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Single-valley-extended solutions for the generalized second type of Feigenbaum-Kadanoff-Shenker equation

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ABSTRACT: In the present paper, the second type of Feigenbaum-Kadanoff-Shenker equation is generalized to a more broader form. Construction methods for single-valley-extended continuous solutions of this generalized equation are given.

KEYWORDS: Feigenbaum-Kadanoff-Shenker equation, single-valley solution, multi-valley solution

MSC2020: 37E20 39B12 39B22

INTRODUCTION

In 1978, Feigenbaum [1, 2] introduced the notion of renormalization for real dynamical systems. It is found that the fixed point of the period-doubling renormalization operator satisfies the Feigenbaum equation

$$\begin{cases} g(g(-\lambda x)) = -\lambda g(x), & 0 < \lambda < 1, \\ g(0) = 1, \ -1 \leqslant g(x) \leqslant 1, & x \in [-1,1], \end{cases} \tag{1}$$

where $g : [-1,1] \to [-1,1]$ is the unknown function. In order to describe the quasi-periodic route to chaos for mappings of the circle, Feigenbaum, Kadanoff and Shenker [3] presented the following equation

$$\begin{cases} g(g(\epsilon^2 x)) = -\epsilon g(x), & 0 < \epsilon < 1, \\ g(0) = 1, \ -1 \leqslant g(x) \leqslant 1, & x \in [-1,1], \end{cases} \tag{2}$$

which is called the Feigenbaum-Kadanoff-Shenker (FKS) equation [4]. For seeking solutions of Eqs. (1) and (2) in some appropriate function spaces, a lot of interesting results were obtained. We refer the reader to Refs. [5]–[16] and references therein.

A function $g : [-1,1] \to [-1,1]$ is called a single-peak even solution of the Eq. (1) (or Eq. (2)) if g is a continuous even solution of Eq. (1) (or Eq. (2)) and g is strictly increasing on $[-1,0]$ and strictly decreasing on $[0,1]$.

For the sake of simplify the research of single-peak even solutions of Eq. (1), Yang and Zhang [17] provided the second type of Feigenbaum equation

$$\begin{cases} f(f(\lambda x)) = \lambda f(x), & \lambda \in (0,1), \\ f(0) = 1, \ 0 \leqslant f(x) \leqslant 1, & x \in [0,1], \end{cases} \tag{3}$$

where $f : [0,1] \to [0,1]$ is the unknown function. For the same reason Shi [18] put forward the second type of FKS equation

$$\begin{cases} f(f(\epsilon^2 x)) = \epsilon f(x), & \epsilon \in (0,1), \\ f(0) = 1, \ 0 \leqslant f(x) \leqslant 1, & x \in [0,1], \end{cases} \tag{4}$$

where $f : [0,1] \to [0,1]$ is the unknown function.

A function $f : [a,b] \to \mathbb{R}$ is called a single-valley function on $[a,b]$ if f is continuous and there exists a $c \in (a,b)$ such that f is strictly decreasing on $[a,c]$ and strictly increasing on $[c,b]$.

A continuous solution $f : [0,1] \to [0,1]$ of Eq. (3) (or Eq. (4)) is called a single-valley solution if f is a single-valley function on $[0,1]$.

Finding a single-peak even solution of Eq. (1) is equivalent to finding a single-valley solution of Eq. (3). The same fact holds for Eqs. (2) and (4). By extending a suitable single-valley function construction methods to obtain all single-valley solutions of corresponding equations were given in [17, 18].

In 1988, Liao [19] found that by extending single-valley functions two different kinds of solutions of Eq. (3) could be obtained. All solutions of the first kind are single-valley solutions. All solutions of the second kind possess infinitely many local extrema. Since all of them are obtained by extending a single-valley function, thus he named them as single-valley-extended solutions.

In 2021, Shi [20] also found that the same fact held for Eq. (4). That is, by extending single-valley functions, two different kinds of solutions of Eq. (4) can also be obtained. He also named them as single-valley-extended solutions. Both Eqs. (3) and (4) are interesting objects to investigate. For the other new results of Eqs. (3) and (4), we refer the reader to Refs. [21]–[24].

In 2011, Zhang and Si [25] put forward a class of generalized Feigenbaum equations

$$\begin{cases} f(f(h(x))) = h(f(x)), \\ f(0) = 1, \ -1 \leqslant f(x) \leqslant 1, & x \in [-1,1], \end{cases} \tag{5}$$

where $f : [-1,1] \to [-1,1]$ is the unknown function. The existence and uniqueness of C^∞ even solutions of

Eq. (5) are considered. It is easy to see that, when $h(x) = -\lambda x$, Eq. (5) is just Eq. (1). In 2014, Zhang [26,27] considered the second type of Feigenbaum equation in a more broader sense

$$\begin{cases} f(f(h(x))) = h(f(x)), \\ f(0) = 1,\ 0 \leqslant f(x) \leqslant 1, \quad x \in [0,1], \end{cases} \tag{6}$$

where $f : [0,1] \to [0,1]$ is the unknown function. The author discussed the existence of single-valley-extended continuous solutions and C^∞ solutions of the above equation (6). In fact, when $h(x) = \lambda x$, Eq. (6) reduces to Eq. (3).

