RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2022.147

ScienceAsia 49 (2023): **212–223**

Inequalities for *m*-polynomial exponentially *s*-type convex functions in fractional calculus

Pshtiwan Othman Mohammed[a], **Artion Kashuri**[b], **Bahaaeldin Abdalla**[c,*], **Y. S. Hamed**[d], **Khadijah M. Abualnaja**[d]

[a] Department of Mathematics, College of Education, University of Sulaimani, Sulaimani, Kurdistan Region, Iraq
[b] Department of Mathematics, Faculty of Technical and Natural Sciences, University "Ismail Qemali", Vlora, Albania
[c] Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586 Saudi Arabia
[d] Department of Mathematics and Statistics, College of Science, Taif University, Taif 21944 Saudi Arabia

*Corresponding author, e-mail: babdallah@psu.edu.sa

Received 29 Mar 2021, Accepted 26 Aug 2022
Available online 5 Dec 2022

ABSTRACT: In this article, we introduce a new class of functions named the m-polynomial exponentially s-type convex and we study some of its algebraic properties. We investigate a new integral inequality of Hermite-Hadamard (H-H) type by using the new introduced definition. Also, we prove a new midpoint identity. By examining this identity we can deduce some integral inequalities of midpoint type for the new introduced definition. Several special cases are discussed in details which emphasize that the results accounted in this paper unify and extend various results in this field of study. Finally, we provide some applications on the Bessel functions and special means of distinct positive real numbers to demonstrate the applicability of the new results.

KEYWORDS: H-H inequality, Hölder and power mean inequalities, m-polynomial exponentially s-type convex function, Bessel function, special mean

MSC2020: 26A33 26A51 26D07 26D10 26D15 26D20

INTRODUCTION

A set $Se \subset \mathbb{R}$ is said to be convex if for all $\wp_1, \wp_2 \in Se$ and $\varrho \in [0,1]$,

$$\varrho \wp_1 + (1-\varrho)\wp_2 \in Se.$$

Then, we say that a function $\phi : Se \to \mathbb{R}$ is convex if for all $\wp_1, \wp_2 \in Se$ and $\varrho \in [0,1]$,

$$\phi(\varrho \wp_1 + (1-\varrho)\wp_2) \leqslant \varrho\, \phi(\wp_1) + (1-\varrho)\phi(\wp_2). \quad (1)$$

Also, we say that ϕ is concave whenever $-\phi$ is convex.

There is an important integral inequality in the literature for the convex function (1), namely the Hermite-Hadamard (H-H) type integral inequality, which is given by [1].

$$\phi\left(\frac{\wp_1+\wp_2}{2}\right) \leqslant \frac{1}{\wp_2-\wp_1}\int_{\wp_1}^{\wp_2}\phi(x)\,\mathrm{d}x \leqslant \frac{\phi(\wp_1)+\phi(\wp_2)}{2}. \quad (2)$$

The H-H integral inequality (2) has been applied to different types of convex functions such as GA-convex functions [2], quasi-convex functions [3], s-geometrically convex functions [4], (α, m)-convex functions [5], and the readers can visit [6, 7] to find other types.

Here some definitions of convex type functions are introduced and used in sequel.

Definition 1 ([8]) A function $\phi : Se \subseteq \mathbb{R} \to \mathbb{R}$ is said to be exponentially convex, if

$$\phi(\varrho \wp_1 + (1-\varrho)\wp_2) \leqslant \varrho \frac{\phi(\wp_1)}{e^{\eta \wp_1}} + (1-\varrho)\frac{\phi(\wp_2)}{e^{\eta \wp_2}} \quad (3)$$

holds for all $\wp_1, \wp_2 \in Se$, $\varrho \in [0,1]$ and $\eta \in \mathbb{R}$.

Toply et al [9] introduced the class of m-polynomial convex functions as follows.

Definition 2 Let $m \in \mathbb{N}$. A function $\phi : Se \subseteq \mathbb{R} \to \mathbb{R}$ is said to be m-polynomial convex, if

$$\phi(\varrho \wp_1 + (1-\varrho)\wp_2) \leqslant \frac{1}{m}\sum_{\ell=1}^{m}\left[1-(1-\varrho)^\ell\right]\phi(\wp_1) + \frac{1}{m}\sum_{\ell=1}^{m}\left[1-\varrho^\ell\right]\phi(\wp_2) \quad (4)$$

holds for all $\wp_1, \wp_2 \in Se$ and $\varrho \in [0,1]$.

Recently, Rashid et al [10] defined the class of m-polynomial s-type convex functions.

Definition 3 Let $s \in [0,1]$ and $m \in \mathbb{N}$. A function $\phi : Se \subseteq \mathbb{R} \to \mathbb{R}$ is said to be m-polynomial s-type convex, if

$$\phi(\varrho \wp_1 + (1-\varrho)\wp_2) \leqslant \frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s(1-\varrho))^\ell\right]\phi(\wp_1) + \frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s\varrho)^\ell\right]\phi(\wp_2) \quad (5)$$

holds for all $\wp_1, \wp_2 \in Se$ and $\varrho \in [0,1]$.

With the help of the above definitions, we introduce a new definition of the class of convex functions.

Definition 4 Let $s \in [0,1]$ and $m \in \mathbb{N}$. The function $\phi : Se \to \mathbb{R}$ is called m-polynomial exponentially s-type convex, if

$$\phi(\varrho\wp_1+(1-\varrho)\wp_2) \leqslant \frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s(1-\varrho))^{\ell}\right]\frac{\phi(\wp_1)}{e^{\eta\wp_1}} + \frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s\varrho)^{\ell}\right]\frac{\phi(\wp_2)}{e^{\eta\wp_2}} \quad (6)$$

holds for each $\wp_1, \wp_2 \in Se$ and $\varrho \in [0,1]$.

Remark 1 From Definition 4, we can observe that:
(i) if $m = s = 1$ and $\eta = 0$, then (6) reduces to (1);
(ii) if $m = s = 1$, then Definition 4 reduces to Definition 1;
(iii) if $s = 1$ and $\eta = 0$, then Definition 4 reduces to Definition 2;
(iv) if $\eta = 0$, then Definition 4 reduces to Definition 3.

The aim of this paper is to investigate new H-H type integral inequalities for the new introduced m-polynomial exponentially s-type convex functions via the κ-fractional integral operators, where the κ-fractional integral operators are defined as follows.

