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The WG inverse and its application in a constrained matrix approximation problem

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ABSTRACT: We study the application of WG inverse to a constrained matrix approximation problem to deduce the unique solution of the problem, and get characterizations of WG inverse by applying matrix decompositions. Moreover, we get one Gauss-Jordan elimination method for the WG inverse.

KEYWORDS: WG inverse, core-EP decomposition, full-rank decomposition, Gauss-Jordan elimination method

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INTRODUCTION

In this paper, $\mathbb{C}_{n,n}$ denotes the set of $n\times n$ matrices with complex entries; M^*, $\mathscr{R}(M)$, and $\operatorname{Ind}(M)$ represent the conjugate transpose, range space, and index of $M\in\mathbb{C}_{n,n}$, respectively. In particular,

$$\mathbb{C}_n^{\text{CM}}=\left\{M\mid M\in\mathbb{C}_{n,n},\ \operatorname{rank}(M^2)=\operatorname{rank}(M)\right\}.$$

The symbols $M^{\#}$, M^d and $M^{\dagger}$ denote the group, Drazin and Moore-Penrose inverses of M, respectively (see [1, 2]). The symbol $M^{\textcircled{\dagger}}$ denotes the core-EP inverse of M, which is the unique matrix satisfying $XMX=X$, $XM^{k+1}=M^k$, $(MX)^*=MX$, and $\mathscr{R}(X)\subseteq\mathscr{R}\left(M^k\right)$, where $k=\operatorname{Ind}(M)$ (see [3]). When $k=1$, X is called the core inverse of M, and is denoted by $X=M^{\textcircled{\#}}$ (see [4]). Obviously, the core-EP inverse is a generalization of the core inverse for matrix of arbitrary index.

Let $M\in\mathbb{C}_{n,n}$, $\operatorname{rank}\left(M^k\right)=r$ and $\operatorname{Ind}(M)=k$. The core-EP decomposition of M is

$$M=M_1+M_2, \tag{1}$$

where $M_1\in\mathbb{C}_n^{\text{CM}}$, $M_2^k=0$, and $M_1^*M_2=M_2M_1=0$. Here one or both of M_1 and M_2 can be null (see [5]). It is easy to check that there exists an n-square unitary matrix U such that

$$M_1=U\begin{bmatrix}T & S\\ 0 & 0\end{bmatrix}U^* \text{ and } M_2=U\begin{bmatrix}0 & 0\\ 0 & N\end{bmatrix}U^*, \tag{2}$$

where $S\in\mathbb{C}_{r,n-r}$, $T\in\mathbb{C}_{r,r}$ is a nonsingular matrix, $N\in\mathbb{C}_{n-r,n-r}$, and $N^k=0$. The well-known C-N decomposition of M is

$$M=\widehat{M_1}+\widehat{M_2},$$

where $\widehat{M_1}\in\mathbb{C}_n^{\text{CM}}$, $\widehat{M_2}^k=0$, and $\widehat{M_1}\widehat{M_2}=\widehat{M_2}\widehat{M_1}=0$. Here one or both of $\widehat{M_1}$ and $\widehat{M_2}$ can be null (see [2]). It is easy to check that there exists a nonsingular matrix P such that

$$\widehat{M_1}=P\begin{bmatrix}\widehat{T} & 0\\ 0 & 0\end{bmatrix}P^{-1} \text{ and } \widehat{M}_2=P\begin{bmatrix}0 & 0\\ 0 & \widehat{N}\end{bmatrix}P^{-1},$$

where $\widehat{T}\in\mathbb{C}_{r,r}$ is invertible, $\widehat{N}\in\mathbb{C}_{n-r,n-r}$ is nilpotent, and $\widehat{N}^k=0$.

In [6, Theorem 2.3], we see that

$$M^{\textcircled{\dagger}}=M^dM^k\left(M^k\right)^{\dagger}. \tag{3}$$

Furthermore, by applying (3), we get

$$M^{\textcircled{\dagger}}=M_1^{\textcircled{\#}}=\widehat{M}_1^{\textcircled{\#}}.$$

It is worth noting that when the group inverse is applied to M_1 and $\widehat{M}_1$, respectively, the two results are different. The well-known Drazin inverse M^d is the group inverse of $\widehat{M}_1$. The weak group inverse (for short, WG inverse) of M is the group of M_1. The WG inverse is introduced by Wang and Chen in [7], which is the unique matrix satisfying

$$\left(2^l\right)\ MX^2=X,\quad \left(3^c\right)\ MX=M^{\textcircled{\dagger}}M. \tag{4}$$

The WG inverse is a new type of generalized group inverse and is different from other generalized group inverses. In [7], by applying the core-EP decomposition, the authors proved that (4) is consistent and the WG inverse $M^{\textcircled{w}}$ is the unique solution of (4),

$$M^{\textcircled{w}}=U\begin{bmatrix}T^{-1} & T^{-2}S\\ 0 & 0\end{bmatrix}U^*. \tag{5}$$

