently, see [23–26].

Inspired by the above results, we consider the deformation theory of associative H-pseudoalgebras and Lie H-pseudoalgebras.

Throughout this paper, we denote by K a fixed algebraically closed field of characteristic zero, and H is a cocommutative Hopf algebra over K. For a comultiplication $\Delta : C \to C\otimes C$ on a linear space C, we use Sweedler's notation [27] $\Delta(c) = c_1\otimes c_2$, $\forall c \in C$. For the composition of two maps f and g, we write either $g\circ f$ or simply gf. For any vector space V, we will define $\sigma(f\otimes g) = g\otimes f$, $(123)(f\otimes g\otimes h) = h\otimes f\otimes g$ and $(23)(f\otimes g\otimes h) = f\otimes h\otimes g$, for all $f, g, h \in V$. Similarly, we have symbols $(132), (12), (13)$ and so on.

PRELIMINARIES

In this section, we recall some basic definitions about associative and Lie H-pseudoalgebras, see, for example [5, 16, 20].

Definition 1 An H-pseudoalgebra $(A, \mu = *)$ is a left H-module A together with an operation

$$\mu : A\otimes A \to (H\otimes H)\otimes_H A, \quad a\otimes b \mapsto \mu(a\otimes b) = a * b,$$

called the pseudoproduct, and satisfying H-bilinearity: for all $a, b \in A$ and $f, g \in H$,

$$fa * gb = (f \otimes g \otimes_H 1)(a * b).$$

If $a*b = \sum_i f_i \otimes g_i \otimes_H e_i$, then $fa*gb = \sum_i ff_i \otimes gg_i \otimes_H e_i$.

In order to define the associativity of an H-pseudoalgebra, we extend the pseudoproduct from $A \otimes A \to H^{\otimes 2} \otimes_H A$ to maps $(H^{\otimes 2} \otimes_H A) \otimes A \to H^{\otimes 3} \otimes_H A$ and $A \otimes (H^{\otimes 2} \otimes_H A) \to H^{\otimes 3} \otimes_H A$ by letting

$$(f \otimes_H a) * b = \sum_i (f \otimes 1)(\Delta \otimes \mathrm{id})(f_i \otimes g_i) \otimes_H e_i,$$

and

$$a * (f \otimes_H b) = \sum_i (1 \otimes f)(\mathrm{id} \otimes \Delta)(f_i \otimes g_i) \otimes_H e_i,$$

where $f \in H^{\otimes 2}$ and $a * b = \sum_i f_i \otimes g_i \otimes_H e_i$.

The H-pseudoalgebra $(A, *)$ is associative if it satisfies $(a*b)*c = a*(b*c)$ for all $a, b, c \in A$, and $(A, *)$ is commutative if $a * b = (\sigma \otimes_H \mathrm{id})(b * a)$ holds.

Definition 2 Let $(A, *)$ and $(B, *)$ be two associative H-pseudoalgebras.

(1) A left A-module is a pair (M, ρ_M), where M is a left H-module, $\rho_M \in \mathrm{Hom}_{H\otimes H}(A \otimes M, (H \otimes H) \otimes_H M)$ (we denote $\rho_M(a \otimes m) = a * m$), satisfying $(a*b)*m = a*(b*m)$, for all $a, b \in A$ and $m \in M$.

(2) A right A-module is a pair (N, ρ_N), where N is a left H-module, $\rho_N \in \mathrm{Hom}_{H\otimes H}(N \otimes A, (H \otimes H) \otimes_H N)$ (we denote $\rho_N(n \otimes a) = n * a$), satisfying $n*(a*b) = (n*a)*b$, for all $a, b \in A$ and $n \in N$.

(3) (M, ρ_M) is called an A-B-bimodule, if M is a left A-module and right B-module such that $(a*m)*b = a*(m*b)$ for all $a \in A$, $b \in B$ and $m \in M$. An A-A-bimodule is simply called an A-bimodule.

Remark 1 An associative H-pseudoalgebra $(A, *)$ is an A-bimodule with the pseudoproduct.

Definition 3 Let $(A, *)$ be an H-pseudoalgebra. A is called a Lie H-pseudoalgebra (in this case, $a * b$ is denoted by $[a*b]$ and called the pseudobracket) if $[\cdot * \cdot]$ satisfies the following conditions:

(1) Skew-commutativity, $[a*b] = -(\sigma \otimes_H \mathrm{id})[b*a]$.

(2) Jacobi identity, $[[a*b]*c] = [a*[b*c]] - ((12) \otimes_H \mathrm{id})[b*[a*c]]$ in $H^{\otimes 3} \otimes_H A$, for all $a, b, c \in A$.

Definition 4 Let $(L, [\cdot * \cdot])$ be a Lie H-pseudoalgebra. A left L-module is a pair (N, ρ_N), where N is a left H-module, $\rho_N \in \mathrm{Hom}_{H\otimes H}(L \otimes N, (H \otimes H) \otimes_H N)$. For all $a, b \in L$ and $n \in N$, we denote $\rho_N(a \otimes n) = a * n$, satisfying $[a*b]*n = a*(b*n) - ((12) \otimes_H \mathrm{id})(b*(a*n))$.

Remark 2 A Lie H-pseudoalgebra $(L, [\cdot * \cdot])$ is a left L-module with the pseudobracket.

The cohomology of associative H-pseudoalgebras

Let $(A, *)$ be an associative H-pseudoalgebra and M an A-bimodule. The space of n-cochains $C^n(A, M)$ consists of all maps

$$\varphi \in \mathrm{Hom}_{H^{\otimes n}}(A^{\otimes n}, H^{\otimes n} \otimes_H M).$$

Explicitly, for all $h_i \in H$ and $a_i \in A$ $(i = 1, 2, \ldots, n)$, φ has the following defining property: H-polylinearity,

$$\varphi(h_1 a_1 \otimes \cdots \otimes h_n a_n) = ((h_1 \otimes \cdots \otimes h_n) \otimes_H 1)\varphi(a_1 \otimes \cdots \otimes a_n).$$

For $n = 0$, we put $C^0(A, M) = K \otimes_H M \simeq M/H_+ M$, where $H_+ = \{h \in H \mid \varepsilon(h) = 0\}$ is the augmentation ideal of H. The differential $d_0 : C^0(A, M) \to C^1(A, M) = \mathrm{Hom}_H(A, M)$ is given by

$$(d_0(1 \otimes_H m))(a) = \sum_i (\mathrm{id} \otimes \varepsilon)(h_i) m_i - \sum_j (\varepsilon \otimes \mathrm{id})(f_j) n_j,$$

for all $a \in A$, $m \in M$, where $a*m = \sum_i h_i \otimes_H m_i \in H^{\otimes 2} \otimes_H M$ and $m * a = \sum_j f_j \otimes_H n_j \in H^{\otimes 2} \otimes_H M$.