Definition 5 ([11]) Let $\nu_1, \kappa > 0$, $\wp_1 < \wp_2$ and $\phi \in \mathscr{L}[\wp_1, \wp_2]$. Then the κ-fractional integrals of order ν_1 are defined by

$$\mathscr{I}_{\wp_1^+}^{\nu_1,\kappa}\phi(x) = \frac{1}{\kappa\Gamma_\kappa(\nu_1)}\int_{\wp_1}^{x}(x-\varrho)^{\frac{\nu_1}{\kappa}-1}\phi(\varrho)\,\mathrm{d}\varrho, \quad \wp_1 < x,$$

and

$$\mathscr{I}_{\wp_2^-}^{\nu_1,\kappa}\phi(x) = \frac{1}{\kappa\Gamma_\kappa(\nu_1)}\int_{x}^{\wp_2}(\varrho-x)^{\frac{\nu_1}{\kappa}-1}\phi(\varrho)\,\mathrm{d}\varrho, \quad \wp_2 > x,$$

where $\Gamma_\kappa(\cdot)$ is the κ–gamma function, defined by

$$\Gamma_\kappa(\nu_1) = \int_0^\infty \varrho^{\nu_1-1}\,\mathrm{e}^{-\frac{\varrho^\kappa}{\kappa}}\,\mathrm{d}\varrho, \quad \Gamma_\kappa(\nu_1+\kappa) = \nu_1\Gamma_\kappa(\nu_1),$$

and $\Gamma_1(\cdot) = \Gamma(\cdot)$.

Remark 2 Definition 5 with $\kappa = 1$ becomes the following standard Riemann-Liouville (RL) fractional integral operators.

$$\mathscr{I}_{\wp_1^+}^{\nu_1}\phi(x) = \frac{1}{\Gamma(\nu_1)}\int_{\wp_1}^{x}(x-\varrho)^{\nu_1-1}\phi(\varrho)\,\mathrm{d}\varrho, \quad \wp_1 < x,$$

and

$$\mathscr{I}_{\wp_2^-}^{\nu_1}\phi(x) = \frac{1}{\Gamma(\nu_1)}\int_{x}^{\wp_2}(\varrho-x)^{\nu_1-1}\phi(\varrho)\,\mathrm{d}\varrho, \quad \wp_2 > x.$$

Definition 6 ([12]) A function $\phi : Se \subseteq \mathbb{R} \to \mathbb{R}$ is said to be MT-convex on Se, if it is nonnegative and satisfies the following inequality: for all $\wp_1, \wp_2 \in Se$ and $\varrho \in (0,1)$,

$$\phi(\varrho\wp_1+(1-\varrho)\wp_2) \leqslant \frac{\sqrt{\varrho}}{2\sqrt{1-\varrho}}\phi(\wp_1) + \frac{\sqrt{1-\varrho}}{2\sqrt{\varrho}}\phi(\wp_2).$$

Definition 7 ([13]) For any function ϕ that is $\mathscr{L}[\wp_1, \wp_2]$ and for any $x \in [\wp_1, \wp_2]$, the left ν-th Atangana-Baleanu (AB) fractional integral of $\phi(x)$ is defined as follow: for $0 < \nu_1 < 1$,

$$\mathscr{I}_{\wp_1^+}^{\nu_1}\phi(x) := \frac{\nu_1}{B(\nu_1)}\mathscr{I}_{\wp_1^+}^{\nu_1}\phi(x) + \frac{1-\nu_1}{B(\nu_1)}\phi(x),$$

where $B(\nu_1)$ is a normalisation function that is real and positive and satisfies $B(0) = B(1) = 1$. It is also possible to define the right AB-integral, in the natural way analogous to Remark 2.

The H-H type integral inequalities are involved in fractional calculus models and they has been applied for different types of convex functions such as MT-type [12], λ_ψ-type [14], a new class of convex functions [15], and the readers can find out the other types in the literature [16–18].

In addition, it has been received a huge amount of attention in applying other models of fractional calculus such as standard Riemann-Liouville fractional operators [19], conformable fractional operators [20], generalized fractional operators [21], generalized discrete operators [22], ψ-RL-fractional operators [23], tempered fractional operators [24], AB- and Prabhakar fractional operators [13], and κ-fractional integral operators [25–28].

ALGEBRAIC PROPERTIES OF THE NEW DEFINED CONVEX FUNCTIONS

Here we derive some algebraic properties of our new defined convex function.

Condition A We say that $\eta \geqslant 0$ if and only if $\wp_1, \wp_2 \leqslant 0$ and $\eta \leqslant 0$ if and only if $\wp_1, \wp_2 \geqslant 0$.

Proposition 1 *If Condition A is satisfied, then each nonnegative m-polynomial convex function ϕ is also an m-polynomial exponentially s-type convex function.*

Proof: By using the assumption, we have for all $\wp_1, \wp_2 \in Se$ and $\varrho \in [0,1]$:

$$\begin{aligned}
&\phi(\varrho\wp_1+(1-\varrho)\wp_2)\\
&\leqslant \frac{1}{m}\sum_{\ell=1}^{m}\left[1-(1-\varrho)^{\ell}\right]\phi(\wp_1) + \frac{1}{m}\sum_{\ell=1}^{m}\left[1-\varrho^{\ell}\right]\phi(\wp_2)\\
&\leqslant \frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s(1-\varrho))^{\ell}\right]\phi(\wp_1) + \frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s\varrho)^{\ell}\right]\phi(\wp_2)\\
&\leqslant \frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s(1-\varrho))^{\ell}\right]\frac{\phi(\wp_1)}{\mathrm{e}^{\eta\wp_1}} + \frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s\varrho)^{\ell}\right]\frac{\phi(\wp_2)}{\mathrm{e}^{\eta\wp_2}}.
\end{aligned}$$

This gives the proof. □

Now, we study some algebraic properties.

Theorem 1 *Let $s\in[0,1]$, $m\in\mathbb{N}$ and $\eta\in\mathbb{R}$. Assume that ϕ, ϕ_1, $\phi_2 : Se\to\mathbb{R}$. If ϕ, ϕ_1, and ϕ_2 are three m-polynomial exponentially s-type convex functions, then*

(1) $\phi_1+\phi_2$ is m-polynomial exponentially s-type convex function;

(2) for nonnegative real number c, $c\phi$ is m-polynomial exponentially s-type convex function.