In addition, the authors gave the following characterizations of WG inverse:

$$\begin{aligned}M^{\textcircled{w}}&=M^k\left(M^{k+2}\right)^{\textcircled{\#}}M=\left(M^2P_{M^k}\right)^{\dagger}M\\ &=(MM^{\textcircled{\dagger}}M)^{\#}=(M^{\textcircled{\dagger}})^2M\\ &=\left(M^2\right)^{\textcircled{\dagger}}M.\end{aligned} \tag{6}$$

Furthermore, when $M \in \mathbb{C}_n^{\text{CM}}$, it is obvious that

$$M^{\textcircled{\dagger}} = M^{\sharp}. \tag{7}$$

In recent years, a number of mathematicians are interested in the WG inverse. Ferreyra et al [8] extended the notion of WG inverse to rectangular matrix $M \in \mathbb{C}_{m,n}$, introduced the W-weighted WG inverse of M and denoted it by $M^{\textcircled{w},\mathrm{W}}$. It is easy to see that when $m = n$ and $W = I_n$, the W-weighted WG inverse is simplified to the WG inverse. Mosić and Stanimirović [9] gave limit representation, integral representation and perturbation formulae for the WG inverse. Yan et al [10] derived some new characteristics and properties of the WG inverse. Xu et al [11] introduced a generalized WG inverse. Mosić and Zhang [12] studied a Weighted WG inverse for Hilbert space operators. Zhou et al [13, 14] considered WG inverse in proper $*$-rings and characterized the inverse by equations. Furthermore, by applying WG inverse, Wang and Liu [15] introduced the WG matrix, denoted the set of all WG matrices by $\mathbb{C}_n^{\text{WG}}$: $\mathbb{C}_n^{\text{WG}} = \left\{M \mid M \in \mathbb{C}_{n,n},\ M^{\textcircled{w}}M = MM^{\textcircled{w}}\right\}$, and proved

$$\mathbb{C}_n^{\text{CM}} \subseteq \mathbb{C}_n^{\text{WG}}.$$

Ferreyra et al [16] introduced the weak core inverse of $M \in \mathbb{C}_{n,n}$, denoted it by $M^{\textcircled{w},\dagger}$, and $M^{\textcircled{w},\dagger} = M^{\textcircled{w}}P_M$, where $P_M = MM^{\dagger}$; and defined the concept of weak core matrix. Denote the set of all weak core matrices by $\mathbb{C}_n^{\text{WC}}$: $\mathbb{C}_n^{\text{WC}} = \left\{M \mid M \in \mathbb{C}_{n,n},\ M^{\textcircled{w},\dagger} = M^{d,\dagger}\right\}$, in which $M^{d,\dagger}$ is the DMP inverse of M, and $M^{d,\dagger} = M^d MM^{\dagger}$. It is noteworthy that $\mathbb{C}_n^{\text{WG}}$ is a proper subset of $\mathbb{C}_n^{\text{WC}}$:

$$\mathbb{C}_n^{\text{WG}} \subseteq \mathbb{C}_n^{\text{WC}}, \qquad \text{(see [16])}.$$

With in-depth research, we see more and more properties, characterizations and applications of WG inverse.

It is widely known that generalized inverse is one of the most often used and the most effective tools in many classes of inconsistent (or consistent) matrix equations. For example, Penrose [17] proved that the minimum-norm least-squares solution of

$$Mx = b$$

is unique and $x = M^{\dagger}b$. Campbell and Meyer [18] showed that $x = M^d b$ is the unique solution of the consistent constrained matrix equation

$$Mx = b \quad \text{subject to} \quad x \in \mathscr{R}\left(M^k\right),$$

where $k = \text{Ind}(M)$ and $b \in \mathscr{R}\left(M^k\right)$. By applying core inverse, Wang and Zhang [19] studied the constrained matrix approximation problem: $\|Mx - b\|_F = \min$ with respect to $x \in \mathscr{R}(M)$, and gave the unique solution $x = M^{\textcircled{\#}}b$, where $M \in \mathbb{C}_n^{\mathtt{CM}}$. Ji, Mosić et al [20, 21] studied the constrained matrix approximation problem:

$$\min \|Mx - b\|_F^2 \ \text{ subject to } \ x \in \mathscr{R}\left(M^k\right), \tag{8}$$

and gave the unique solution

$$x = M^{\textcircled{\dagger}}b, \tag{9}$$

where k is the index of M. In this paper, we consider the application of WG inverse to a constrained best approximation problem, give the unique solution by applying WG inverse, and get several characterizations of the WG inverse.

APPLICATION OF WG INVERSE TO A CONSTRAINED MATRIX EQUATION

Let $M \in \mathbb{C}_{n,n}$ with $\text{rank}\left(M^k\right) = r$ and $\text{Ind}(M) = k$. And let the core-EP decomposition of M have the form described in (1), and M_1 and M_2 be as given in (2), then

$$M^k = U\begin{bmatrix} T^k & \widehat{T} \\ 0 & 0 \end{bmatrix}U^* \tag{10}$$

and

$$M^{k+2} = U\begin{bmatrix} T^{k+2} & \widetilde{T} \\ 0 & 0 \end{bmatrix}U^*, \tag{11}$$

where $\widehat{T} = T^{k-1}S + T^{k-2}SN + \cdots + TSN^{k-2} + SN^{k-1}$ and $\widetilde{T} = T^2\widehat{T}$.