Inspired by the above work we will consider the second type of FKS equation in a more broader sense

$$\begin{cases} f(f(h^2(x))) = h(f(x)), \\ f(0) = 1,\ 0 \leqslant f(x) \leqslant 1, \quad x \in [0,1], \end{cases} \tag{7}$$

where h is a strictly increasing continuous function on $[0,1]$ with

$$h(0) = 0,\ h(x) < x, \quad x \in (0,1],$$

and h^2 is the 2-fold iteration of h. When $h(x) = \epsilon x$, Eq. (7) is just Eq. (4).

We find that by extending single-valley functions two different classes of solutions of Eq. (7) can be obtained. All solutions of the first class are also single-valley solutions. All solutions of the second class possess infinitely many local minimum points, thus we call them multi-valley solutions. Since all of them are obtained by extending a single-valley function, thus we also name them as single-valley-extended solutions. The concrete definition is as following.

Definition 1 A function $f : [0,1] \to [0,1]$ is called a single-valley-extended solution of Eq. (7) if f is a continuous solution of Eq. (7) and $f|_{[h^2(1),1]}$ is a single-valley function.

By extending a suitable strictly decreasing function, Yang and Zhang [17] provided another construction method to obtain the single-valley solution of Eq. (3). Thus it is an interesting problem that by extending a suitable strictly decreasing function can we get a single-valley solution of Eq. (7). We will give a positive answer to the above problem under some suitable conditions.

PROPERTIES OF SINGLE-VALLEY-EXTENDED SOLUTIONS

Theorem 1 *If f is a single-valley-extended solution of Eq. (7), then the following facts hold:*

(i) $f(1) = h(1)$, $f(f(h^2(1))) = h^2(1)$*;*

(ii) *if the unique global minimum point of f on $[0,1]$ is α, then $f(\alpha) = 0$;*

(iii) *for $x \in \left[0, h^2(1)\right]$, $f(x) = \alpha \Leftrightarrow x = h^2(\alpha)$;*

(iv) *the unique $x \in [\alpha, 1]$ satisfying $f(x) = h(x)$ is $x = 1$;*

(v) $f(h^2(1)) \neq \alpha$.

Proof: (i) Put $x = 0$ in Eq. (7), we have $f(1) = h(1)$. Then

$$f(f(h^2(1))) = h(f(1)) = h^2(1).$$

(ii) Let $\mu \in (0,1]$ be the global minimum point of f, we obtain that

$$h(f(\mu)) = f(f(h^2(\mu))) \geqslant f(\mu).$$

Since $h(0) = 0$ and $h(x) < x$ for $x \in (0,1]$, we get $f(\mu) = 0$. Assume that $\mu \leqslant h^2(1)$, then $0 < h^{-2}(\mu) \leqslant 1$. Note that

$$h(f(h^{-2}(\mu))) = f(f(\mu)) = f(0) = 1$$

contradicts the fact that $h(x) < x$, $x \in (0,1]$, so $\mu > h^2(1)$ is proved. Since μ is the global minimum point and $f|_{[h^2(1),1]}$ is a single-valley function, we get $\mu = \alpha$.

(iii) If $x \in [0, h^2(1)]$, by the uniqueness of the point α, we have

$$\begin{aligned} f(x) = \alpha &\Leftrightarrow f^2(x) = 0 \\ &\Leftrightarrow h(f(h^{-2}(x))) = 0 \\ &\Leftrightarrow f(h^{-2}(x)) = 0 \\ &\Leftrightarrow h^{-2}(x) = \alpha \\ &\Leftrightarrow x = h^2(\alpha). \end{aligned}$$

(iv) Obviously, $f(1) = h(1)$ and $f(\alpha) = h(\alpha)$. If $\xi \in (\alpha, 1]$ satisfies $f(x) = h(x)$, there exists a $\beta \in [0, h^2(\alpha)) \subset [0, \alpha)$ such that $f(\beta) = \xi$ because $f(0) = 1$, $f(h^2(\alpha)) = \alpha$. Thus, we have

$$f(f(h^2(\beta))) = h(f(\beta)) = h(\xi) = f(\xi).$$

Since $0 < h^2(\beta) < h^2(\alpha)$, by means of (iii) we know that $f(h^2(\beta)) > \alpha$. Note that $f|_{[\alpha,1]}$ is strictly increasing, then $f(h^2(\beta)) = \xi$. By induction, we get $f(h^{2n}(\beta)) = \xi$ $(n = 1, 2, \ldots)$, implying that $\xi = 1$. This shows the uniqueness of the solution.