Proof: The proof is evident, so we omit here. □

Theorem 2 *Let $s\in[0,1]$, $m\in\mathbb{N}$ and $\eta\in\mathbb{R}$. Assume that $\phi_1 : Se\to\mathbb{R}$ be a convex function and $\phi_2:\mathbb{R}\to\mathbb{R}$ is a non-decreasing and m-polynomial exponentially s-type convex function. Then, the function $\phi_2\circ\phi_1 : Se\to\mathbb{R}$ is an m-polynomial exponentially s-type convex.*

Proof: For all $\wp_1,\wp_2\in Se$ and $\varrho\in[0,1]$, we have

$$\begin{aligned}(\phi_2\circ\phi_1)(\varrho\wp_1+(1-\varrho)\wp_2) &= \phi_2(\phi_1(\varrho\wp_1+(1-\varrho)\wp_2))\\ &\leqslant \phi_2(\varrho\phi_1(\wp_1)+(1-\varrho)\phi_1(\wp_2))\\ &\leqslant \frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s(1-\varrho))^{\ell}\right]\frac{\phi_2(\phi_1(\wp_1))}{e^{\eta\wp_1}}\\ &\quad+\frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s\varrho)^{\ell}\right]\frac{\phi_2(\phi_1(\wp_2))}{e^{\eta\wp_2}}\\ &= \frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s(1-\varrho))^{\ell}\right]\frac{(\phi_2\circ\phi_1)(\wp_1)}{e^{\eta\wp_1}}\\ &\quad+\frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s\varrho)^{\ell}\right]\frac{(\phi_2\circ\phi_1)(\wp_2)}{e^{\eta\wp_2}},\end{aligned}$$

which gives the proof. □

Theorem 3 *Let $s\in[0,1]$, $m\in\mathbb{N},\eta\in\mathbb{R}$ and $\mathscr{U}=\{\wp\in[\wp_1,\wp_2]:\phi(\wp)<\infty\}$. Assume that $\phi_j:[\wp_1,\wp_2]\to\mathbb{R}$ is a family of m-polynomial exponentially s-type convex functions with $\phi(\wp)=\sup_j\phi_j(\wp)$. Then, ϕ is an m-polynomial exponentially s-type convex function on $\mathscr{U}$.*

Proof: Let $\wp_1,\wp_2\in\mathscr{U}$ and $\varrho\in[0,1]$, then we have

$$\begin{aligned}\phi(\varrho\wp_1+(1-\varrho)\wp_2) &= \sup_j\phi_j(\varrho\wp_1+(1-\varrho)\wp_2)\\ &\leqslant \frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s(1-\varrho))^{\ell}\right]\frac{\sup_j\phi_j(\wp_1)}{e^{\eta\wp_1}}\\ &\quad+\frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s\varrho)^{\ell}\right]\frac{\sup_j\phi_j(\wp_2)}{e^{\eta\wp_2}}\\ &= \frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s(1-\varrho))^{\ell}\right]\frac{\phi(\wp_1)}{e^{\eta\wp_1}}\\ &\quad+\frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s\varrho)^{\ell}\right]\frac{\phi(\wp_2)}{e^{\eta\wp_2}}<\infty,\end{aligned}$$

which completes the proof. □

THE H-H INEQUALITIES FOR THE NEW CONVEX FUNCTION

The aim of this section is to find some κ-fractional integral inequalities of H-H type for m-polynomial exponentially s-type convex functions.

Theorem 4 *Let $s\in[0,1]$, $\nu_1,\kappa>0$, $m\in\mathbb{N}$ and $\phi:[\wp_1,\wp_2]\to\mathbb{R}$ be a m-polynomial exponentially s-type convex function. If $\phi\in\mathscr{L}[\wp_1,\wp_2]$ and $\eta\in\mathbb{R}$, then we have*

$$\begin{aligned}&\frac{\kappa}{\nu_1}\left(\frac{m(2-s)2^m}{2^m(2m-s(m+1))+s^{m+1}}\right)\phi\left(\frac{\wp_1+\wp_2}{2}\right)\\ &\leqslant\frac{1}{(\wp_2-\wp_1)^{\frac{\nu_1}{\kappa}}}\left[A^{\nu_1,\kappa}_{\phi,1}(\eta;\wp_1,\wp_2)+A^{\nu_1,\kappa}_{\phi,2}(\eta;\wp_1,\wp_2)\right]\\ &\leqslant\frac{1}{(\wp_2-\wp_1)^{\frac{\nu_1}{\kappa}}}\Bigg\{\left[B^{\nu_1,\kappa}_{1,m}(\eta;s,\wp_1,\wp_2)+B^{\nu_1,\kappa}_{4,m}(\eta;s,\wp_1,\wp_2)\right]\frac{\phi(\wp_1)}{e^{\eta\wp_1}}\\ &\quad+\left[B^{\nu_1,\kappa}_{2,m}(\eta;s,\wp_1,\wp_2)+B^{\nu_1,\kappa}_{3,m}(\eta;s,\wp_1,\wp_2)\right]\frac{\phi(\wp_2)}{e^{\eta\wp_2}}\Bigg\},\end{aligned}\tag{7}$$

where

$$A^{\nu_1,\kappa}_{\phi,1}(\eta;\wp_1,\wp_2):=\int_{\wp_1}^{\wp_2}(\wp_2-x)^{\frac{\nu_1}{\kappa}-1}\frac{\phi(x)}{e^{\eta x}}\,dx,$$
$$A^{\nu_1,\kappa}_{\phi,2}(\eta;\wp_1,\wp_2):=\int_{\wp_1}^{\wp_2}(x-\wp_1)^{\frac{\nu_1}{\kappa}-1}\frac{\phi(x)}{e^{\eta x}}\,dx,$$

and

$$B^{\nu_1,\kappa}_{1,m}(\eta;s,\wp_1,\wp_2)=\frac{1}{m}\sum_{\ell=1}^{m}\int_{\wp_1}^{\wp_2}\frac{(\wp_2-x)^{\frac{\nu_1}{\kappa}-1}}{e^{\eta x}}\left[1-\left(s\frac{x-\wp_1}{\wp_2-\wp_1}\right)^{\ell}\right]dx;$$
$$B^{\nu_1,\kappa}_{2,m}(\eta;s,\wp_1,\wp_2)=\frac{1}{m}\sum_{\ell=1}^{m}\int_{\wp_1}^{\wp_2}\frac{(\wp_2-x)^{\frac{\nu_1}{\kappa}-1}}{e^{\eta x}}\left[1-\left(s\frac{\wp_2-x}{\wp_2-\wp_1}\right)^{\ell}\right]dx;$$
$$B^{\nu_1,\kappa}_{3,m}(\eta;s,\wp_1,\wp_2)=\frac{1}{m}\sum_{\ell=1}^{m}\int_{\wp_1}^{\wp_2}\frac{(x-\wp_1)^{\frac{\nu_1}{\kappa}-1}}{e^{\eta x}}\left[1-\left(s\frac{\wp_2-x}{\wp_2-\wp_1}\right)^{\ell}\right]dx;$$
$$B^{\nu_1,\kappa}_{4,m}(\eta;s,\wp_1,\wp_2)=\frac{1}{m}\sum_{\ell=1}^{m}\int_{\wp_1}^{\wp_2}\frac{(x-\wp_1)^{\frac{\nu_1}{\kappa}-1}}{e^{\eta x}}\left[1-\left(s\frac{x-\wp_1}{\wp_2-\wp_1}\right)^{\ell}\right]dx.$$