Consider the constrained matrix equation

$$M^2X = MD \text{ with respect to } \mathscr{R}(X) \subseteq \mathscr{R}\left(M^k\right), \tag{12}$$

in which X and D are both n-by-m matrices, and $\text{Ind}(M) = k$. Since the rank of M^2 is less than or equal to the rank of M, we know that the constrained matrix equation (12) is not always consistent. Therefore, we study the least-squares solution of it in the Frobenius norm.

Theorem 1 *Let $M \in \mathbb{C}_{n,n}$, $\text{Ind}(M) = k$ and $\text{rank}\left(M^k\right) = r$. Then the least-squares solution of* (12) *exists uniquely, and*

$$X = M^{\textcircled{w}}D. \tag{13}$$

Proof: Since $\mathscr{R}(X) \subseteq \mathscr{R}\left(M^k\right)$, there is an $n \times m$ matrix Y satisfying $X = M^kY$. Then X is the least-squares solution of (12) if and only if Y is the solution of

$$\|M^{k+2}Y - MD\|_F^2 = \min. \tag{14}$$

Denote

$$U^*Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \quad \text{and} \quad U^*D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}, \tag{15}$$

where $Y_1 \in \mathbb{C}_{r,m}$ and $D_1 \in \mathbb{C}_{r,m}$.

By applying the core-EP decomposition of M, (2) and (11), we have

$$\begin{aligned}&\left\|M^{k+2}Y-MD\right\|_F^2\\&=\left\|U\begin{bmatrix}T^{k+2}&\widetilde{T}\\0&0\end{bmatrix}\begin{bmatrix}Y_1\\Y_2\end{bmatrix}-U\begin{bmatrix}T&S\\0&N\end{bmatrix}\begin{bmatrix}D_1\\D_2\end{bmatrix}\right\|_F^2\\&=\left\|\begin{bmatrix}T^{k+2}Y_1+\widetilde{T}Y_2-TD_1-SD_2\\-ND_2\end{bmatrix}\right\|_F^2\\&=\left\|T^{k+2}Y_1+\widetilde{T}Y_2-TD_1-SD_2\right\|_F^2+\|ND_2\|_F^2.\end{aligned}\tag{16}$$

Since T is invertible, it follows that

$$\min_Y\|M^{k+2}Y-MD\|_F^2=\|ND_2\|_F^2\tag{17}$$

and

$$Y_1=-T^{-(k+2)}\left(\widetilde{T}Y_2-TD_1-SD_2\right),\tag{18}$$

where $Y_2\in\mathbb{C}_{n-r,m}$ is arbitrary. Therefore, we see that the least-squares solution of (12) exists. By applying (5), (15), (19) and $X=M^kY$, we get

$$\begin{aligned}X=M^kY&=U\begin{bmatrix}T^k&\widehat{T}\\0&0\end{bmatrix}\begin{bmatrix}-T^{-(k+2)}\left(T^2\widehat{T}Y_2-TD_1-SD_2\right)\\Y_2\end{bmatrix}\\&=U\begin{bmatrix}-T^kT^{-(k+2)}\left(T^2\widehat{T}Y_2-TD_1-SD_2\right)+\widehat{T}Y_2\\0\end{bmatrix}\\&=U\begin{bmatrix}T^{-1}D_1+T^{-2}SD_2\\0\end{bmatrix}=U\begin{bmatrix}T^{-1}&T^{-2}S\\0&0\end{bmatrix}U^*U\begin{bmatrix}D_1\\D_2\end{bmatrix}\\&=M^{Ⓦ}D,\end{aligned}\tag{19}$$

that is, (13) is the unique solution of (12). □

Remark 1 When $m=1$, from (8) and (9), we conclude that the unique solution to (12) is

$$X=\left(\left(M^2\right)^{\textcircled{\dagger}}\right)(MD)=\left(\left(M^2\right)^{\textcircled{\dagger}}M\right)D.$$

By substituting (6) into the above equation, we get $X=M^{Ⓦ}D$.

Remark 2 When $k=1$, we get that (12) is consistent:

$$M^2X=MD\ \text{ with respect to }\ \mathscr{R}(X)\subseteq\mathscr{R}(M).$$

It is easy to give the unique solution: $X=\left(M^2\right)^{\#}MD$. Since the index of M is 1, we know that M^2 is also group invertible and $\left(M^2\right)^{\#}=\left(M^{\#}\right)^2$. It follows that $X=M^{\#}D$. On the other hand, since the index of M is 1, by using Theorem 1, we get $X=M^{Ⓦ}D=M^{\#}D$.