For $n \geqslant 1$, the differential $d_n : C^n(A, M) \to C^{n+1}(A, M)$ is defined by

$$\begin{aligned} &d_n\varphi(a_1 \otimes \cdots \otimes a_{n+1}) \\ &= a_1 * \varphi(a_2 \otimes \cdots \otimes a_{n+1}) + (-1)^{n+1} \varphi(a_1 \otimes \cdots \otimes a_n) * a_{n+1} \\ &+ \sum_{i=1}^{n} (-1)^i \varphi(a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i * a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_{n+1}). \quad (1) \end{aligned}$$

We use the following convention in the above equation. If $a * m = \sum_i f_i \otimes_H m_i \in H^{\otimes 2} \otimes_H M$, $m * a = \sum_j g_j \otimes_H n_j \in H^{\otimes 2} \otimes_H M$ for all $a \in A$ and $m \in M$, then for all $f \in H^{\otimes n}$, we set

$$a * (f \otimes m) = \sum_i (1 \otimes f)(\mathrm{id} \otimes \Delta^{(n-1)})(f_i) \otimes_H m_i \in H^{\otimes(n+1)} \otimes_H M,$$

and

$$(f \otimes m) * a = \sum_j (f \otimes 1)(\Delta^{(n-1)} \otimes \mathrm{id})(g_j) \otimes_H n_j \in H^{\otimes(n+1)} \otimes_H M,$$

where $\Delta^{(n-1)} = (\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \Delta) \cdots (\mathrm{id} \otimes \Delta)\Delta : H \to H^{\otimes n}$ is the iterated comultiplication for $n > 1$, and $\Delta^{(0)} := \mathrm{id}$. Note that (1) holds also for $n = 0$ if we define $\Delta^{(-1)} = \varepsilon$.

For $g \in H^{\otimes 2}$ and $\varphi \in C^n(A, M)$, we set

$$\begin{aligned} &\varphi(a_1 \otimes \cdots \otimes a_{i-1} \otimes (g \otimes_H a_i) \otimes a_{i+1} \otimes \cdots \otimes a_n) \\ &\quad = ((\mathrm{id}^{\otimes(i-1)} \otimes g\Delta \otimes \mathrm{id}^{\otimes(n-i)}) \otimes_H \mathrm{id})\varphi(a_1 \otimes \cdots \otimes a_n) \\ &\quad \in H^{\otimes(n+1)} \otimes_H M. \end{aligned}$$

In particular, by setting $n = 1$ and $n = 2$ in (1), respectively, we have

$$d_1\varphi(a \otimes b) = a * \varphi(b) - \varphi(a * b) + \varphi(a) * b,$$

and

$$d_2\varphi(a\otimes b\otimes c)=a*\varphi(b,c)-\varphi(a*b,c)+\varphi(a,b*c)-\varphi(a,b)*c.$$

It can be shown that $d_{n+1}d_n=0$ by a direct computation. A $\varphi\in C^n(A,M)$ is called an n-cocycle if $d_n\varphi=0$, and φ is called an n-coboundary if there exists an $(n-1)$-cochain ψ such that $d_{n-1}\psi=\varphi$. Denote by $Z^n(A,M)$ and $B^n(A,M)$ the subspaces of n-cocycles and n-coboundaries, respectively. The quotient space $H^n(A,M)=Z^n(A,M)/B^n(A,M)$ is called the n-th cohomology group of A with coefficients in M.

The cohomology of Lie H-pseudoalgebras

Let $(L,[\cdot*\cdot])$ be a Lie H-pseudoalgebra and M a left L-module. For $n\geqslant 1$, $C^n(L,M)$ consists of all $\gamma\in \mathrm{Hom}_{H^{\otimes n}}(L^{\otimes n},H^{\otimes n}\otimes_H M)$. Explicitly, γ has the following defining properties:

(1) H-polylinearity,

$$\gamma(h_1a_1\otimes\cdots\otimes h_na_n)=((h_1\otimes\cdots\otimes h_n)\otimes_H 1)\gamma(a_1\otimes\cdots\otimes a_n).$$

(2) Skew-symmetry,

$$\begin{aligned}&\gamma(a_1\otimes\cdots\otimes a_{i+1}\otimes a_i\otimes\cdots\otimes a_n)\\&\quad=-(\sigma_{i,i+1}\otimes_H \mathrm{id})\gamma(a_1\otimes\cdots\otimes a_i\otimes a_{i+1}\otimes\cdots\otimes a_n),\end{aligned}$$

where $\sigma_{i,i+1}:H^{\otimes n}\to H^{\otimes n}$ is the transposition of the i-th and $(i+1)$-st factors.

For $n\geqslant 1$, the differential $d_n: C^n(L,M)\to C^{n+1}(L,M)$ is defined as follows:

$$\begin{aligned}&(d_n\gamma)(a_1\otimes a_2\otimes\cdots\otimes a_n\otimes a_{n+1})\\&=\sum_{1\leqslant i\leqslant n+1}(-1)^{i+1}(\sigma_{1\to i}\otimes_H\mathrm{id})a_i*\gamma(a_1\otimes\cdots\otimes\hat{a}_i\otimes\cdots\otimes a_{n+1})\\&\quad+\sum_{1\leqslant i<j\leqslant n+1}(-1)^{i+j}(\sigma_{1\to i,2\to j}\otimes_H\mathrm{id})\\&\quad\times\gamma([a_i*a_j]\otimes a_1\otimes\cdots\otimes\hat{a}_i\otimes\cdots\otimes\hat{a}_j\otimes\cdots\otimes a_{n+1}),\end{aligned}$$

where $\sigma_{1\to i}$ is the permutation $h_i\otimes h_1\otimes\cdots\otimes h_{i-1}\otimes h_{i+1}\otimes\cdots\otimes h_{n+1}\mapsto h_1\otimes\cdots\otimes h_{n+1}$, $\sigma_{1\to i,2\to j}$ is the permutation $h_i\otimes h_j\otimes h_1\otimes\cdots\otimes h_{i-1}\otimes h_{i+1}\otimes\cdots\otimes h_{j-1}\otimes h_{j+1}\otimes\cdots\otimes h_{n+1}\mapsto h_1\otimes\cdots\otimes h_{n+1}$, and $\hat{a}_i$ means omitted. In particular, for $n=1$, we have

$$\begin{aligned}&(d_1\gamma)(a_1\otimes a_2)\\&\quad=a_1*\gamma(a_2)-((12)\otimes_H\mathrm{id})a_2*\gamma(a_1)-\gamma([a_1*a_2]),\end{aligned}$$

and for $n=2$, we get

$$\begin{aligned}&(d_2\gamma)(a_1\otimes a_2\otimes a_3)=a_1*\gamma(a_2\otimes a_3)-((12)\otimes_H\mathrm{id})a_2*\gamma(a_1\otimes a_3)\\&+((132)\otimes_H\mathrm{id})a_3*\gamma(a_1\otimes a_2)-\gamma([a_1*a_2]\otimes a_3)\\&+((23)\otimes_H\mathrm{id})\gamma([a_1*a_3]*a_2)-((123)\otimes_H\mathrm{id})\gamma([a_2*a_3]\otimes a_1).\end{aligned}$$

One can routinely verify that $d_{n+1}d_n=0$. A $\gamma\in C^n(L,M)$ is called an n-cocycle if $d_n\gamma=0$, and γ is called an n-coboundary if there exists an $(n-1)$-cochain ψ such that $d_{n-1}\psi=\gamma$. Denote by $Z^n(L,M)$ and $B^n(L,M)$ the subspaces of n-cocycles and n-coboundaries, respectively. The quotient space $H^n(L,M)=Z^n(L,M)/B^n(L,M)$ is called the n-th cohomology group of L with coefficients in M.