(v) If $f(h^2(1)) = \alpha$, we have

$$f(f(h^2(1))) = f(\alpha) = 0,$$

which contradicts the fact (i) of Theorem 1. □

Let h be a self-mapping. For integers $n \geqslant 0$, define the n-th iteration of h by $h^n = h \circ h^{n-1}$ and $h^0 = id$, where id denotes the identity mapping and $\circ$ denotes the composition of mappings. If f be a single-valley-extended solution of equation (7), we see that there exists a unique minimum point $\alpha \in (h^2(1), 1)$. For convenience, denote

$$f_- = f|_{[h^2(1),\alpha]}, \qquad f_+ = f|_{[\alpha,1]}.$$

Clearly, f_- is strictly decreasing and f_+ is strictly increasing.

Theorem 2 *Let f be a single-valley-extended solution of Eq. (7). If $f(h^2(1)) < \alpha$ holds, then $f|_{[0,\alpha]}$ is strictly decreasing.*

Proof: Since $f(0) = 1 > \alpha$, $f(h^2(1)) < \alpha$ and f is continuous, by virtue of (iii) of Theorem 1, we know that

$$f([0,h^2(\alpha))) \subset (\alpha,1], \quad f((h^2(\alpha),h^2(1)]) \subset [0,\alpha).$$

For each integer $n \geqslant 1$, $f|_{[h^{2n+2}(1),h^{2n}(1)]}$ is strictly decreasing holds. When $n = 1$, for $x \in [h^2(\alpha),h^2(1)]$ we see that $h^{-2}(x) \in [\alpha,1]$ and

$$f(f|_{[h^2(\alpha),h^2(1)]}(x)) = h(f_+(h^{-2}(x))).$$

Since f_+ is strictly increasing, $f|_{[h^2(\alpha),h^2(1)]}$ must be strictly monotonic. Note that $f(h^2(1)) < \alpha = f(h^2(\alpha))$, then $f|_{[h^2(\alpha),h^2(1)]}$ is strictly decreasing. Similarly, when $n = 1$ and $x \in [h^4(1),h^2(\alpha)]$, we have $h^{-2}(x) \in [h^2(1),\alpha]$ and

$$f_+(f|_{[h^4(1),h^2(\alpha)]}(x)) = h(f_-(h^{-2}(x))),$$

implying that $f|_{[h^4(1),h^2(\alpha)]}$ is strictly decreasing. Thus, $f|_{[h^4(1),h^2(1)]}$ is strictly decreasing.

By induction, we can prove that $f|_{[h^{2n+2}(1),h^{2n}(1)]}$ is strictly decreasing for each integer $n \geqslant 2$. Note that

$$f(0) = 1, \quad (0,\alpha] = \bigcup_{n=1}^{\infty}[h^{2n+2}(1),h^{2n}(1)]\bigcup[h^2(1),\alpha]$$

and f is continuous, we say that $f|_{[0,\alpha]}$ is strictly decreasing. □

Theorem 3 *Let f be a single-valley-extended solution of Eq. (7). If $f(h^2(1)) > \alpha$, then*

(1) $f|_{[h^{2n}(\alpha),h^{2n}(1)]}$ is strictly increasing;

(2) $f|_{[h^{2n+2}(1),h^{2n}(\alpha)]}$ is strictly decreasing

for any integer $n \geqslant 0$.

Proof: When $n = 0$, with the help of Definition 1 we see that $f|_{[\alpha,1]}$ is strictly increasing and $f|_{[h^2(1),\alpha]}$ is strictly decreasing. For $x \in [h^4(1),h^2(1)]$, by the continuity of f and $f(h^2(1)) > \alpha$ and (iii) of Theorem 1, we get that $f([h^4(1),h^2(1)]) \subset [\alpha,1]$.

When $n = 1$, for $x \in [h^2(\alpha),h^2(1)]$ we have

$$f_+(f(x)) = h(f_+(h^{-2}(x))),$$

i.e.,

$$f(x) = f_+^{-1}(h(f_+(h^{-2}(x)))).$$

Since $h^{-2}(x) \in [\alpha,1]$ and f_+ is strictly increasing, we see that $f|_{[h^2(\alpha),h^2(1)]}$ is strictly increasing.