In Theorem 1, we see that $X=M^{Ⓦ}D$ is the unique least-squares solution of $M^2X=MD$ with respect to $\mathscr{R}(X)\subseteq\mathscr{R}\left(M^k\right)$. Then, it is obvious that, when $D=I_n$, the WG inverse $M^{Ⓦ}$ of M is the unique least-squares solution of $M^2X=M$ with respect to $\mathscr{R}(X)\subseteq\mathscr{R}\left(M^k\right)$.

Corollary 1 *Let $M\in\mathbb{C}_{n,n}$, $\text{Ind}(M)=k$ and $\text{rank}\left(M^k\right)=r$. Then the WG inverse $M^{Ⓦ}$ is the unique least-squares solution of*

$$M^2X=M\ \textit{ with respect to }\ \mathscr{R}(X)\subseteq\mathscr{R}\left(M^k\right).\tag{20}$$

Next, for $\mathscr{R}(X)\in\mathscr{R}\left(M^k\right)$, we obtain that X is the least-squares solution of $M^2X=M$ with respect to $\mathscr{R}(X)\subseteq\mathscr{R}\left(M^k\right)$ if and only if Y is the least-squares solution of $M^{k+2}Y=M$. It is easy to check that $Y=\left(M^{k+2}\right)^{\dagger}M+\left(I_n-\left(M^{k+2}\right)^{\dagger}M^{k+2}\right)Z$, where $Z\in\mathbb{C}_{n,n}$ is arbitrary. Therefore, we get Theorem 2, by $M^k\left(I_n-\left(M^{k+2}\right)^{\dagger}M^{k+2}\right)=0$ and $X=M^kY$.

Theorem 2 *Let $M\in\mathbb{C}_{n,n}$ with $\text{Ind}(M)=k$. Then*

$$M^{Ⓦ}=M^k\left(M^{k+2}\right)^{\dagger}M.\tag{21}$$

In particular, when the index of M is 1, by applying Theorem 2, we get the well-known result: $M^{\#}=M\left(M^3\right)^{\dagger}M$. The corresponding one is

$$M^{\#}=M\left(M^3\right)^{(1)}M,\tag{22}$$

where $M^{(1)}$ is any element in the set $S=\{X|MXM=M\}$. It is obvious that the Moore-Penrose of M is in the set S. Therefore, when the index of M is 1, (21) is a special case of (22).

In Theorem 2, since the index of M is k, we have $\left(M^d\right)^2M^{k+2}=M^k$ and $P_{M^k}=P_{M^{l+2}}$, where l is a positive integer and is greater than or equal to k. Therefore, we get Theorem 3.

Theorem 3 *Let $M\in\mathbb{C}_{n,n}$, $\text{Ind}(M)=k$ and $\text{rank}\left(M^k\right)=r$. Then*

$$\begin{aligned}M^{Ⓦ}&=\left(M^d\right)^2P_{\mathscr{R}(M^k)}M&(23)\\&=M^l\left(M^{l+2}\right)^{\dagger}M,&(24)\end{aligned}$$

where l is greater than or equal to k.

Example 1 Let

$$M=\begin{bmatrix}0&4&-1\\-1&3&-1\\-2&-2&0\end{bmatrix}.$$

Then $\text{Ind}(M)=2$, $\text{rank}\left(M^2\right)=1$, and

$$M^2=\begin{bmatrix}-2&14&-4\\-1&7&-2\\2&-14&4\end{bmatrix},\ M^4=\begin{bmatrix}-18&126&-36\\-9&63&-18\\18&-126&36\end{bmatrix},$$

$$\left(M^4\right)^{\dagger}=\begin{bmatrix}-1/2187&-1/4374&1/2187\\7/2187&7/4374&-7/2187\\-2/2187&-1/2187&2/2187\end{bmatrix},$$

$$M^d=\begin{bmatrix}-2/27&14/27&-4/27\\-1/27&7/27&-2/27\\2/27&-14/27&4/27\end{bmatrix},$$

and

$$P_{\mathscr{R}(M^2)} = \begin{bmatrix} 4/9 & 2/9 & -4/9 \\ 2/9 & 1/9 & -2/9 \\ -4/9 & -2/9 & 4/9 \end{bmatrix}.$$

By applying (21) and (23), we get

$$\begin{aligned} M^{ⓦ} &= \begin{bmatrix} 2/27 & 10/27 & -2/27 \\ 1/27 & 5/27 & -1/27 \\ -2/27 & -10/27 & 2/27 \end{bmatrix} \\ &= M^2 \left(M^4\right)^{\dagger} M = \left(M^d\right)^2 P_{\mathscr{R}(M^2)} M. \end{aligned}$$

Furthermore, let

$$D = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 0 & -1 \end{bmatrix}.$$

Then

$$MD = \begin{bmatrix} -4 & 13 \\ -4 & 8 \\ 0 & -10 \end{bmatrix}.$$

By applying Theorem 1, we get the least-squares solution of

$$M^2 X = MD \text{ with respect to } \mathscr{R}(X) \subseteq \mathscr{R}\left(M^2\right)$$

as

$$X = \begin{bmatrix} -8/27 & 4/3 \\ -4/27 & 2/3 \\ 8/27 & -4/3 \end{bmatrix}.$$

CHARACTERIZATION AND ALGORITHM OF WG INVERSE

In this section, we deduce several characterizations of the WG inverse by using matrix decompositions, matrix equations and rank equalities.