DEFORMATIONS OF ASSOCIATIVE AND LIE H-PSEUDOALGEBRAS

In this section, we define the deformations of associative and Lie H-pseudoalgebras and characterize them in terms of cohomology groups.

The deformations of associative H-pseudoalgebras

Definition 5 Let $(A,*)$ be an associative H-pseudoalgebra and

$$\begin{aligned}G_t(x,y)&=\sum_{i\geqslant 0}g_i(x,y)t^i\\&=g_0(x,y)+g_1(x,y)t+g_2(x,y)t^2+\cdots,\end{aligned}$$

where t is a formal variable, each $g_i:A\otimes A\to H^{\otimes 2}\otimes_H A$ is an H-bilinear map and $g_0(x,y)=x*y$. We call G_t the 1-parameter formal deformation (we simply call it deformation) of $(A,*)$ if (A,G_t) is a family of associative H-pseudoalgebras. More precisely, we have

$$G_t(x,G_t(y,z))=G_t(G_t(x,y),z),\quad \forall x,y,z\in A, \tag{2}$$

which is called the deformation equation of associative H-pseudoalgebra.

Example 1 $A=H\{e_1,e_2\}$ is an associative H-pseudoalgebra with the pseudoproduct given by $e_1*e_1=e_1*e_2=0$, $e_2*e_1=1\otimes 1\otimes_H e_1$, $e_2*e_2=1\otimes 1\otimes_H e_2$. Define

$$\begin{aligned}&G_1(e_1,e_1)=G_1(e_1,e_2)=0,\\&G_1(e_2,e_1)=1\otimes 1\otimes_H(1+t)e_1,\\&G_1(e_2,e_2)=1\otimes 1\otimes_H(1+t)e_2,\end{aligned}$$

where $t\in K$. It is easy to check that (2) holds. Hence G_1 is a deformation of A.

Now we focus on the deformation equation of associative H-pseudoalgebra. Note that (2) is equivalent to

$$\sum_{i,j\geqslant 0}(g_i(x,g_j(y,z))-g_i(g_j(x,y),z))t^{i+j}=0.$$

Here we denote $g_i\hat{\circ}g_j(x,y,z)=g_i(x,g_j(y,z))-g_i(g_j(x,y),z)$, then the above equation may be written as

$$\sum_{i,j\geqslant 0}(g_i\hat{\circ}g_j)t^{i+j}=0,$$

which is equivalent to

$$\sum_{k\geqslant 0}t^k\sum_{i=0}^{k}g_i\hat{\circ}g_{k-i}=0\quad\text{or}$$

$$\sum_{i=0}^{k}g_i\hat{\circ}g_{k-i}=0,\qquad k=0,1,2,\ldots.$$

For $k=0$, $g_0\hat{\circ}g_0=0$, this corresponds to the associativity of A.

For $k=1$, $g_0\hat{\circ}g_1+g_1\hat{\circ}g_0=0$, i.e.,

$$a*g_1(b,c)-g_1(a,b)*c+g_1(a,b*c)-g_1(a*b,c)=0, \quad (3)$$

For $k=2$, $g_0\hat{\circ}g_2+g_1\hat{\circ}g_1+g_2\hat{\circ}g_0=0$, i.e., $d_2g_2=g_0\hat{\circ}g_2+g_2\hat{\circ}g_0=-g_1\hat{\circ}g_1$.

$\vdots$

For $k=n$ $(n=3,4,\dots)$, $d_2g_n=g_0\hat{\circ}g_n+g_n\hat{\circ}g_0=-(g_0\hat{\circ}g_{n-1}+g_2\hat{\circ}g_{n-2}+\cdots+g_{n-1}\hat{\circ}g_0)$.

Condition (3) implies that $d_2g_1=0$. By the definition of n-cocycle, we have the following result.

Proposition 1 *Let $(A,*)$ be an associative H-pseudoalgebra and $G_t(x,y)=\sum_{i\geqslant 0}g_i(x,y)t^i$ the deformation of A. Then g_1 is a 2-cocycle of the cohomology of A.*

Definition 6 Let $(A,*)$ be an associative H-pseudoalgebra. Two deformations $G_t(x,y)=\sum_{i\geqslant 0}g_i(x,y)t^i$ and $G'_t(x,y)=\sum_{i\geqslant 0}g'_i(x,y)t^i$ of $(A,*)$ are equivalent if there exists a family $\Phi_t: A\to A[[t]]$ of H-linear maps of the form $\Phi_t=\sum_{i\geqslant 0}\phi_i t^i=\phi_0+\phi_1t^1+\phi_2t^2+\cdots$, where $\phi_i: A\to A$ is an H-linear map for $i=1,2,\dots$ and $\phi_0=\mathrm{id}$ such that

$$\Phi_t(G_t(x,y))=G'_t(\Phi_t(x),\Phi_t(y)).$$

More precisely,

$$\sum_{i\geqslant 0}\phi_i\Big(\sum_{j\geqslant 0}g_j(x,y)t^j\Big)t^i=\sum_{i\geqslant 0}g'_i\Big(\sum_{j\geqslant 0}\phi_j(x)t^j,\sum_{k\geqslant 0}\phi_k(y)t^k\Big)t^i.$$

The deformation G_t is called trivial if G_t is equivalent to g_0. An associative H-pseudoalgebra $(A,*)$ is called rigid, if every 1-parameter formal deformation G_t is trivial.

Theorem 1 ([28]) *Let $(A,*)$ be an associative H-pseudoalgebra. There is a one-to-one correspondence between the elements of $H^2(A,A)$ and the infinitesimal deformation of A defined by $G_t(x,y)=g_0(x,y)+g_1(x,y)t$.*

Theorem 2 *Let $(A,*)$ be an associative H-pseudoalgebra and $H^2(A,A)=0$. Then $(A,*)$ is rigid.*

Proof: Let G_t be the deformation of $(A,*)$. Suppose that $G_t=g_0+\sum_{i\geqslant n}g_it^i$, then

$$d_2g_n=-(g_1\hat{\circ}g_{n-1}+g_2\hat{\circ}g_{n-2}+\cdots+g_{n-1}\hat{\circ}g_1)=0.$$

Hence $g_n\in Z^2(A,A)=B^2(A,A)$. It follows that there exists $\phi\in C^1(A,A)$ such that $g_n=d_1\phi$.