Similarly, for $x \in [h^4(1),h^2(\alpha)]$ we have

$$f(x) = f_+^{-1}(h(f_-(h^{-2}(x)))),$$

implying $f|_{[h^4(1),h^2(\alpha)]}$ is strictly decreasing. By induction we can prove that (1) and (2) hold for all integers $n \geqslant 2$. □

Remark 1 For a single-valley-extended solution f, we see that f is a single-valley solution if $f(h^2(1)) < \alpha$ and it is a multi-valley solution if $f(h^2(1)) > \alpha$.

THE FIRST CONSTRUCTION METHOD FOR SINGLE-VALLEY SOLUTIONS

Lemma 1 *If f is a single-valley solution of Eq. (7), then*

$$h^2(\alpha) < h^2(1) \leqslant f(h^2(1)) < \alpha.$$

Proof: By Theorem 2, it suffices to consider $h^2(\alpha) < f(h^2(1))$ and $h^2(1) \leqslant f(h^2(1))$, respectively.

Since $h^2(\alpha) < h^2(1) < \alpha$, we have

$$f(h^2(\alpha)) = \alpha > h^2(1) = f(f(h^2(1))).$$

From $f(h^2(1)) < \alpha$ and $f|_{[0,\alpha]}$ being strictly decreasing, we get $f|_{[0,\alpha]}(h^2(\alpha)) > f|_{[0,\alpha]}(f(h^2(1)))$ and $h^2(\alpha) < f(h^2(1))$, respectively.

If $f(h^2(1)) < h^2(1)$, i.e., $h^{-2}(f(h^2(1))) < 1$. By Eq. (7) we have

$$h \circ f(h^{-2}(f(h^2(1)))) = f(f(f(h^2(1)))) = f(h^2(1)),$$

implying

$$f(h^{-2}(f(h^2(1)))) = h(h^{-2}(f(h^2(1)))).$$

Since $h^2(\alpha) < f(h^2(1)) < h^2(1)$, we know that $\alpha < h^{-2}(f(h^2(1))) < 1$. By the above discussion we say that $h^{-2}(f(h^2(1)))$ is also a solution of the equation $f(x) = h(x)$ on $[\alpha,1]$, which contradicts the item (iv) of Theorem 1. □

As in [17], for any given constants X and Y, let $[X;Y]$ denote the closed interval whose endpoints are X and Y, i.e. when $X < Y$, $[X;Y] = [X,Y]$ and when $X > Y$, $[X;Y] = [Y,X]$.

Theorem 4 *Suppose that the given constants α, a and the strictly decreasing continuous function $\psi : [0,a] \to [h^2(1),1]$ satisfy $h^2(1) \leqslant a < \alpha < 1$ and*

P_1. $\psi(0) = 1$, $\psi(a) = h^2(1)$, $\psi(h^2(1)) = a$ and $\psi(h^2(\alpha)) = \alpha$,

P_2. the function $\phi(x) = \psi(h^2(x))$ has the unique periodic point α on the interval $[0,1]$,

there exists a single-valley solution f of Eq. (7) satisfies $f|_{[0,a]} = \psi$ and $f(\alpha) = 0$.

Proof: Clearly, ϕ^2 is strictly increasing on $[0,1]$. We have

$$\phi^0(0) = 0, \ \ \phi^1(0) = \psi(0) = 1, \ \ \phi^2(0) = \psi(h^2(1)) = a$$

and

$$\phi^3(0) = \phi(\phi^2(0)) = \psi(h^2(a)).$$

It is easy to verify that

$$\phi^0(0) < \phi^2(0) < \phi^3(0) < \phi^1(0).$$

Thus, the sequence $\{\phi^{2k}(0)\}_{k=0}^{\infty}$ is strictly increasing and $\{\phi^{2k+1}(0)\}_{k=0}^{\infty}$ is strictly decreasing. From P_2 the two series possess the common limit α.

Denote

$$\Delta_k := [\phi^k(0); \phi^{k+2}(0)], \quad k = 0, 1, 2, \dots.$$

The solution f will be defined on Δ_k by induction. Let

$$f_0(x) = \psi(x), \quad x \in \Delta_0.$$

For $k = 1, 2, \dots$ we can define the continuous and monotonic function f_k on Δ_k by

$$f_k(x) = h(f_{k-1}(\phi^{-1}(x))). \tag{8}$$

Putting $x = \phi^k(0)$ and $x = \phi^{k+2}(0)$ in (8), respectively, we have

$$f_k(\phi^k(0)) = h(f_{k-1}(\phi^{-1}(\phi^k(0)))) = h(f_{k-1}(\phi^{k-1}(0)))$$

and

$$\begin{aligned} f_k(\phi^{k+2}(0)) &= h(f_{k-1}(\phi^{-1}(\phi^{k+2}(0)))) \\ &= h(f_{k-1}(\phi^{k+1}(0))). \end{aligned}$$