The matrix equation plays an important role in characterizing generalized inverses. In Theorem 4, we will derive a characterization of WG inverse by using matrix equations.

Theorem 4 *Let $M \in \mathbb{C}_{n,n}$ with* $\operatorname{Ind}(M) = k$, $\operatorname{rank}\left(M^k\right) = r$. *Then the WG inverse of M is the unique matrix $X \in \mathbb{C}_{n,n}$ satisfying the following equations:*

$$\begin{cases} (1)\ \left(M^k\right)^* M^2 X = \left(M^k\right)^* M, \\ (2)\ \mathscr{R}(X) \subseteq \mathscr{R}\left(M^k\right). \end{cases} \tag{25}$$

Proof: Let $M \in \mathbb{C}_{n,n}$ be as given in (1) and (2). Suppose that X satisfies the above equations, and is denoted by

$$X = U \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} U^*,$$

in which $X_{11} \in \mathbb{C}_{r,r}$. Since $\mathscr{R}(X) \subseteq \mathscr{R}(M^k)$, we obtain $X = M^k Y$. Denote

$$Y = U \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} U^*,$$

in which $Y_{11} \in \mathbb{C}_{r,r}$. Then, by applying (10), we get

$$\begin{aligned} U \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} U^* &= U \begin{bmatrix} T^k & \widehat{T} \\ 0 & 0 \end{bmatrix} U^* U \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} U^* \\ &= U \begin{bmatrix} T^k Y_{11} + \widehat{T} Y_{21} & T^k Y_{12} + \widehat{T} Y_{22} \\ 0 & 0 \end{bmatrix} U^*. \end{aligned}$$

Therefore, we obtain

$$X_{21} = 0 \text{ and } X_{22} = 0. \tag{26}$$

By applying (10) and (26) we have

$$\begin{aligned} (M^k)^* M^2 X &= U \begin{bmatrix} (T^k)^* & 0 \\ \widehat{T}^* & 0 \end{bmatrix} \begin{bmatrix} T & S \\ 0 & N \end{bmatrix}^2 \begin{bmatrix} X_{11} & X_{12} \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} (T^k)^* T^2 X_{11} & (T^k)^* T^2 X_{12} \\ \widehat{T}^* T^2 X_{11} & \widehat{T}^* T^2 X_{12} \end{bmatrix} U^*, \\ (M^k)^* M &= U \begin{bmatrix} (T^k)^* & 0 \\ \widehat{T}^* & 0 \end{bmatrix} \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^* \\ &= U \begin{bmatrix} (T^k)^* T & (T^k)^* S \\ \widehat{T}^* T & \widehat{T}^* S \end{bmatrix} U^*. \end{aligned} \tag{27}$$

Since $(M^k)^* M^2 X = (M^k)^* M$ and T is invertible, we obtain that both X_{11} and X_{12} are unique, and $X_{11} = T^{-1}$ and $X_{12} = T^{-2} S$. Thus, it follows from (26) that

$$X = U \begin{bmatrix} T^{-1} & T^{-2} S \\ 0 & 0 \end{bmatrix} U^*.$$

Therefore, by using (5), the WG inverse of M is the unique solution of (25). □

In [22], by applying the full-rank decomposition and elementary matrix operation, Sheng and Xin presented one Gauss-Jordan elimination method for core inverse. In [20] Ji and Wei generalized the algorithm and gave an algorithm for core-EP inverse. In the following discussion, by applying the full-rank decomposition, we give a characterization of WG inverse, and consider one Gauss-Jordan elimination method for WG inverse. More details of the Gauss-Jordan elimination method for generalized inverses can be seen in [20, 22–25].

Theorem 5 *Let $M \in \mathbb{C}_{n,n}$,* $\operatorname{Ind}(M) = k$ *and* $\operatorname{rank}\left(M^k\right) = r$, *and let $M^k = PQ$ be a full-rank decomposition of M^k. Then*

$$M^{ⓦ} = P\left(P^* M^2 P\right)^{-1} P^* M. \tag{28}$$

Proof: Let $M^k = PQ$ be a full-rank decomposition of M^k, where P is a full column matrix, and Q is a full row matrix. Since the index of M is k, we have $\operatorname{rank}(M^k) = \operatorname{rank}((M^k)^2)$, that is, M^k is group invertible. Therefore, QP is an r-by-r invertible matrix.