Let $\Phi_t=\mathrm{id}-\phi t^n:(A,G'_t)\to(A,G_t)$. Note that Φ_t is an isomorphism of H-modules with

$$\Phi_t\circ\sum_{i\geqslant 0}\phi^i t^{in}=\sum_{i\geqslant 0}\phi^i t^{in}\circ\Phi_t=\mathrm{id}_A.$$

Define $G'_t(x,y)=\Phi_t^{-1}G_t(\Phi_t(x),\Phi_t(y))$. It is immediate that G'_t is a formal deformation of $(A,*)$ and G_t is equivalent to G'_t. Suppose that $G'_t(x,y)=g'_0(x,y)+\sum_{i>0}g'_i(x,y)t^i$. Then

$$(\mathrm{id}-\phi t^n)\Big(g'_0(x,y)+\sum_{i>0}g'_i(x,y)t^i\Big)=\Big(g_0(x,y)+\sum_{i\geqslant n}g_i(x,y)t^i\Big)(x-\phi(x)t^n,y-\phi(y)t^n),$$

that is,

$$\begin{aligned}x*y+\sum_{i>0}g'_i(x,y)t^i-\phi\Big(x*y+\sum_{i>0}g'_i(x,y)t^i\Big)t^n\\=x*y-(x*\phi(y)+\phi(x)*y)t^n+(\phi(x)*\phi(y))t^{2n}\\+\sum_{i\geqslant n}g_i(x,y)t^i-\sum_{i\geqslant n}(g_i(\phi(x),y)+g_i(x,\phi(y)))t^{i+n}\\+\sum_{i\geqslant n}g_i(\phi(x),\phi(y))t^{i+2n}.\end{aligned}$$

Then we have $g'_1=g'_2=\cdots=g'_{n-1}=0$ and the coefficients of t^n are identical, i.e.,

$$g'_n-\phi(x*y)=-(\phi(x)*y+x*\phi(y))+g_n(x,y).$$

It follows that $g'_n=g_n-d_1\phi=0$ and $G'_t(x,y)=g'_0(x,y)+\sum_{i\geqslant n}g'_i(x,y)t^i$. By induction, this procedure ends with $G'_t=g'_0$. So G_t is equivalent to g_0, as desired. □

The deformations of Lie H-pseudoalgebras

Definition 7 Let $(L,[\cdot*\cdot])$ be a Lie H-pseudoalgebra. A 1-parameter formal deformation (we simply call it deformation) of L is given by

$$[x*y]_t=\sum_{i\geqslant 0}[x,y]_it^i=[x,y]_0+[x,y]_1t+[x,y]_2t^2+\cdots,$$

where t is a formal variable, $[\cdot,\cdot]_i: L\otimes L\to H^{\otimes 2}\otimes_H L$ is an H-bilinear map and $[x,y]_0=[x*y]$ such that $(L,[\cdot*\cdot]_t)$ is a family of Lie H-pseudoalgebras, that is, for all $x,y,z\in L$, the following conditions hold:

(skew-commutativity):

$$[x*y]_t=-[y*x]_t, \quad (4)$$

(Jacobi identity):

$$[x*[y*z]_t]_t-((12)_H\mathrm{id})[y*[x*z]_t]_t=[[x*y]_t*z]_t. \quad (5)$$

Conditions (4) and (5) are called the deformation equations of a Lie H-pseudoalgebra. Note that the skew-commutativity of $[\cdot*\cdot]_t$ is equivalent to the skew-symmetry of all $[\cdot,\cdot]_i$ for $i=0,1,2,\dots$.

Example 2 We consider the 3-dimensional Lie algebra L with the bracket

$$[x,y]_t = 2y - kty, \qquad [x,z]_t = -2z + (ly + kz)t,$$
$$[y,z]_t = x - \frac{k}{2}xt, \quad \forall k,l \in K.$$

Suppose that $H = sp\{1,g\}$ is the group Hopf algebra with $g^2 = 1, \Delta(g) = g \otimes g, S(g) = g = g^{-1}$. Then $[\cdot * \cdot]_1$ defines a deformation of Lie H-pseudoalgebra $H \otimes L$ with the action

$$[(f \otimes x) * (h \otimes y)]_1 = f \otimes g \otimes_H (1 \otimes [x,y]_t),$$
$$[(f \otimes x) * (h \otimes z)]_1 = f \otimes g \otimes_H (1 \otimes [x,z]_t),$$
$$[(f \otimes y) * (h \otimes z)]_1 = f \otimes g \otimes_H (1 \otimes [y,z]_t),$$

where $f,h \in \{1,g\}$.

Note that the deformation equation (5) can be rewritten as

$$\sum_{i=0}^{k}\Big([x,[y,z]_i]_{k-i} - ((12)\otimes_H \mathrm{id})[y,[x,z]_i]_{k-i} - [[x,y]_i,z]_{k-i}\Big) = 0, \qquad k = 0,1,2,\dots.$$

For $k = 0$, it is nothing else but the Jacobi identity of $(L,[\cdot * \cdot])$.

For $k = 1$, we have

$$[x,[y*z]]_1 - ((12)\otimes_H \mathrm{id})[y,[x*z]]_1 - [[x*y],z]_1 + [x*[y,z]_1] - ((12)\otimes_H \mathrm{id})[y*[x,z]_1] - [[x,y]_1*z] = 0,$$

which implies that $d_2[\cdot,\cdot]_1 = 0$. Hence we have the following result.

Proposition 2 *The first-order term* $[\cdot,\cdot]_1$ *of the deformation* $[\cdot * \cdot]_t = \sum_{i\geqslant 0}[\cdot,\cdot]_i t^i$ *of Lie H-pseudoalgebra* $(L,[\cdot * \cdot])$ *is a 2-cocycle of the cohomology of L.*

Definition 8 Let $(L,[\cdot * \cdot])$ be a Lie H-pseudoalgebra. Two deformations $[x*y]_t = \sum_{i\geqslant 0}[x,y]_i t^i$ and $[x*y]'_t = \sum_{i\geqslant 0}[x,y]'_i t^i$ of L are equivalent if there exists a family $\Phi_t : L \to L[[t]]$ of H-linear maps of the form $\Phi_t = \sum_{i\geqslant 0}\phi_i t^i = \phi_0 + \phi_1 t^1 + \phi_2 t^2 + \cdots$, where $\phi_i : L \to L$ is an H-linear map for $i \in \mathbb{N}_+$ and $\phi_0 = \mathrm{id}$ such that

$$\Phi_t([x*y]_t) = [\Phi_t(x), \Phi_t(y)]'_t.$$

More precisely,

$$\sum_{i\geqslant 0}\phi_i\Big(\sum_{j\geqslant 0}[x,y]_j t^j\Big)t^i = \sum_{i\geqslant 0}\Big[\sum_{j\geqslant 0}\phi_j(x)t^j, \sum_{k\geqslant 0}\phi_k(y)t^k\Big]'_i t^i.$$

A formal deformation $[\cdot*\cdot]_t$ is called trivial if $[\cdot*\cdot]_t$ is equivalent to $[\cdot*\cdot]$. A Lie H-pseudoalgebra $(L,[\cdot*\cdot])$ is called rigid, if every 1-parameter formal deformation $[\cdot * \cdot]_t$ is trivial.

Similar to the results obtained in Theorems 1 and 2, we have the following results for Lie H-pseudoalgebras.