Consequently, we know that

$$\begin{aligned} f_k(\phi^k(0)) &= h(f_{k-1}(\phi^{k-1}(0))) = h^2(f_{k-2}(\phi^{k-2}(0))) \\ &= \cdots = h^k(f_0(0)) = h^k(1) \end{aligned} \tag{9}$$

and

$$\begin{aligned} f_k(\phi^{k+2}(0)) &= h(f_{k-1}(\phi^{k+1}(0))) = h^2(f_{k-2}(\phi^k(0))) \\ &= \cdots = h^k(f_0(\phi^2(0))) = h^k(f_0(a)) = h^{k+2}(1). \end{aligned} \tag{10}$$

From (9) and (10) we get

$$f_k(\phi^{k+2}(0)) = f_{k+2}(\phi^{k+2}(0)) = h^{k+2}(1), \tag{11}$$

implying that f_k and f_{k+2} coincide at the common endpoint of Δ_k and Δ_{k+2}.

Define

$$f(x) := \begin{cases} f_k(x), & x \in \Delta_k, \\ 0, x = \alpha. \end{cases}$$

From the strictly monotonicity of f_k and Eq. (11), we know that f is continuous on $[0, \alpha)$ and $(\alpha, 1]$. Moreover, by using (9) and the fact

$$\lim_{x\to\alpha} f(x) = \lim_{k\to\infty} f_k(\phi^k(0)) = \lim_{k\to\infty} h^k(1) = 0,$$

we say that f is also continuous at α. Thus, f is a single-valley solution of Eq. (7). □

Lemma 2 *([17]) Let continuous function $g : [a, b] \to [a, b]$ be strictly decreasing. If $c \in [a, b]$ is a periodic point, then c is either a fixed point or a 2-periodic point.*

Example 1 Let $h(x) = x/2$, $\alpha = 1/2$, $a = 2/5$, we have $h^2(1) < a < \alpha$. Define

$$\psi(x) = \begin{cases} -4x + 1, & x \in [0, \frac{1}{8}]; \\ -\frac{4}{5}x + \frac{3}{5}, & x \in [\frac{1}{8}, \frac{1}{4}]; \\ -x + \frac{13}{20}, & x \in [\frac{1}{4}, \frac{2}{5}]. \end{cases}$$

By simple calculation, we get

$$\psi(0) = 1,\ \psi(\frac{1}{4}) = \frac{2}{5},\ \psi(\frac{2}{5}) = \frac{1}{4},\ \psi(\frac{1}{8}) = \frac{1}{2},$$

implying P_1 holds.

Define

$$\phi(x) = \begin{cases} -x + 1, & x \in [0, \frac{1}{2}]; \\ -\frac{1}{5}x + \frac{3}{5}, & x \in [\frac{1}{2}, 1]. \end{cases}$$

Obviously, ϕ has a unique fixed point $\alpha = 1/2$. Since ϕ is strictly decreasing, the other periodic points must be 2-periodic points. If ϕ has a 2-periodic point x, without any loss of generality, let $x \in [0, 1/2]$. Then

$$x = -\frac{1}{5}(-x + 1) + \frac{3}{5}$$

yields that $x = 1/2$, implying P_2 holds.

By Theorem 4, there exists a unique single-valley solution f of Eq. (7) satisfying $f|_{[0,2/5]} = \psi$ (see Fig. 1).

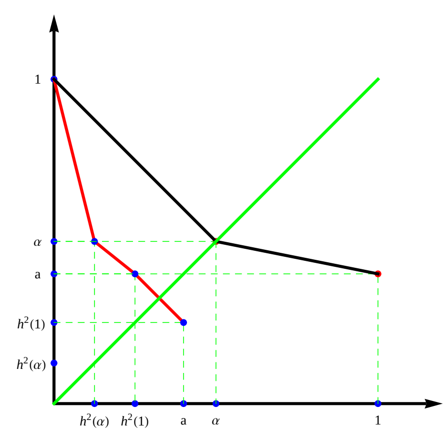

Fig. 1 The graphs of ψ and ϕ.

Example 2 Let $h(x) = \sqrt{2/5}x$, $alpha = 1/2$, $a = 2/5$, we have $h^2(1) = a < \alpha$. Define

$$\psi_1(x) = \begin{cases} -\frac{5}{2}x + 1, & x \in [0, \frac{1}{5}]; \\ -\frac{1}{2}x + \frac{3}{5}, & x \in [\frac{1}{5}, \frac{2}{5}], \end{cases}$$

and

$$\phi_1(x)=\begin{cases}-x+1, & x\in[0,\frac{1}{2}];\\ -\frac{1}{5}x+\frac{3}{5}, & x\in[\frac{1}{2},1].\end{cases}$$

By simple calculation, we know that ϕ_1 has the unique fixed point $\alpha=1/2$ and P_1 and P_2 hold by the similar arguments as that of Example 1. So, there exists a unique single-valley solution $\hat{f}$ of Eq. (7) satisfying $\hat{f}|_{[0,2/5]}=\psi_1$ from Theorem 4 (Fig. 2).