Let the core-EP decomposition of M be of the same form as (1). Then M_1 and M_2 are of the forms as in (2). By applying (10), we obtain a decomposition of M^k:

$$M^k = \left(U\begin{bmatrix} T^k \\ 0 \end{bmatrix}\right)\left(\begin{bmatrix} I_r & T^{-k}\widehat{T} \end{bmatrix} U^*\right). \tag{29}$$

It is obvious that the above decomposition is a full-rank decomposition of M^k, too. Write $L = \left(\begin{bmatrix} I_r & T^{-k}\widehat{T} \end{bmatrix} U^*\right) P \in \mathbb{C}_{r,r}$. Since M^k is group invertible, we have

$$\begin{aligned} r &= \operatorname{rank}(M^k) = \operatorname{rank}\left((M^k)^2\right) \\ &= \operatorname{rank}\left(\left(U\begin{bmatrix} T^k \\ 0 \end{bmatrix}\right) LQ\right) \leqslant \operatorname{rank}(L) \leqslant r. \end{aligned}$$

It follows that L is invertible. Therefore, there exists Y which satisfies

$$P = U\begin{bmatrix} T^k \\ 0 \end{bmatrix} Y = U\begin{bmatrix} T^k Y \\ 0 \end{bmatrix}, \tag{30}$$

where $Y = L(QP)^{-1} = \left(\begin{bmatrix} I_r & T^{-k}\widehat{T} \end{bmatrix} U^*\right) P(QP)^{-1} \in \mathbb{C}_{r,r}$. Since L and QP are invertible, Y is invertible.

By applying (30), we have

$$\begin{aligned} P^*M^2P &= \begin{bmatrix} Y^*(T^k)^* & 0 \end{bmatrix} U^*U \begin{bmatrix} T^2 & TS+SN \\ 0 & N^2 \end{bmatrix} U^*U \begin{bmatrix} T^kY \\ 0 \end{bmatrix} \\ &= Y^*(T^k)^*T^2T^kY, \end{aligned} \tag{31}$$

and

$$\begin{aligned} P^*M &= \begin{bmatrix} Y^*(T^k)^* & 0 \end{bmatrix} \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^* \\ &= \begin{bmatrix} Y^*(T^k)^*T & Y^*(T^k)^*S \end{bmatrix} U^*. \end{aligned}$$

Since Y and T are invertible, from (31), we conclude that

$$P^*M^2P \text{ is invertible.} \tag{32}$$

Therefore, we get

$$\begin{aligned} & P\left(P^*M^2P\right)^{-1} P^*M \\ &= U\begin{bmatrix} T^kY \\ 0 \end{bmatrix}\left(Y^*\left(T^k\right)^* T^2T^kY\right)^{-1}\begin{bmatrix} Y^*(T^k)^*T & Y^*(T^k)^*S \end{bmatrix} U^* \\ &= U\begin{bmatrix} I_r \\ 0 \end{bmatrix}(T^2)^{-1}\begin{bmatrix} T & S \end{bmatrix} U^* = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^*. \end{aligned}$$

It follows that we get (28). □

Example 1′ Let M be as in Example 1. We choose

$$P = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}.$$

By applying Theorem 5, we get

$$P^*M^2P = 81, \quad P^*M = \begin{bmatrix} 3 & -15 & 3 \end{bmatrix},$$

and

$$\begin{aligned} M^{ⓦ} &= P\left(P^*M^2P\right)^{-1} P^*M \\ &= \begin{bmatrix} 2/27 & 10/27 & -2/27 \\ 1/27 & 5/27 & -1/27 \\ -2/27 & -10/27 & 2/27 \end{bmatrix}. \end{aligned}$$

Based on Theorem 5, by applying elementary transformation, we give an algorithm for the WG inverse. By elementary column operation on M^k, we get $\begin{bmatrix} P & 0 \end{bmatrix}$. Then write

$$\mathscr{M} = \begin{bmatrix} P^*M^2P & P^*M \\ P & 0 \end{bmatrix}.$$

By elementary row operation on $\mathscr{M}$, we get $\mathscr{M}_1$:

$$\mathscr{M} = \begin{bmatrix} P^*M^2P & P^*M \\ P & 0 \end{bmatrix} \to \mathscr{M}_1 = \begin{bmatrix} I_r & \left(P^*M^2P\right)^{-1}P^*M \\ P & 0 \end{bmatrix}.$$

Furthermore, by elementary row operation on $\mathscr{M}_1$, we get $\mathscr{M}_2$:

$$\begin{aligned} \mathscr{M}_1 &= \begin{bmatrix} I_r & \left(P^*M^2P\right)^{-1} P^*M \\ P & 0 \end{bmatrix} \\ &\downarrow \\ \mathscr{M}_2 &= \begin{bmatrix} I_r & \left(P^*M^2P\right)^{-1} P^*M \\ 0 & -P\left(P^*M^2P\right)^{-1} P^*M \end{bmatrix}. \end{aligned}$$

Therefore, we have $M^{ⓦ} = P\left(P^*M^2P\right)^{-1} P^*M$.