Theorem 3 ([5]) *Let* $(L,[\cdot * \cdot])$ *be a Lie H-pseudoalgebra. There is a one-to-one correspondence between the elements of* $H^2(L,L)$ *and the infinitesimal deformation of L defined by* $[x*y]_t = [x*y] + [x,y]_1 t$.

Theorem 4 *Let* $(L,[\cdot*\cdot])$ *be a Lie H-pseudoalgebra and* $H^2(L,L) = 0$. *Then* $(L,[\cdot*\cdot])$ *is rigid.*

POISSON *H*-PSEUDOALGEBRAS

In this section we mainly construct Poisson H-pseudoalgebra from the first-order term of deformation for a commutative associative H-pseudoalgebra. First we give the definition of Poisson H-pseudoalgebras.

Definition 9 A Poisson H-pseudoalgebra is a triple $(P,[\cdot*\cdot],*)$, such that $(P,[\cdot*\cdot])$ is a Lie H-pseudoalgebra, $(P,*)$ is a commutative associative H-pseudoalgebra and they satisfy the compatible condition

$$[x*(y*z)] = [x*y]*z + ((12)\otimes_H \mathrm{id})y*[x*z], \quad (6)$$

for all $x,y,z \in P$.

Remark 3

(1) When $H = k$, a Poisson H-pseudoalgebra is actually a Poisson algebra.

(2) The compatible condition (6) is equivalent to

$$[(x*y)*z] = x*[y*z] + ((23)\otimes_H \mathrm{id})[x*z]*y.$$

In fact,

$$\begin{aligned}
[(x*y)*z] &= -((132)\otimes_H \mathrm{id})[z*(x*y)] \\
&\overset{(6)}{=} -((132)\otimes_H \mathrm{id})([z*x]*y + ((12)\otimes_H \mathrm{id})x*[z*y]) \\
&= ((132)(12)\otimes_H \mathrm{id})[x*z]*y \\
&\quad - ((132)(12)\otimes_H \mathrm{id})x*[z*y] \\
&= ((23)\otimes_H \mathrm{id})[x*z]*y - ((23)\otimes_H \mathrm{id})x*[z*y] \\
&= x*[y*z] + ((23)\otimes_H \mathrm{id})[x*z]*y.
\end{aligned}$$

Example 3 Let V be an ordinary Poisson algebra. Then $P = H \otimes V$ equipped with operations

$$(f\otimes a)*(g\otimes b) = f\otimes g \otimes_H (1\otimes ab) \quad \text{and}$$
$$[(f\otimes a)*(g\otimes b)] = f\otimes g\otimes_H (1\otimes [a,b]),$$

for all $f,g \in H$ and $a,b \in V$ is a Poisson H-pseudoalgebra.

Proposition 3 *Let $(A,*)$ be a commutative associative H-pseudoalgebra. Define* $[x*y]_A = 0$ *or* $[x*y]_A = x*y - (\sigma\otimes_H id)(y*x)$ *for all $x,y \in A$, then* $(A,*,[*]_A)$ *is a Poisson H-pseudoalgebra.*

Proof: A commutative associative H-pseudoalgebra $(A,*)$ is naturally a Poisson H-pseudoalgebra with the trivial pseudobracket $[x*y]_A = 0$. Suppose $[x*y]_A =$

$x*y-(\sigma\otimes_H \mathrm{id})(y*x)$, then $(A,[*]_A)$ is a Lie H-pseudoalgebra [5]. We only need to check the compatible condition (6). For all $x,y,z\in A$, we have

$$\begin{aligned}
&[x*y]_A*z+((12)\otimes_H \mathrm{id})y*[x*z]_A\\
&=(x*y-(\sigma\otimes_H \mathrm{id})(y*x))*z\\
&\quad+((12)\otimes_H \mathrm{id})(y*(x*z-(\sigma\otimes_H \mathrm{id})(z*x)))\\
&=(x*y)*z-((12)\otimes_H \mathrm{id})((y*x)*z)\\
&\quad+((12)\otimes_H \mathrm{id})(y*(x*z)-((23)\otimes_H \mathrm{id})y*(z*x))\\
&=(x*y)*z-((12)\otimes_H \mathrm{id})((y*x)*z)\\
&\quad+((12)\otimes_H \mathrm{id})(y*(x*z))-((123)\otimes_H \mathrm{id})y*(z*x))\\
&=x*(y*z)-((123)\otimes_H \mathrm{id})(y*z)*x\\
&=[x*(y*z)]_A.
\end{aligned}$$

This completes the proof. □

As in the case of Poisson algebras, the following proposition shows that the tensor product of two Poisson H-pseudoalgebras is still a Poisson H-pseudoalgebra. Since the proof is a straightforward computation, we omit the details.

Proposition 4 *Let H_i be the cocommutative Hopf algebras for $i=1,2$. Suppose $(A_i,*_i,[\cdot *_i \cdot])$ is a Poisson H_i-pseudoalgebra, then the tensor product $(A_1\otimes A_2,*,[\cdot*\cdot])$ is a Poisson $H_1\otimes H_2$-pseudoalgebra with the following structures:*

$$\begin{aligned}
(a\otimes b)*(a'\otimes b')&=a*_1 a'\otimes b*_2 b'\\
&=\sum_{i,j}(f_i\otimes f_j')\otimes(g_i\otimes g_j')\otimes_H(e_i\otimes e_j'),
\end{aligned}$$

and

$$\begin{aligned}
[(a\otimes b)*(a'\otimes b')]&=a*_1 a'\otimes[b*_2 b']+[a*_1 a']\otimes b*_2 b'\\
&=\sum_{i,j}(f_i\otimes p_j')\otimes(g_i\otimes q_j')\otimes_H(e_i\otimes t_j')\\
&\quad+\sum_{i,j}(p_i\otimes f_j')\otimes(q_i\otimes g_j')\otimes_H(t_i\otimes e_j'),
\end{aligned}$$

where $a*_1 a'=\sum_i f_i\otimes g_i\otimes_{H_1} e_i$, $[a*_1 a']=\sum_i p_i\otimes q_i\otimes_{H_1} t_i$, $b*_2 b'=\sum_j f_j'\otimes g_j'\otimes_{H_2} e_j'$ and $[b*_2 b']=\sum_j p_j'\otimes q_j'\otimes_{H_2} t_j'$ for all $a,a'\in A_1$ and $b,b'\in A_2$.

Next we show that the first-order element of a deformation induces a Poisson H-pseudoalgebra. We start with some useful lemmas.

Lemma 1 *Let $(A,*)$ be a commutative associative H-pseudoalgebra and ϕ a skew-commutative 2-cochain such that $d_2\phi=0$. Then*

$$\phi(x,y*z)=\phi(x,y)*z+((12)\otimes_H \mathrm{id})(y*\phi(x,z)),$$

for all $x,y,z\in A$.