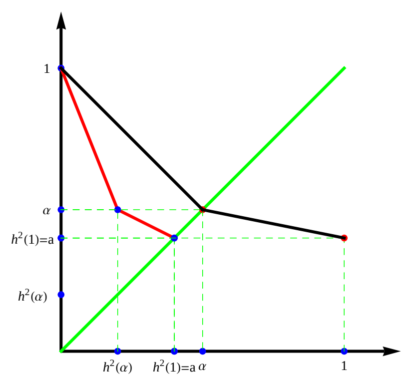

Fig. 2 The graphs of ψ_1 and ϕ_1.

THE SECOND CONSTRUCTION METHOD FOR SINGLE-VALLEY SOLUTIONS

Theorem 5 *Suppose that the given constants α, a and the single-valley function $\psi:[h^2(1),1]\to[0,1]$ satisfy $h^2(1)\leqslant a<\alpha<1$ and*

- P_3. *$\psi(h^2(1))=a$, $\psi(a)=h^2(1)$, $\psi(\alpha)=0$ and $\psi(1)=h(1)$,*
- P_4. *the equation $\psi(x)=h(x)$ has the unique solution $x=1$ on $[\alpha,1]$,*

there exists a single-valley solution f of Eq. (7) satisfying $f|_{[h^2(1),1]}=\psi$.

Proof: It is easy to verify that $(0,1]=\bigcup_{n=0}^{\infty}[h^{2n+2}(1),h^{2n}(1)]$. Denote

$$\psi_-=\psi|_{[h^2(1),\alpha]},\quad \psi_+:=\psi|_{[\alpha,1]},$$

$$\varphi_0(x)=\psi(x),\quad x\in[h^2(1),1],\tag{12}$$

and

$$\varphi_1(x)=\begin{cases}\psi_-^{-1}\circ h\circ\psi_+\circ h^{-2}(x), & x\in[h^2(\alpha),h^2(1)],\\ \psi_+^{-1}\circ h\circ\psi_-\circ h^{-2}(x), & x\in[h^4(1),h^2(\alpha)],\end{cases}\tag{13}$$

and for $k=2,3,\dots,$

$$\varphi_k(x)=\psi_+^{-1}\circ h\circ\varphi_{k-1}\circ h^{-2}(x),\ x\in[h^{2k+2}(1),h^{2k}(1)].\tag{14}$$

Let

$$\varphi(x)=\begin{cases}1, & x=0,\\ \varphi_k(x), & x\in[h^{2k+2}(1),h^{2k}(1)],\ k=0,1,2,\dots.\end{cases}$$

It is easy to see that φ_k is strictly decreasing and continuous on $[h^{2k+2}(1),h^{2k}(1)]$. For each $k\geqslant 1$, φ_k is strictly decreasing. By means of (12) and (13), we know that

$$\varphi_1(h^2(1))=\psi_-^{-1}\circ h\circ\varphi_0(1)=\psi_-^{-1}(h^2(1))$$
$$=a=\psi(h^2(1))=\varphi_0(h^2(1)).$$

By induction, we get that

$$\varphi_k(h^{2k}(1))=\varphi_{k-1}(h^{2k}(1)),\quad k=1,2,\dots,$$

implying that φ is continuous on $(0,1]$.

Since $\varphi|_{(0,\alpha]}$ is strictly decreasing and bounded, we know that $\lim_{x\to 0^+}\varphi(x)=\xi$ exists. By virtue of (14), we get

$$\xi=\lim_{k\to\infty}\varphi_k(h^{2k+2}(1))=\lim_{k\to\infty}\psi_+^{-1}\circ h\circ\varphi_{k-1}(h^{2k}(1))$$
$$=\psi_+^{-1}\circ h(\xi),$$

then

$$\psi_+(\xi)=h(\xi).$$

From P_4 we have $\xi=1$. Thus φ is a single-valley solution of Eq. (7). □

Example 3 Let $h(x)=x^2/2$, $\alpha=1/2$, $a=1/4$ and

$$\psi(x)=\begin{cases}-x+\frac{3}{8}, & x\in[\frac{1}{8},\frac{1}{4}];\\ -\frac{x}{2}+\frac{1}{4}, & x\in[\frac{1}{4},\frac{1}{2}];\\ x-\frac{1}{2}, & x\in[\frac{1}{2},1].\end{cases}$$

Since

$$h^2(1)<a<\alpha,\quad h^2(1)=1/8,$$

there exists a unique single-valley solution f of Eq. (7) by using Theorem 5, which satisfies $f|_{[1/8,1]}=\psi$ (see Fig. 3).