In summary, we have the Gauss-Jordan elimination method for WG inverse:

Algorithm 1

Step 1: Input M, $k = \operatorname{Ind}(M)$ and $r = \operatorname{rank}(M^k)$;

Step 2: Perform elementary column operations on M^k to get $[P \mid 0]$, where $P \in \mathbb{C}_{n,r}$ and $\operatorname{rank}(P) = r$;

Step 3: Compute P^*M and P^*M^2P, form the block matrix $\mathscr{M}$, perform elementary row operations on the first s rows and convert it to $\mathscr{M}_1$;

Step 4: Perform elementary row operations on the block matrix in $\mathscr{M}_1$ to getrid of all the entries below the identity matrix I_r : $\mathscr{M}_2$'

Step 5: Output the WG inverse $M^{ⓦ} = P\left(P^*M^2P\right)^{-1}P^*M$.

Example 2 ([3, 20]) Compute the WG inverse $M^{ⓦ}$ for M, where

$$M = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix}.$$

Then $\mathrm{Ind}(M)=2$ and $\mathrm{rank}\left(M^2\right)=1$. Perform elementary column operations on

$$M^2=\begin{bmatrix}0&0&0\\5&3&1\\5&3&1\end{bmatrix},$$

then we get

$$P=\begin{bmatrix}0\\5\\5\end{bmatrix}.$$

Compute $P^*M^2P = 200$ and $P^*M = \begin{bmatrix}15 & 5 & 15\end{bmatrix}$, then

$$\mathscr{M}=\begin{bmatrix}P^*M^2P & P^*M\\ P & 0\end{bmatrix}=\left[\begin{array}{c|ccc}200&15&5&15\\\hline 0&0&0&0\\5&0&0&0\\5&0&0&0\end{array}\right].$$

In the blockmatrix $\mathscr{M}$, multiply the first row by 1/200:

$$\mathscr{M}=\begin{bmatrix}1&3/40&1/40&3/40\\0&0&0&0\\5&0&0&0\\5&0&0&0\end{bmatrix}.$$

Add the first row multiplied by −5 to the third row and fourth row, respectively:

$$\mathscr{M}=\begin{bmatrix}1&3/40&1/40&3/40\\0&0&0&0\\0&-3/8&-1/8&-3/8\\0&-3/8&-1/8&-3/8\end{bmatrix}.$$

Therefore, we get

$$M^{Ⓦ}=\begin{bmatrix}0&0&0\\3/8&1/8&3/8\\3/8&1/8&3/8\end{bmatrix}.$$

In the following Example 3, based on Theorem 5 and the above algorithm, we give an example to explain how to calculate the least-squares solution of the constrained matrix equation (12) by applying the Gauss-Jordan elimination method for WG inverse.

Example 3 Let

$$M=\begin{bmatrix}0&-1&2&-2\\-1&2&0&1\\1&-3&3&-4\\1&-3&2&-3\end{bmatrix}\quad\text{and}\quad D=\begin{bmatrix}1&2\\0&2\\1&0\\2&1\end{bmatrix}.$$

Then $\mathrm{Ind}(M)=2$, $\mathrm{rank}(M)=3$, $\mathrm{rank}\left(M^2\right)=2$,

$$M^2=\begin{bmatrix}1&-2&2&-3\\-1&2&0&1\\2&-4&3&-5\\2&-4&2&-4\end{bmatrix}\quad\text{and}\quad MD=\begin{bmatrix}-2&-4\\1&3\\-4&-8\\-3&-7\end{bmatrix}.$$

Perform elementary column operations on M^2, we get

$$P=\begin{bmatrix}1&2\\-1&0\\2&3\\2&2\end{bmatrix}.$$

Compute $P^*M^2P = \begin{bmatrix}10&12\\12&17\end{bmatrix}$ and $P^*MD = \begin{bmatrix}-17&-37\\-22&-46\end{bmatrix}$, then

$$\mathscr{M}=\begin{bmatrix}P^*M^2P & P^*Mb\\ P & 0\end{bmatrix}=\left[\begin{array}{cc|cc}10&12&-17&-37\\12&17&-22&-46\\\hline 1&2&0&0\\-1&0&0&0\\2&3&0&0\\2&2&0&0\end{array}\right].$$

In the blockmatrix $\mathscr{M}$, multiply the first block row by the inverse of P^*M^2P,

$$\left(P^*M^2P\right)^{-1}=\begin{bmatrix}17/26&-6/13\\-6/13&5/13\end{bmatrix}$$

we get

$$\mathscr{M}_1=\left[\begin{array}{cc|cc}1&0&-25/26&-77/26\\0&1&-8/13&-8/13\\\hline 1&2&0&0\\-1&0&0&0\\2&3&0&0\\2&2&0&0\end{array}\right].$$

Add the first block row multiplied by $-P$ to the second block row:

$$\mathscr{M}_2=\left[\begin{array}{cc|cc}1&0&-25/26&-77/26\\0&1&-8/13&-8/13\\\hline 0&0&57/26&109/26\\0&0&-25/26&-77/26\\0&0&49/13&101/13\\0&0&41/13&93/13\end{array}\right].$$