Proof: Since ϕ is skew-commutative (i.e., $\phi(x,y)=-(\sigma\otimes_H \mathrm{id})\phi(y,x)$) and A is commutative, we have

$$\phi(y,x*z)=-((123)\otimes_H \mathrm{id})\phi(x*z,y), \tag{7}$$
$$x*\phi(y,z)=-((23)\otimes_H \mathrm{id})x*\phi(z,y), \tag{8}$$
$$\phi(x,z)*y=((132)\otimes_H \mathrm{id})(y*\phi(x,z)), \tag{9}$$
$$\phi(x,y)*z=-((12)\otimes_H \mathrm{id})\phi(y,x)*z, \tag{10}$$

for all $x,y,z\in A$. The condition $d_2\phi=0$ implies that

$$\phi(x,y*z)=\phi(x*y,z)-x*\phi(y,z)+\phi(x,y)*z. \tag{11}$$

Then we obtain

$$\begin{aligned}
&((23)\otimes_H \mathrm{id})\phi(x,z*y)=\\
&((23)\otimes_H \mathrm{id})(\phi(x*z,y)-x*\phi(z,y)+\phi(x,z)*y),
\end{aligned}\tag{12}$$

and

$$\begin{aligned}
&((12)\otimes_H \mathrm{id})(\phi(y*x,z)-\phi(y,x*z))\\
&\qquad=((12)\otimes_H \mathrm{id})(y*\phi(x,z)-\phi(y,x)*z).
\end{aligned}\tag{13}$$

By adding (11)–(13) and considering the fact that ϕ is skew-commutative and A is commutative, we have

$$\begin{aligned}
\mathrm{LHS}&=\phi(x,y*z)+((23)\otimes_H \mathrm{id})\phi(x,z*y)\\
&\qquad+((12)\otimes_H \mathrm{id})(\phi(y*x,z)-\phi(y,x*z))\\
&=\phi(x,y*z)+\phi(x,y*z)+\phi(x*y,z)\\
&\quad-((12)\otimes_H \mathrm{id})\phi(y,x*z)\\
&\overset{(7)}{=}2\phi(x,y*z)+\phi(x*y,z)+((12)(123)\otimes_H \mathrm{id})\phi(x*z,y)\\
&=2\phi(x,y*z)+\phi(x*y,z)+((23)\otimes_H \mathrm{id})\phi(x*z,y),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{RHS}&=\phi(x*y,z)-x*\phi(y,z)+\phi(x,y)*z\\
&\quad+((23)\otimes_H \mathrm{id})(\phi(x*z,y)-x*\phi(z,y)+\phi(x,z)*y)\\
&\quad+((12)\otimes_H \mathrm{id})(y*\phi(x,z)-\phi(y,x)*z)\\
&\overset{(8)-(10)}{=}\phi(x*y,z)+((23)\otimes_H \mathrm{id})(x*\phi(z,y))\\
&\quad+\phi(x,y)*z+((23)\otimes_H \mathrm{id})\phi(x*z,y)\\
&-((23)\otimes_H \mathrm{id})(x*\phi(z,y))+((23)(132)\otimes_H \mathrm{id})(y*\phi(x,z))\\
&\quad+((12)\otimes_H \mathrm{id})(y*\phi(x,z))+\phi(x,y)*z\\
&\quad=\phi(x*y,z)+((23)\otimes_H \mathrm{id})\phi(x*z,y)+2\phi(x,y)*z\\
&\qquad+2((12)\otimes_H \mathrm{id})(y*\phi(x,z)).
\end{aligned}$$

It follows that

$$\begin{aligned}
&2\phi(x,y*z)+\phi(x*y,z)+((23)\otimes_H \mathrm{id})\phi(x*z,y)\\
&\qquad=\phi(x*y,z)+((23)\otimes_H \mathrm{id})\phi(x*z,y)\\
&\qquad\quad+2\phi(x,y)*z+2((12)\otimes_H \mathrm{id})(y*\phi(x,z)),
\end{aligned}$$

that is,

$$\phi(x,y*z)=\phi(x,y)*z+((12)\otimes_H \mathrm{id})(y*\phi(x,z)).$$

This completes the proof. □

Lemma 2 *Let $(A,*)$ be a commutative associative H-pseudoalgebra and G_t the deformation of A. Then*

$$x*g_2(y,z)=((123)\otimes_H \mathrm{id})(g_2(y,z)*x), \quad (14)$$
$$g_2(y,z)*x=((132)\otimes_H \mathrm{id})(x*g_2(y,z)), \quad (15)$$
$$g_2(y,z*x)=((23)\otimes_H \mathrm{id})g_2(y,x*z), \quad (16)$$
$$g_2(z*x,y)=((12)\otimes_H \mathrm{id})g_2(x*z,y), \quad (17)$$

for all $x,y,z\in A$.

Proof: It can be verified by a direct computation and we omit the details. □

Lemma 3 *Let $(A,*)$ be a commutative associative H-pseudoalgebra and G_t the deformation of A. Then*

$$\begin{aligned}&-g_1\hat{\circ}g_1(x,y,z)-((123)\otimes_H \mathrm{id})g_1\hat{\circ}g_1(y,z,x)\\&\quad-((132)\otimes_H \mathrm{id})g_1\hat{\circ}g_1(z,x,y)\\&=g_2\hat{\circ}g_0(x,y,z)+((123)\otimes_H \mathrm{id})g_2\hat{\circ}g_0(y,z,x)\\&\quad+((132)\otimes_H \mathrm{id})g_2\hat{\circ}g_0(z,x,y).\end{aligned}$$

for all $x,y,z\in A$.

Proof: Since $g_0\hat{\circ}g_2+g_1\hat{\circ}g_1+g_2\hat{\circ}g_0=0$, we have

$$\begin{aligned}&-g_1\hat{\circ}g_1(x,y,z)=g_0\hat{\circ}g_2(x,y,z)+g_2\hat{\circ}g_0(x,y,z)\\&=x*g_2(y,z)-g_2(x,y)*z+g_2(x,y*z)-g_2(x*y,z). \quad (18)\end{aligned}$$

Now we compute

$$\begin{aligned}&-g_1\hat{\circ}g_1(x,y,z)-((123)\otimes_H \mathrm{id})g_1\hat{\circ}g_1(y,z,x)\\&\quad-((132)\otimes_H \mathrm{id})g_1\hat{\circ}g_1(z,x,y)\\&\overset{(18)}{=}x*g_2(y,z)-g_2(x,y)*z+g_2(x,y*z)-g_2(x*y,z)\\&\quad+((123)\otimes_H \mathrm{id})\big(y*g_2(z,x)-g_2(y,z)*x+g_2(y,z*x)\\&\quad-g_2(y*z,x)\big)+((132)\otimes_H \mathrm{id})\big(z*g_2(x,y)-g_2(z,x)*y\\&\quad+g_2(z,x*y)-g_2(z*x,y)\big)\\&\overset{(14),(15)}{=}x*g_2(y,z)-g_2(x,y)*z+g_2(x,y*z)\\&\quad-g_2(x*y,z)+((123)(123)\otimes_H \mathrm{id})g_2(z,x)*y\\&-((123)(132)\otimes_H\mathrm{id})(x*g_2(y,z))+((123)\otimes_H\mathrm{id})g_2(y,z*x)\\&-((123)\otimes_H\mathrm{id})g_2(y*z,x))+((132)(123)\otimes_H\mathrm{id})g_2(x,y)*z\\&\quad-((132)\otimes_H \mathrm{id})g_2(z,x)*y+((132)\otimes_H \mathrm{id})g_2(z,x*y)\\&\quad-((132)\otimes_H \mathrm{id})g_2(z*x,y))\\&=g_2(x,y*z)-g_2(x*y,z)\\&\quad+((123)\otimes_H \mathrm{id})(g_2(y,z*x)-g_2(y*z,x))\\&\quad+((132)\otimes_H \mathrm{id})(g_2(z,x*y)-g_2(z*x,y))\\&=g_2\hat{\circ}g_0(x,y,z)+((123)\otimes_H \mathrm{id})g_2\hat{\circ}g_0(y,z,x)\\&\quad+((132)\otimes_H \mathrm{id})g_2\hat{\circ}g_0(z,x,y).\end{aligned}$$