Example 4 Let $h(x)=x^2/2$, $\alpha=1/2$, $a=1/8$ and

$$\tilde{\psi}(x)=\begin{cases}-\frac{x}{3}+\frac{1}{6}, & x\in[\frac{1}{8},\frac{1}{2}];\\ x-\frac{1}{2}, & x\in[\frac{1}{2},1].\end{cases}$$

Note that

$$h^2(1)=a<\alpha,\quad h^2(1)=1/8,$$

By Theorem 5, there exists a unique single-valley solution $\tilde{f}$ of Eq. (7) satisfying $\tilde{f}|_{[1/8,1]}=\tilde{\psi}$ (see Fig. 4).

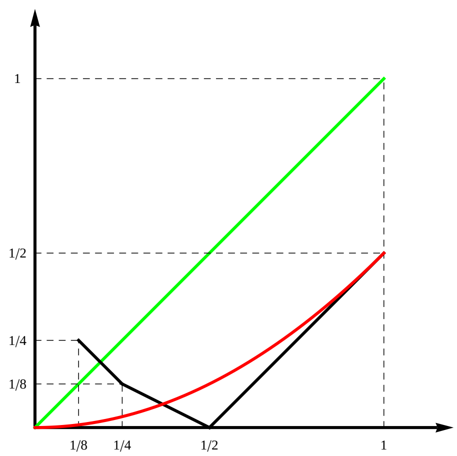

Fig. 3 The graphs of ψ and h.

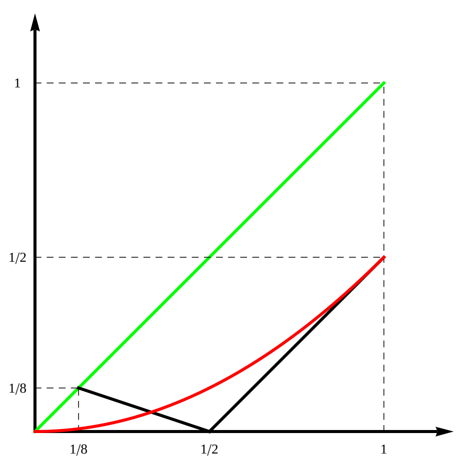

Fig. 4 The graphs of $\tilde{\psi}$ and h.

THE CONSTRUCTION METHOD FOR MULTI-VALLEY SOLUTIONS

Theorem 6 *Let $\varphi : [h^2(1), 1] \to [0, 1]$ be a single-valley function and $\alpha \in (h^2(1), 1)$ the unique minimum point of φ. If φ satisfies the following conditions*

P_8. *$\varphi(1) = h(1)$, $\varphi(\varphi(h^2(1))) = h^2(1)$, $\varphi(h^2(1)) > \alpha$;*

P_9. *$\varphi(\alpha) = 0$;*

P_{10}. *the unique solution $x \in [\alpha, 1]$ of the equation $\varphi(x) = h(x)$ is $x = 1$,*

there exists a multi-valley solution f of Eq. (7) satisfying $f|_{[h^2(1),1]} = \varphi$.

Proof: Denote

$$\varphi_- := \varphi|_{[h^2(1),\alpha]}, \quad \varphi_+ := \varphi|_{[\alpha,1]}$$

and

$$I_n := [h^{2n+2}(1), h^{2n}(1)], \quad n = 0, 1, 2, \dots.$$

Let $f_0 = \varphi$, then f_0 is well defined and continuous. For $n \geqslant 1$, define inductively

$$f_n(x) := \varphi_+^{-1}(h(f_{n-1}(h^{-2}(x)))). \tag{15}$$

Since φ_+^{-1} is also continuous, then each f_n is well defined and continuous on I_n. Let

$$f(x) = \begin{cases} 1, & x = 0, \\ f_n(x), & x \in I_n. \end{cases}$$

First, we show that f_{n+1} and f_n coincide at the point $\{h^{2n+2}(1)\} = I_{n+1} \bigcap I_n$ for any $n \geqslant 0$. From P_8 we have

$$\varphi(h^2(1)) < 1.$$

Otherwise, if $\varphi(h^2(1)) = 1$ we get

$$\varphi(1) = h^2(1) = h(1)$$

from $\varphi(\varphi(h^2(1))) = h^2(1)$, which contradicts $0 < h(1) < 1$. Thus, we get

$$\alpha < \varphi(h^2(1)) < 1,$$

consequently, P_8 can be rewrite as

$$f(f(h^2(1))) = \varphi_+(f_0(h^2(1))) = h^2(1). \tag{16}$$

When $n = 0$, by (16) we have

$$\begin{aligned} f_1(h^2(1)) &= \varphi_+^{-1}(h(f_0(h^2(1)/h^2(1)))) = \varphi_+^{-1}(h^2(1)) \\ &= \varphi_+^{-1}(\varphi_+(f_0(h^2(1)))) = f_0(h^2(1)). \end{aligned}$$