Therefore, we get

$$X=\begin{bmatrix}-57/26&-109/26\\25/26&77/26\\-49/13&-101/13\\-41/13&-93/13\end{bmatrix}.$$

On the other hand, by applying Theorem 1 and Theorem 2, we also get

$$M^{Ⓦ}=M^k\left(M^{k+2}\right)^{\dagger}M$$
$$=\begin{bmatrix}5/26&-31/26&2&-57/26\\-25/26&51/26&0&25/26\\10/13&-36/13&3&-49/13\\15/13&-41/13&2&-41/13\end{bmatrix}$$

and

$$X = M^{ⓦ} D = \begin{bmatrix} -57/26 & -109/26 \\ 25/26 & 77/26 \\ -49/13 & -101/13 \\ -41/13 & -93/13 \end{bmatrix}.$$

In Theorem 5, we see that the characterization (28) of WG inverse is based on the full-rank decomposition. But the interesting bit is that only the column part is used in the characterization (28). Based on those results, we give a new characterization in the following theorem.

Theorem 6 *Let* $M \in \mathbb{C}_{n,n}$, $\mathrm{Ind}(M) = k$ *and* $\mathrm{rank}\left(M^k\right) = r$. *Then*

$$M^{ⓦ} = T\left(T^* M^2 T\right)^{\dagger} T^* M, \tag{33}$$

where T *is an* $n \times n$ *matrix with* $\mathscr{R}(M^k) = \mathscr{R}(T)$.

Proof: Let $M^k = PQ$ and $T = T_1 T_2$ be full-rank decompositions of M^k and T, respectively. Since $\mathscr{R}(M^k) = \mathscr{R}(T)$, there exists Y satisfying $T_1 = PY$, in which Y is invertible. Therefore, we get a full-rank decomposition of T:

$$T = P(YT_2). \tag{34}$$

It follows that

$$\begin{aligned} T^* M^2 T &= (P(YT_2))^* M^2 (P(YT_2)) \\ &= \left(T_2^* Y^* P^* M^2 P\right) Y T_2. \end{aligned} \tag{35}$$

Since Y is invertible, by applying (32), we conclude that (35) is a full-rank decomposition of $T^* M^2 T$. By applying (32), (34), and (35), we obtain

$$\begin{aligned} &T\left(T^* M^2 T\right)^{\dagger} T^* M \\ &\quad = P(YT_2)\left(\left(T_2^* Y^* P^* M^2 P\right) Y T_2\right)^{\dagger} T_2^* Y^* P^* M \\ &\quad = P(YT_2)(YT_2)^* ((YT_2)(YT_2)^*)^{-1} \\ &\qquad \times \left(\left(T_2^* Y^* P^* M^2 P\right)^* \left(T_2^* Y^* P^* M^2 P\right)\right)^{-1} \\ &\qquad \times \left(T_2^* Y^* P^* M^2 P\right)^* T_2^* Y^* \left(P^* M^2 P\right)\left(P^* M^2 P\right)^{-1} P^* M \\ &\quad = P\left(P^* M^2 P\right)^{-1} P^* M. \end{aligned}$$

By applying Theorem 5, it follows that we deduce (33). □

Example 3′ Let M and D be as given in Example 3, $\mathrm{Ind}(M) = 2$, and

$$T = \begin{bmatrix} -5 & 0 & -10 & 2 \\ 1 & 2 & 14 & 2 \\ -8 & -1 & -22 & 2 \\ -6 & -2 & -24 & 0 \end{bmatrix}.$$

It is obvious that $\mathscr{R}(M^2) = \mathscr{R}(T)$. By applying Theorem 2, we get

$$\begin{aligned} M^{ⓦ} &= M^2\left(M^4\right)^{\dagger} M \\ &= \begin{bmatrix} 5/26 & -31/26 & 2 & -57/26 \\ -25/26 & 51/26 & 0 & 25/26 \\ 10/13 & -36/13 & 3 & -49/13 \\ 15/13 & -41/13 & 2 & -41/13 \end{bmatrix}. \end{aligned} \tag{36}$$

On the other hand, by applying Theorem 6, we have

$$\begin{aligned} &\left(T^* M^2 T\right)^{\dagger} \\ &= \begin{bmatrix} 152/8599 & -27/4031 & -236/48805 & -143/9465 \\ -27/4031 & 62/24235 & 19/9726 & 129/22438 \\ -236/48805 & 19/9726 & 5/2439 & 59/13790 \\ -143/9465 & 129/22438 & 59/13790 & 79/6104 \end{bmatrix} \end{aligned}$$

and

$$T\left(T^* M^2 T\right)^{\dagger} T^* M = \begin{bmatrix} 5/26 & -31/26 & 2 & -57/26 \\ -25/26 & 51/26 & 0 & 25/26 \\ 10/13 & -36/13 & 3 & -49/13 \\ 15/13 & -41/13 & 2 & -41/13 \end{bmatrix}.$$

By applying (36), it follows that $M^{ⓦ} = T\left(T^* M^2 T\right)^{\dagger} T^* M$.

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