This completes the proof. □

Lemma 4 *Let $(A,*)$ be a commutative associative H-pseudoalgebra and G_t the deformation of A. Then*

$$\begin{aligned}&((12)\otimes_H id)g_1\hat{\circ}g_1(y,x,z)+((13)\otimes_H id)g_1\hat{\circ}g_1(z,y,x)\\&\qquad+((23)\otimes_H id)g_1\hat{\circ}g_1(x,z,y)\\&=-((12)\otimes_H id)g_2\hat{\circ}g_0(y,x,z)-((13)\otimes_H id)g_2\hat{\circ}g_0(z,y,x)\\&\quad-((23)\otimes_H id)g_2\hat{\circ}g_0(x,z,y).\end{aligned}$$

for all $x,y,z\in A$.

Proof: For all $x,y,z\in A$, we have

$$\begin{aligned}&((12)\otimes_H \mathrm{id})g_1\hat{\circ}g_1(y,x,z)+((13)\otimes_H \mathrm{id})g_1\hat{\circ}g_1(z,y,x)\\&\qquad+((23)\otimes_H \mathrm{id})g_1\hat{\circ}g_1(x,z,y)\\&=-((12)\otimes_H \mathrm{id})(g_0\hat{\circ}g_2(y,x,z)+g_2\hat{\circ}g_0(y,x,z))\\&\quad-((13)\otimes_H \mathrm{id})(g_0\hat{\circ}g_2(z,y,x)+g_2\hat{\circ}g_0(z,y,x))\\&\quad-((23)\otimes_H \mathrm{id})(g_0\hat{\circ}g_2(x,z,y)+g_2\hat{\circ}g_0(x,z,y))\\&=((12)\otimes_H\mathrm{id})(g_2(y,x)*z-y*g_2(x,z)+g_2(y*x,z)-g_2(y,x*z))\\&+((13)\otimes_H\mathrm{id})(g_2(z,y)*x-z*g_2(y,x)+g_2(z*y,x)-g_2(z,y*x))\\&+((23)\otimes_H\mathrm{id})(g_2(x,z)*y-x*g_2(z,y)+g_2(x*z,y)-g_2(x,z*y))\\&\overset{(14),(15)}{=}((12)(132)\otimes_H\mathrm{id})z*g_2(y,x)-((12)(123)\otimes_H\mathrm{id})g_2(x,z)*y\\&\quad+((12)\otimes_H \mathrm{id})g_2(y*x,z)-((12)\otimes_H \mathrm{id})g_2(y,x*z)\\&\quad+((13)\otimes_H \mathrm{id})g_2(z,y)*x-((13)\otimes_H \mathrm{id})z*g_2(y,x)\\&\quad+((13)\otimes_H \mathrm{id})g_2(z*y,x)-((13)\otimes_H \mathrm{id})g_2(z,y*x)\\&\quad+((23)\otimes_H \mathrm{id})g_2(x,z)*y-((23)(123)\otimes_H \mathrm{id})g_2(z,y)*x\\&\quad+((23)\otimes_H \mathrm{id})g_2(x*z,y)-((23)\otimes_H \mathrm{id})g_2(x,z*y)\\&=((12)\otimes_H \mathrm{id})g_2(y*x,z)-((12)\otimes_H \mathrm{id})g_2(y,x*z)\\&\quad+((13)\otimes_H \mathrm{id})g_2(z*y,x)-((13)\otimes_H \mathrm{id})g_2(z,y*x)\\&\quad+((23)\otimes_H \mathrm{id})g_2(x*z,y)-((23)\otimes_H \mathrm{id})g_2(x,z*y)\\&=-((12)\otimes_H \mathrm{id})g_2\hat{\circ}g_0(y,x,z)-((13)\otimes_H \mathrm{id})g_2\hat{\circ}g_0(z,y,x)\\&\quad-((23)\otimes_H \mathrm{id})g_2\hat{\circ}g_0(x,z,y).\end{aligned}$$

So we complete the proof. □

Now we give the main result of this section.

Theorem 5 *Let $G_t=\sum_{i\geqslant 0}g_it^i$ be the deformation of a commutative associative H-pseudoalgebra $(A,*)$. Define $\{x*y\}=g_1(x,y)-(\sigma\otimes_H \mathrm{id})g_1(y,x)$, for all $x,y\in A$. Then $(A,*,\{*\})$ is a Poisson H-pseudoalgebra.*

Proof: By Lemma 1, the compatible condition (6) holds. It is easy to prove that the pseudobracket $\{\cdot*\cdot\}$ is skew-commutative. It suffices to prove that $\{\cdot*\cdot\}$ satisfies the Jacobi identity. Using Lemmas 2–4, for all $x,y,z\in A$, we have

$$\begin{aligned}&\{\{x*y\}*z\}-\{x*\{y*z\}\}+((12)\otimes_H \mathrm{id})\{y*\{x*z\}\}\\&=g_1(g_1(x,y),z)-((132)\otimes_H \mathrm{id})g_1(z,g_1(x,y))\\&-((12)\otimes_H \mathrm{id})g_1(g_1(y,x),z)+((13)\otimes_H \mathrm{id})g_1(z,g_1(y,x))\\&-g_1(x,g_1(y,z))+((123)\otimes_H \mathrm{id})g_1(g_1(y,z),x)\\&+((23)\otimes_H \mathrm{id})g_1(x,g_1(z,y))-((13)\otimes_H \mathrm{id})g_1(g_1(z,y),x)\\&+((12)\otimes_H \mathrm{id})g_1(y,g_1(x,z))-((23)\otimes_H \mathrm{id})g_1(g_1(x,z),y)\\&-((123)\otimes_H\mathrm{id})g_1(y,g_1(z,x))+((132)\otimes_H\mathrm{id})g_1(g_1(z,x),y)\end{aligned}$$