Proof: Let $w_1,w_2\in[\wp_1,\wp_2]$. Applying m-polynomial exponentially s-type convexity of ϕ on $[\wp_1,\wp_2]$ to get

$$\phi\left(\frac{w_1+w_2}{2}\right)\leqslant\frac{1}{m}\sum_{\ell=1}^{m}\left[1-\left(\frac{s}{2}\right)^{\ell}\right]\left[\frac{\phi(w_1)}{e^{\eta w_1}}+\frac{\phi(w_2)}{e^{\eta w_2}}\right].\tag{8}$$

By making use of (8) with $w_1=\varrho\wp_2+(1-\varrho)\wp_1$ and $w_2=\varrho\wp_1+(1-\varrho)\wp_2$, we get

$$\phi\left(\frac{\wp_1+\wp_2}{2}\right)\leqslant\frac{1}{m}\sum_{\ell=1}^{m}\left[1-\left(\frac{s}{2}\right)^{\ell}\right]\times\left[\frac{\phi(\varrho\wp_2+(1-\varrho)\wp_1)}{e^{\eta(\varrho\wp_2+(1-\varrho)\wp_1)}}+\frac{\phi(\varrho\wp_1+(1-\varrho)\wp_2)}{e^{\eta(\varrho\wp_1+(1-\varrho)\wp_2)}}\right].\tag{9}$$

Multiplying both sides of (9) by $\varrho^{\frac{\nu_1}{\kappa}-1}$ and then integrating the result with respect to ϱ over $[0,1]$, we obtain

$$\begin{aligned}\frac{\kappa}{\nu_1}&\left(\frac{m(2-s)2^m}{2^m(2m-s(m+1))+s^{m+1}}\right)\phi\left(\frac{\wp_1+\wp_2}{2}\right)\\&\leqslant\int_0^1\varrho^{\frac{\nu_1}{\kappa}-1}\frac{\phi(\varrho\wp_2+(1-\varrho)\wp_1)}{e^{\eta(\varrho\wp_2+(1-\varrho)\wp_1)}}\,d\varrho\\&\quad+\int_0^1\varrho^{\frac{\nu_1}{\kappa}-1}\frac{\phi(\varrho\wp_1+(1-\varrho)\wp_2)}{e^{\eta(\varrho\wp_1+(1-\varrho)\wp_2)}}\,d\varrho\\&=\frac{1}{(\wp_2-\wp_1)^{\frac{\nu_1}{\kappa}}}\left[\int_{\wp_1}^{\wp_2}(\wp_2-x)^{\frac{\nu_1}{\kappa}-1}\frac{\phi(x)}{e^{\eta x}}\,dx\right.\\&\quad\left.+\int_{\wp_1}^{\wp_2}(x-\wp_1)^{\frac{\nu_1}{\kappa}-1}\frac{\phi(x)}{e^{\eta x}}\,dx\right]\\&=\frac{1}{(\wp_2-\wp_1)^{\frac{\nu_1}{\kappa}}}\left[A_{\phi,1}^{\nu_1,\kappa}(\eta;\wp_1,\wp_2)+A_{\phi,2}^{\nu_1,\kappa}(\eta;\wp_1,\wp_2)\right],\end{aligned}$$

which gives the first half inequality of (7). In order to prove the second half inequality of (7), we use the definition of m-polynomial exponentially s-type convexity of ϕ to get

$$\frac{\phi(\varrho\wp_2+(1-\varrho)\wp_1)}{e^{\eta(\varrho\wp_2+(1-\varrho)\wp_1)}}\leqslant\frac{1}{e^{\eta(\varrho\wp_2+(1-\varrho)\wp_1)}}\times\left\{\frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s\varrho)^\ell\right]\frac{\phi(\wp_1)}{e^{\eta\wp_1}}+\frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s(1-\varrho))^\ell\right]\frac{\phi(\wp_2)}{e^{\eta\wp_2}}\right\},$$

and

$$\frac{\phi(\varrho\wp_1+(1-\varrho)\wp_2)}{e^{\eta(\varrho\wp_1+(1-\varrho)\wp_2)}}\leqslant\frac{1}{e^{\eta(\varrho\wp_1+(1-\varrho)\wp_2)}}\times\left\{\frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s\varrho)^\ell\right]\frac{\phi(\wp_2)}{e^{\eta\wp_2}}+\frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s(1-\varrho))^\ell\right]\frac{\phi(\wp_1)}{e^{\eta\wp_1}}\right\},$$

where $\varrho\in[0,1]$. Then, by adding the above inequalities, we have

$$\begin{aligned}&\frac{\phi(\varrho\wp_2+(1-\varrho)\wp_1)}{e^{\eta(\varrho\wp_2+(1-\varrho)\wp_1)}}+\frac{\phi(\varrho\wp_1+(1-\varrho)\wp_2)}{e^{\eta(\varrho\wp_1+(1-\varrho)\wp_2)}}\\&\leqslant\frac{1}{e^{\eta(\varrho\wp_2+(1-\varrho)\wp_1)}}\left\{\frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s\varrho)^\ell\right]\frac{\phi(\wp_1)}{e^{\eta\wp_1}}\right.\\&\quad\left.+\frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s(1-\varrho))^\ell\right]\frac{\phi(\wp_2)}{e^{\eta\wp_2}}\right\}\\&\quad+\frac{1}{e^{\eta(\varrho\wp_1+(1-\varrho)\wp_2)}}\left\{\frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s\varrho)^\ell\right]\frac{\phi(\wp_2)}{e^{\eta\wp_2}}\right.\\&\quad\left.+\frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s(1-\varrho))^\ell\right]\frac{\phi(\wp_1)}{e^{\eta\wp_1}}\right\}.\end{aligned}\tag{10}$$