Assume that (15) holds for $n = k$, i.e.,

$$f_k(h^{2k}(1)) = f_{k-1}(h^{2k}(1)).$$

When $n = k + 1$, from the definitions of f_{k+1} and f_k we have

$$\begin{aligned} f_{k+1}(h^{2k+2}(1)) &= \varphi_+^{-1}(h(f_k(h^{-2}(h^{2k+2}(1))))) \\ &= \varphi_+^{-1}(h(f_k(h^{2k}(1)))) \\ &= \varphi_+^{-1}(h(f_{k-1}(h^{2k}(1)))) \\ &= \varphi_+^{-1}(h(f_{k-1}(h^{-2}(h^{2k+2}(1))))) \\ &= f_k(h^{2k+2}(1)). \end{aligned}$$

Thus, we say that f is continuous on $(0, 1]$.

Second, we prove that f is right continuous at $x = 0$. Consider the sequence $\{f_n(h^{2n}(\alpha))\}_{n=1}^{\infty}$. Clearly,

$$f_1(h^2(\alpha)) = \varphi_+^{-1}(h(f_0(h^{-2}(h^2(\alpha))))) = \varphi_+^{-1}(0) = \alpha$$

and

$$f_2(h^4(\alpha)) = \varphi_+^{-1}(h(f_1(h^{-2}(h^4(\alpha))))) = \varphi_+^{-1}(h(\alpha)) > \varphi_+^{-1}(0) = \alpha = f_1(h^2(\alpha)).$$

From the above discussion we see that

$$\alpha = f_1(h^2(\alpha)) < f_2(h^4(\alpha)) = \varphi_+^{-1}(h(\alpha)) < \varphi_+^{-1}(h(1)) = 1.$$

By induction we get

$$\alpha = f_1(h^2(\alpha)) < f_2(h^4(\alpha)) < \cdots < f_n(h^{2n}(\alpha)) < \cdots < 1$$

and

$$\lim_{n\to\infty} f_n(h^{2n}(\alpha)) = A.$$

Then, we from the definition of f_n have

$$\varphi_+(f_n(h^{2n}(\alpha))) = h(f_{n-1}(h^{2n-2}(\alpha))).$$

As $n \to \infty$ we get

$$\varphi_+(A) = h(A), \quad A \in [\alpha, 1].$$

From P_{10} we know that

$$A = 1 = f(0).$$

Note that $f_n(h^{2n}(\alpha))$ is the unique global minimum of f_n, so

$$\lim_{x\to 0^+} f(x) = f(0).$$

Finally, from the construction of f and assumptions P_8 and P_9, we know that that f is just a multi-valley solution. □

Example 5 Let $h(x) = x/2$, $\alpha = 1/2$ and

$$\varphi(x) = \begin{cases} -3x + \frac{3}{2}, & x \in [\frac{1}{4}, \frac{1}{2}]; \\ x - \frac{1}{2}, & x \in [\frac{1}{2}, 1]. \end{cases}$$

By simple calculation, we have

$$\varphi(1) = 1/2 = h(1), \quad \varphi(\varphi(h^2(1))) = 1/4 = h^2(1)$$

and

$$\varphi(\alpha) = \varphi(1/2) = 0, \quad \varphi(h^2(1)) = \varphi(1/4) > \alpha$$

and $x = 1$ is the unique solution of equation $\varphi(x) = h(x)$ on $[\alpha, 1]$. Then, Eq. (7) has a unique multi-valley solution f together with $f|_{[1/4,1]} = \varphi$ from Theorem 6 (see Fig. 5, in fact it contains the graphs of f_0, f_1 and f_2 defined by (15)).

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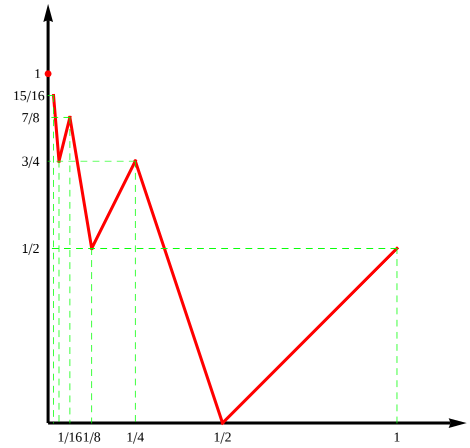

Fig. 5 The graphs of functions f_0, f_1 and f_2.

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