$$
\begin{aligned}
&= g_1(g_1(x,y),z) - g_1(x,g_1(y,z)) \\
&\quad - ((123)\otimes_H \mathrm{id})(g_1(y,g_1(z,x)) - g_1(g_1(y,z),x)) \\
&\quad - ((132)\otimes_H \mathrm{id})(g_1(z,g_1(x,y)) - g_1(g_1(z,x),y)) \\
&\quad + ((12)\otimes_H \mathrm{id})(g_1(y,g_1(x,z)) - g_1(g_1(y,x),z)) \\
&\quad + ((13)\otimes_H \mathrm{id})(g_1(g_1(z,g_1(y,x)) - g_1(z,y),x)) \\
&\quad + ((23)\otimes_H \mathrm{id})(g_1(x,g_1(z,y)) - g_1(g_1(x,z),y)) \\
&= g_2 \hat{\circ} g_0(x,y,z) + ((123)\otimes_H \mathrm{id}) g_2 \hat{\circ} g_0(y,z,x) \\
&\quad + ((132)\otimes_H \mathrm{id}) g_2 \hat{\circ} g_0(z,x,y) - ((12)\otimes_H \mathrm{id}) g_2 \hat{\circ} g_0(y,x,z) \\
&\quad - ((13)\otimes_H \mathrm{id}) g_2 \hat{\circ} g_0(z,y,x) - ((23)\otimes_H \mathrm{id}) g_2 \hat{\circ} g_0(x,z,y) \\
&= g_2(x,y*z) - g_2(x*y,z) + ((123)\otimes_H \mathrm{id}) g_2(y,z*x) \\
&\quad - ((123)\otimes_H \mathrm{id}) g_2(y*z,x) + ((132)\otimes_H \mathrm{id}) g_2(z,x*y) \\
&\quad - ((132)\otimes_H \mathrm{id}) g_2(z*x,y) - ((12)\otimes_H \mathrm{id}) g_2(y,x*z) \\
&\quad + ((12)\otimes_H \mathrm{id}) g_2(y*x,z) - ((13)\otimes_H \mathrm{id}) g_2(z,y*x) \\
&\quad + ((13)\otimes_H \mathrm{id}) g_2(z*y,x) - ((23)\otimes_H \mathrm{id}) g_2(x,z*y) \\
&\quad + ((23)\otimes_H \mathrm{id}) g_2(x*z,y) \\
&\overset{(16),(17)}{=} g_2(x,y*z) - g_2(x*y,z) + ((123)(23)\otimes_H \mathrm{id}) g_2(y,x*z) \\
&\quad - ((123)\otimes_H \mathrm{id}) g_2(y*z,x) + ((132)\otimes_H \mathrm{id}) g_2(z,x*y) \\
&\quad - ((132)(12)\otimes_H \mathrm{id}) g_2(x*z,y) - ((12)\otimes_H \mathrm{id}) g_2(y,x*z) \\
&\quad + ((12)\otimes_H \mathrm{id}) g_2(y*x,z) - ((13)(23)\otimes_H \mathrm{id}) g_2(z,x*y) \\
&\quad + ((13)(12)\otimes_H \mathrm{id}) g_2(y*z,x) - ((23)\otimes_H \mathrm{id}) g_2(x,z*y) \\
&\quad + ((23)\otimes_H \mathrm{id}) g_2(x*z,y) = 0.
\end{aligned}
$$

This completes the proof. □

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REFERENCES

1. Beilinson A, Drinfeld V (2004) *Chiral Algebras*, American Mathematical Society, Providence, RI.
2. D'Andrea A, Kac VG (1998) Structure theory of finite conformal algebras. *Selecta Math* **4**, 377–418.
3. Kac VG, Retakh A (2008) Simple Jordan conformal superalgebras. *J Algebra Appl* **7**, 517–533.
4. Xu MS, Hong YY, Wu ZX (2021) Finite irreducible conformal modules of rank two Lie conformal algebras. *J Algebra Appl* **20**, 2150145.
5. Bakalov B, D'Andrea A, Kac VG (2001) Theory of finite pseudoalgebras. *Adv Math* **162**, 1–140.
6. Dorfman I (1993) *Dirac Structures and Integrability of Nonlinear Evolution Equations*, John Wiley and Sons, Chichester.
7. Gel'Fand IM, Dorfman IY (1981) Hamiltonian operators and infinite-dimensional Lie algebras. *Funct Anal Its Appl* **15**, 173–187.
8. Gel'Fand IM, Dorfman IY (1982) Hamiltonian operators and the classical Yang-Baxter equation. *Funct Anal Its Appl* **16**, 241–248.
9. Bakalov B, D'Andrea A, Kac VG (2006) Irreducible modules over finite simple Lie pseudoalgebras I. Primitive pseudoalgebras of type W and S. *Adv Math* **204**, 278–346.
10. Bakalov B, D'Andrea A, Kac VG (2013) Irreducible modules over finite simple Lie pseudoalgebras II. Primitive pseudoalgebras of type K. *Adv Math* **232**, 188–237.
11. Bakalov B, D'Andrea A, Kac VG (2021) Irreducible modules over finite simple Lie pseudoalgebras III. Primitive pseudoalgebras of type H. *Adv Math* **392**, 107963.
12. Kolesnikov P (2009) Simple finite Jordan pseudoalgebras. *Symmetry Integr Geom Methods Appl* **5**, 014.
13. Wu ZX (2015) Leibniz H-pseudoalgebras. *J Algebra* **437**, 1–33.
14. Wu ZX (2015) Graded left symmetric pseudo-algebras. *Comm Algebra* **43**, 3869–3897.
15. Boyallian C, Liberati J (2012) On pseudo-bialgebras. *J Algebra* **372**, 1–34.
16. Liu LL, Wang SH, Shi GD, Guo ZT (2022) Infinitesimal H-pseudobialgebra and associative pseudo-Yang-Baxter equation. *Comm Algebra* **50**, 1071–1091.
17. D'Andrea A, Marchei G (2011) Representations of Lie pseudoalgebras with coefficients. *J Algebra* **329**, 92–106.
18. Retakh A (2004) Unital associative pseudoalgebras and their representations. *J Algebra* **277**, 769–805.
19. Shulman MA (2012) Not every pseudoalgebra is equivalent to a strict one. *Adv Math* **229**, 2024–2041.
20. Sun QX (2012) Generalization of H-pseudoalgebraic structures. *J Math Phys* **53**, 012105.
21. Gerstenhaber M (1964) On the deformation of rings and algebras. *Ann Math* **79**, 59–103.
22. Nijenhuis A, Richardson RW (1966) Cohomology and deformations in graded Lie algebras. *Bull Amer Math Soc* **72**, 1–29.
23. Bai CM, Meng DJ (2001) The realization of non-transitive Novikov algebras. *J Phys A Math Gen* **34**, 6435–6442.
24. Kubo F, Taniguchi Y (2004) A controlling cohomology of the deformation theory of Lie triple systems. *J Algebra* **278**, 242–250.
25. Ma Y, Chen LY, Lin J (2015) One-parameter formal deformations of Hom-Lie-Yamaguti algebras. *J Math Phys* **56**, 011701.
26. Makhlouf A, Silvestrov S (2010) Notes on 1-parameter formal deformations of Hom-associative and Hom-Lie algebras. *Forum Math* **22**, 715–739.
27. Sweedler ME (1969) *Hopf Algebras*, W. A. Benjamin, New York.
28. Liberati JI (2022) Cohomology of associative H-pseudoalgebras. *arXiv*, 13832v2.