Multiplying both sides of (10) by $\varrho^{\frac{\nu_1}{\kappa}-1}$ and integrating the obtained inequality with respect to ϱ from 0 to 1 and then making the change of the variable, we get

$$\begin{aligned}&\int_0^1\varrho^{\frac{\nu_1}{\kappa}-1}\frac{\phi(\varrho\wp_2+(1-\varrho)\wp_1)}{e^{\eta(\varrho\wp_2+(1-\varrho)\wp_1)}}\,d\varrho+\int_0^1\varrho^{\frac{\nu_1}{\kappa}-1}\frac{\phi(\varrho\wp_2+(1-\varrho)\wp_1)}{e^{\eta(\varrho\wp_1+(1-\varrho)\wp_2)}}\,d\varrho\\&\leqslant\int_0^1\varrho^{\frac{\nu_1}{\kappa}-1}\frac{1}{e^{\eta(\varrho\wp_2+(1-\varrho)\wp_1)}}\left\{\frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s\varrho)^\ell\right]\frac{\phi(\wp_1)}{e^{\eta\wp_1}}\right.\\&\quad\left.+\frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s(1-\varrho))^\ell\right]\frac{\phi(\wp_2)}{e^{\eta\wp_2}}\right\}d\varrho\\&\quad+\int_0^1\varrho^{\frac{\nu_1}{\kappa}-1}\frac{1}{e^{\eta(\varrho\wp_1+(1-\varrho)\wp_2)}}\left\{\frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s\varrho)^\ell\right]\frac{\phi(\wp_2)}{e^{\eta\wp_2}}\right.\\&\quad\left.+\frac{1}{m}\sum_{\ell=1}^{m}\left[1-(s(1-\varrho))^\ell\right]\frac{\phi(\wp_1)}{e^{\eta\wp_1}}\right\}d\varrho.\end{aligned}$$

By simplifying it, we get

$$\begin{aligned}&\frac{1}{(\wp_2-\wp_1)^{\frac{\nu_1}{\kappa}}}\left[A_{\phi,1}^{\nu_1,\kappa}(\eta;\wp_1,\wp_2)+A_{\phi,2}^{\nu_1,\kappa}(\eta;\wp_1,\wp_2)\right]\\&\leqslant\frac{1}{(\wp_2-\wp_1)^{\frac{\nu_1}{\kappa}}}\left\{\left[B_{1,m}^{\nu_1,\kappa}(\eta;s,\wp_1,\wp_2)+B_{4,m}^{\nu_1,\kappa}(\eta;s,\wp_1,\wp_2)\right]\frac{\phi(\wp_1)}{e^{\eta\wp_1}}\right.\\&\quad\left.+\left[B_{2,m}^{\nu_1,\kappa}(\eta;s,\wp_1,\wp_2)+B_{3,m}^{\nu_1,\kappa}(\eta;s,\wp_1,\wp_2)\right]\frac{\phi(\wp_2)}{e^{\eta\wp_2}}\right\}.\end{aligned}$$

The proof of Theorem 4 is thus completed. □

Corollary 1 *Theorem 4 with $\eta=0$ becomes [10, Theorem 2.1].*

Corollary 2 *Theorem 4 with $s=0$ leads to*

$$\begin{aligned}\phi&\left(\frac{\wp_1+\wp_2}{2}\right)\\&\leqslant\frac{\nu_1}{\kappa(\wp_2-\wp_1)^{\frac{\nu_1}{\kappa}}}\left[A_{\phi,1}^{\nu_1,\kappa}(\eta;\wp_1,\wp_2)+A_{\phi,2}^{\nu_1,\kappa}(\eta;\wp_1,\wp_2)\right]\\&\leqslant\frac{\nu_1}{\kappa(\wp_2-\wp_1)^{\frac{\nu_1}{\kappa}}}\left[B_1^{\nu_1,\kappa}(\eta;\wp_1,\wp_2)\right.\\&\quad\left.+B_2^{\nu_1,\kappa}(\eta;\wp_1,\wp_2)\right]\left[\frac{\phi(\wp_1)}{e^{\eta\wp_1}}+\frac{\phi(\wp_2)}{e^{\eta\wp_2}}\right],\end{aligned}\tag{11}$$

where $A_{\phi,1}^{\nu_1,\kappa}(\eta;\wp_1,\wp_2)$ and $A_{\phi,2}^{\nu_1,\kappa}(\eta;\wp_1,\wp_2)$ are as given in Theorem 4, and

$$B_1^{\nu_1,\kappa}(\eta;\wp_1,\wp_2)=\int_{\wp_1}^{\wp_2}\frac{(x-\wp_1)^{\frac{\nu_1}{\kappa}-1}}{e^{\eta x}}\,dx,$$

$$B_2^{\nu_1,\kappa}(\eta;\wp_1,\wp_2)=\int_{\wp_1}^{\wp_2}\frac{(\wp_2-x)^{\frac{\nu_1}{\kappa}-1}}{e^{\eta x}}\,dx.$$

Corollary 3 *Theorem 4 with $s=1$ leads to*

$$\begin{aligned}&\frac{\kappa}{\nu_1}\left(\frac{m2^m}{2^m(m-1)+1}\right)\phi\left(\frac{\wp_1+\wp_2}{2}\right)\\&\leqslant\frac{1}{(\wp_2-\wp_1)^{\frac{\nu_1}{\kappa}}}\left[A^{\nu_1,\kappa}_{\phi,1}(\eta;\wp_1,\wp_2)+A^{\nu_1,\kappa}_{\phi,2}(\eta;\wp_1,\wp_2)\right]\\&\leqslant\frac{1}{(\wp_2-\wp_1)^{\frac{\nu_1}{\kappa}}}\left\{\left[B^{\nu_1,\kappa}_{1,m}(\eta;\wp_1,\wp_2)+B^{\nu_1,\kappa}_{4,m}(\eta;\wp_1,\wp_2)\right]\frac{\phi(\wp_1)}{e^{\eta\wp_1}}\right.\\&\quad\left.+\left[B^{\nu_1,\kappa}_{2,m}(\eta;\wp_1,\wp_2)+B^{\nu_1,\kappa}_{3,m}(\eta;\wp_1,\wp_2)\right]\frac{\phi(\wp_2)}{e^{\eta\wp_2}}\right\},\quad(12)\end{aligned}$$

where $A^{\nu_1,\kappa}_{\phi,1}(\eta;\wp_1,\wp_2)$, $A^{\nu_1,\kappa}_{\phi,2}(\eta;\wp_1,\wp_2)$ and $B^{\nu_1,\kappa}_{\ell,m}(\eta;\wp_1,\wp_2)=B^{\nu_1,\kappa}_{\ell,m}(\eta;1,\wp_1,\wp_2)$ for all $\ell=1,2,3,4$ are as given in Theorem 4.

Corollary 4 *Theorem 4 with $s=\kappa=\nu_1=1$ leads to*

$$\begin{aligned}&\left(\frac{m2^{m-1}}{2^m(m-1)+1}\right)\phi\left(\frac{\wp_1+\wp_2}{2}\right)\leqslant\frac{1}{\wp_2-\wp_1}A_\phi(\eta;\wp_1,\wp_2)\\&\leqslant\frac{1}{\wp_2-\wp_1}\left\{B_{1,m}(\eta;\wp_1,\wp_2)\frac{\phi(\wp_1)}{e^{\eta\wp_1}}+B_{2,m}(\eta;\wp_1,\wp_2)\frac{\phi(\wp_2)}{e^{\eta\wp_2}}\right\},\quad(13)\end{aligned}$$

where

$$A_\phi(\eta;\wp_1,\wp_2):=\int_{\wp_1}^{\wp_2}\frac{\phi(x)}{e^{\eta x}}\,dx,$$

and

$$B_{1,m}(\eta;\wp_1,\wp_2):=\frac{1}{m}\sum_{\ell=1}^{m}\int_{\wp_1}^{\wp_2}\frac{1}{e^{\eta x}}\left(\frac{\wp_2-x}{\wp_2-\wp_1}\right)^{\ell}dx;$$

$$B_{2,m}(\eta;\wp_1,\wp_2):=\frac{1}{m}\sum_{\ell=1}^{m}\int_{\wp_1}^{\wp_2}\frac{1}{e^{\eta x}}\left(\frac{x-\wp_1}{\wp_2-\wp_1}\right)^{\ell}dx.$$

Corollary 5 *Theorem 4 with $\eta=0$ and $s=\kappa=\nu_1=1$ becomes [9, Theorem 4].*

FURTHER CONSEQUENCES

We need the following lemma in order to proceed.

Lemma 1 *Let $\phi:Se\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on Se with $\wp_1,\wp_2\in Se$ and $\wp_1<\wp_2$. Also, let $\nu_1,\kappa_1>0$ and m be an arbitrary positive integer number. If $\phi'\in\mathscr{L}[\wp_1,\wp_2]$, then we have*

$$\begin{aligned}&S^{\nu_1,\kappa_1}_m(\phi;\wp_1,\wp_2):=\left(\frac{\wp_2-\wp_1}{4m}\right)\Gamma_{\kappa_1}(\nu_1+\kappa_1)\times\\&\sum_{j=0}^{m-1}\left\{\left(\frac{2m}{\wp_2-\wp_1}\right)^{\frac{\nu_1}{\kappa_1}+1}\mathscr{J}^{\nu_1,\kappa_1}_{\left(\frac{(2(m-j)-1)\wp_1+(2j+1)\wp_2}{2m}\right)^-}\phi\left(\frac{(m-j)\wp_1+j\wp_2}{m}\right)\right.\\&\left.-\left(\frac{2m}{\wp_1-\wp_2}\right)^{\frac{\nu_1}{\kappa_1}+1}\mathscr{J}^{\nu_1,\kappa_1}_{\left(\frac{(2(m-j)-1)\wp_1+(2j+1)\wp_2}{2m}\right)^-}\phi\left(\frac{(m-j-1)\wp_1+(j+1)\wp_2}{m}\right)\right\}\\&\quad-\sum_{j=0}^{m-1}\phi\left(\frac{(2(m-j)-1)\wp_1+(2j+1)\wp_2}{2m}\right)\end{aligned}$$

$$\begin{aligned}&=\left(\frac{\wp_2-\wp_1}{4m}\right)\sum_{j=0}^{m-1}\left\{\int_0^1\varrho^{\frac{\nu_1}{\kappa_1}}\phi'\left(\frac{\varrho}{2}\frac{(m-j)\wp_1+j\wp_2}{m}+\frac{(2-\varrho)}{2}\frac{(m-j-1)\wp_1+(j+1)\wp_2}{m}\right)d\varrho\right.\\&\quad\left.-\int_0^1\varrho^{\frac{\nu_1}{\kappa_1}}\phi'\left(\frac{\varrho}{2}\frac{(m-j-1)\wp_1+(j+1)\wp_2}{m}+\frac{(2-\varrho)}{2}\frac{(m-j)\wp_1+j\wp_2}{m}\right)d\varrho\right\}.\quad(14)\end{aligned}$$

Proof: Setting

$$\mathscr{I}_1:=\int_0^1\varrho^{\frac{\nu_1}{\kappa_1}}\phi'\left(\frac{\varrho}{2}\frac{(m-j)\wp_1+j\wp_2}{m}+\frac{(2-\varrho)}{2}\frac{(m-j-1)\wp_1+(j+1)\wp_2}{m}\right)d\varrho,\quad(15)$$

and

$$\mathscr{I}_2:=\int_0^1\varrho^{\frac{\nu_1}{\kappa_1}}\phi'\left(\frac{\varrho}{2}\frac{(m-j-1)\wp_1+(j+1)\wp_2}{m}+\frac{(2-\varrho)}{2}\frac{(m-j)\wp_1+j\wp_2}{m}\right)d\varrho.\quad(16)$$

By applying integration by parts on equality (15), we have

$$\begin{aligned}\mathscr{I}_1&=\left(\frac{2m}{\wp_1-\wp_2}\right)\left[\varrho^{\frac{\nu_1}{\kappa_1}}\phi\left(\frac{\varrho}{2}\frac{(m-j)\wp_1+j\wp_2}{m}+\frac{(2-\varrho)}{2}\frac{(m-j-1)\wp_1+(j+1)\wp_2}{m}\right)\Bigg|_0^1\right.\\&\quad\left.-\frac{\nu_1}{\kappa_1}\int_0^1\varrho^{\frac{\nu_1}{\kappa_1}-1}\phi\left(\frac{\varrho}{2}\frac{(m-j)\wp_1+j\wp_2}{m}+\frac{(2-\varrho)}{2}\frac{(m-j-1)\wp_1+(j+1)\wp_2}{m}\right)d\varrho\right]\\&=\left(\frac{2m}{\wp_1-\wp_2}\right)\left[\phi\left(\frac{(2(m-j)-1)\wp_1+(2j+1)\wp_2}{2m}\right)\right.\\&\quad-\left(\frac{2m}{\wp_1-\wp_2}\right)^{\frac{\nu_1}{\kappa_1}}\Gamma_{\kappa_1}(\nu_1+\kappa_1)\mathscr{J}^{\nu_1,\kappa_1}_{\left(\frac{(2(m-j)-1)\wp_1+(2j+1)\wp_2}{2m}\right)^-}\\&\quad\left.\times\phi\left(\frac{(m-j-1)\wp_1+(j+1)\wp_2}{m}\right)\right].\quad(17)\end{aligned}$$

Similarly, from equality (16), we obtain

$$\begin{aligned}\mathscr{I}_2&=\left(\frac{2m}{\wp_2-\wp_1}\right)\left[\phi\left(\frac{(2(m-j)-1)\wp_1+(2j+1)\wp_2}{2m}\right)\right.\\&\quad-\left(\frac{2m}{\wp_2-\wp_1}\right)^{\frac{\nu_1}{\kappa_1}}\Gamma_{\kappa_1}(\nu_1+\kappa_1)\mathscr{J}^{\nu_1,\kappa_1}_{\left(\frac{(2(m-j)-1)\wp_1+(2j+1)\wp_2}{2m}\right)^-}\\&\quad\left.\times\phi\left(\frac{(m-j)\wp_1+j\wp_2}{m}\right)\right],\quad(18)\end{aligned}$$

for all $\jmath = 0, 1, 2, \dots, m-1$. Then, by subtracting equality (18) from (17), multiplying by the factor $\left(\frac{\wp_2-\wp_1}{4m}\right)$ and summing over $\jmath$ from 0 to $m-1$, we can easily attain the desired identity (14). □

Remark 3 Lemma 1 with $m = \kappa_1 = 1$ leads to

$$\frac{2^{\nu_1-1}\Gamma(\nu_1+1)}{(\wp_2-\wp_1)^{\nu_1}}\left\{\mathscr{J}^{\nu_1}_{\left(\frac{\wp_1+\wp_2}{2}\right)^+}\phi(\wp_2)+\mathscr{J}^{\nu_1}_{\left(\frac{\wp_1+\wp_2}{2}\right)^-}\phi(\wp_1)\right\} -\phi\left(\frac{\wp_1+\wp_2}{2}\right)=\frac{(\wp_2-\wp_1)}{4}\left\{\int_0^1\varrho^{\nu_1}\phi'\left(\frac{\varrho}{2}\wp_1+\frac{(2-\varrho)}{2}\wp_2\right)\mathrm{d}\varrho -\int_0^1\varrho^{\nu_1}\phi'\left(\frac{\varrho}{2}\wp_2+\frac{(2-\varrho)}{2}\wp_1\right)\mathrm{d}\varrho\right\}, \quad (19)$$

which is established in [29, Lemma 3].

Throughout the rest of this study, we consider

$$\mathbf{u}_{m,\jmath} := \frac{(m-\jmath)\wp_1+\jmath\wp_2}{m} \quad\text{and}\quad \mathbf{u}_{m,\jmath+1} := \frac{(m-\jmath-1)\wp_1+(\jmath+1)\wp_2}{m}.$$

Theorem 5 *Let $s \in [0,1]$, $\nu_1, \kappa_1 > 0$, $m \in \mathbb{N}$ and $\phi : [\wp_1, \wp_2] \to \mathbb{R}$ be a differentiable function on $(\wp_1, \wp_2)$ such that $\phi' \in \mathscr{L}[\wp_1, \wp_2]$. If $|\phi'|$ is m-polynomial exponentially s-type convex function on $[\wp_1, \wp_2]$ and $\eta \in \mathbb{R}$, then we have*

$$\left|S_m^{\nu_1,\kappa_1}(\phi;\wp_1,\wp_2)\right| \leqslant \left(\frac{\wp_2-\wp_1}{4m}\right)\left[C_m^{\nu_1,\kappa_1}(s)+D_m^{\nu_1,\kappa_1}(s)\right] \times\sum_{\jmath=0}^{m-1}\left[\frac{|\phi'(\boldsymbol{u}_{m,\jmath})|}{\mathrm{e}^{\eta\boldsymbol{u}_{m,\jmath}}}+\frac{|\phi'(\boldsymbol{u}_{m,\jmath+1})|}{\mathrm{e}^{\eta\boldsymbol{u}_{m,\jmath+1}}}\right], \quad (20)$$

where

$$C_m^{\nu_1,\kappa_1}(s) := \frac{1}{m}\sum_{\ell=1}^{m}\int_0^1 \varrho^{\frac{\nu_1}{\kappa_1}}\left[1-\left(s\left(1-\frac{\varrho}{2}\right)\right)^{\ell}\right]\mathrm{d}\varrho,$$
$$D_m^{\nu_1,\kappa_1}(s) := \frac{1}{m}\sum_{\ell=1}^{m}\int_0^1 \varrho^{\frac{\nu_1}{\kappa_1}}\left[1-\left(s\frac{\varrho}{2}\right)^{\ell}\right]\mathrm{d}\varrho.$$

Proof: By making use of Lemma 1 and properties of modulus, we can deduce

$$\left|S_m^{\nu_1,\kappa_1}(\phi;\wp_1,\wp_2)\right| \leqslant \left(\frac{\wp_2-\wp_1}{4m}\right) \times\sum_{\jmath=0}^{m-1}\left\{\int_0^1\varrho^{\frac{\nu_1}{\kappa_1}}\left|\phi'\left(\frac{\varrho}{2}\mathbf{u}_{m,\jmath}+\frac{(2-\varrho)}{2}\mathbf{u}_{m,\jmath+1}\right)\right|\mathrm{d}\varrho +\int_0^1\varrho^{\frac{\nu_1}{\kappa_1}}\left|\phi'\left(\frac{(2-\varrho)}{2}\mathbf{u}_{m,\jmath}+\frac{\varrho}{2}\mathbf{u}_{m,\jmath+1}\right)\right|\mathrm{d}\varrho\right\}.$$

Then, by making use of m-polynomial exponentially s-type convexity of $|\phi'|$, we get

$$\begin{aligned}\left|S_m^{\nu_1,\kappa_1}(\phi;\wp_1,\wp_2)\right| &\leqslant \left(\frac{\wp_2-\wp_1}{4m}\right) \times\sum_{\jmath=0}^{m-1}\left\{\int_0^1\varrho^{\frac{\nu_1}{\kappa_1}}\left[\frac{1}{m}\sum_{\ell=1}^{m}\left[1-\left(s\left(1-\frac{\varrho}{2}\right)\right)^{\ell}\right]\frac{|\phi'(\mathbf{u}_{m,\jmath})|}{\mathrm{e}^{\eta\mathbf{u}_{m,\jmath}}}\right.\right.\\&\quad\left.+\frac{1}{m}\sum_{\ell=1}^{m}\left[1-\left(s\frac{\varrho}{2}\right)^{\ell}\right]\frac{|\phi'(\mathbf{u}_{m,\jmath+1})|}{\mathrm{e}^{\eta\mathbf{u}_{m,\jmath+1}}}\right]\mathrm{d}\varrho\\&\quad+\int_0^1\varrho^{\frac{\nu_1}{\kappa_1}}\left[\frac{1}{m}\sum_{\ell=1}^{m}\left[1-\left(s\left(1-\frac{\varrho}{2}\right)\right)^{\ell}\right]\frac{|\phi'(\mathbf{u}_{m,\jmath+1})|}{\mathrm{e}^{\eta\mathbf{u}_{m,\jmath+1}}}\right.\\&\quad\left.\left.+\frac{1}{m}\sum_{\ell=1}^{m}\left[1-\left(s\frac{\varrho}{2}\right)^{\ell}\right]\frac{|\phi'(\mathbf{u}_{m,\jmath})|}{\mathrm{e}^{\eta\mathbf{u}_{m,\jmath}}}\right]\mathrm{d}\varrho\right\}\\&=\left(\frac{\wp_2-\wp_1}{4m}\right)\left[C_m^{\nu_1,\kappa_1}(s)+D_m^{\nu_1,\kappa_1}(s)\right]\\&\quad\times\sum_{\jmath=0}^{m-1}\left[\frac{|\phi'(\mathbf{u}_{m,\jmath})|}{\mathrm{e}^{\eta\mathbf{u}_{m,\jmath}}}+\frac{|\phi'(\mathbf{u}_{m,\jmath+1})|}{\mathrm{e}^{\eta\mathbf{u}_{m,\jmath+1}}}\right],\end{aligned}$$

which completes the proof. □

Corollary 6 *Theorem 5 with $\eta = 0$ leads to*

$$\left|S_m^{\nu_1,\kappa_1}(\phi;\wp_1,\wp_2)\right| \leqslant \left(\frac{\wp_2-\wp_1}{4m}\right)\left[C_m^{\nu_1,\kappa_1}(s)+D_m^{\nu_1,\kappa_1}(s)\right] \times\sum_{\jmath=0}^{m-1}\left[|\phi'(\boldsymbol{u}_{m,\jmath})|+|\phi'(\boldsymbol{u}_{m,\jmath+1})|\right]. \quad (21)$$

Corollary 7 *Theorem 5 with $s = 0$ leads to*

$$\left|S_m^{\nu_1,\kappa_1}(\phi;\wp_1,\wp_2)\right| \leqslant \left(\frac{\wp_2-\wp_1}{2m}\right)\left(\frac{\kappa_1}{\nu_1+\kappa_1}\right) \times\sum_{\jmath=0}^{m-1}\left[\frac{|\phi'(\boldsymbol{u}_{m,\jmath})|}{\mathrm{e}^{\eta\boldsymbol{u}_{m,\jmath}}}+\frac{|\phi'(\boldsymbol{u}_{m,\jmath+1})|}{\mathrm{e}^{\eta\boldsymbol{u}_{m,\jmath+1}}}\right]. \quad (22)$$

Corollary 8 *Theorem 5 with $s = 1$ leads to*

$$\left|S_m^{\nu_1,\kappa_1}(\phi;\wp_1,\wp_2)\right| \leqslant \left(\frac{\wp_2-\wp_1}{4m}\right)\left[C_m^{\nu_1,\kappa_1}+D_m^{\nu_1,\kappa_1}\right] \times\sum_{\jmath=0}^{m-1}\left[\frac{|\phi'(\boldsymbol{u}_{m,\jmath})|}{\mathrm{e}^{\eta\boldsymbol{u}_{m,\jmath}}}+\frac{|\phi'(\boldsymbol{u}_{m,\jmath+1})|}{\mathrm{e}^{\eta\boldsymbol{u}_{m,\jmath+1}}}\right], \quad (23)$$

where

$$C_m^{\nu_1,\kappa_1} := \frac{\kappa_1}{\nu_1+\kappa_1}-\frac{2^{\frac{\nu_1}{\kappa_1}+1}}{m}\sum_{\ell=0}^{m}\left[\beta\left(\frac{\nu_1}{\kappa_1}+1,\ell+1\right) -\beta_{\frac{1}{2}}\left(\frac{\nu_1}{\kappa_1}+1,\ell+1\right)\right],$$
$$D_m^{\nu_1,\kappa_1} := \frac{\kappa_1}{\nu_1+\kappa_1}-\frac{1}{m}\sum_{\ell=0}^{m}\frac{\kappa_1}{2^{\ell}\left[\nu_1+\kappa_1(\ell+1)\right]},$$

where $\beta(\cdot,\cdot)$ and $\beta_{1/2}(\cdot,\cdot)$ are the beta function and incomplete beta function, respectively, i.e.,

$$\beta(x,y)=\int_0^1 \varrho^{x-1}(1-\varrho)^{y-1}\,\mathrm{d}\varrho;$$

$$\beta_a(x,y)=\int_0^a \varrho^{x-1}(1-\varrho)^{y-1}\,\mathrm{d}\varrho, \quad 0<a\leqslant 1.$$

Corollary 9 *Theorem 5 with $m=\kappa_1=1$ leads to*

$$\left|\frac{2^{\nu_1-1}\Gamma(\nu_1+1)}{(\wp_2-\wp_1)^{\nu_1}}\left\{\mathscr{I}^{\nu_1}_{\left(\frac{\wp_1+\wp_2}{2}\right)^+}\phi(\wp_2)+\mathscr{I}^{\nu_1}_{\left(\frac{\wp_1+\wp_2}{2}\right)^-}\phi(\wp_1)\right\}-\phi\left(\frac{\wp_1+\wp_2}{2}\right)\right|\leqslant\frac{(2-s)(\wp_2-\wp_1)}{4(\nu_1+1)}\times\left[\frac{|\phi'(\wp_1)|}{\mathrm{e}^{\eta\wp_1}}+\frac{|\phi'(\wp_2)|}{\mathrm{e}^{\eta\wp_2}}\right]. \quad (24)$$

Moreover, if $\eta=0$ and $s=1$, we get

$$\left|\frac{2^{\nu_1-1}\Gamma(\nu_1+1)}{(\wp_2-\wp_1)^{\nu_1}}\left\{\mathscr{I}^{\nu_1}_{\left(\frac{\wp_1+\wp_2}{2}\right)^+}\phi(\wp_2)+\mathscr{I}^{\nu_1}_{\left(\frac{\wp_1+\wp_2}{2}\right)^-}\phi(\wp_1)\right\}-\phi\left(\frac{\wp_1+\wp_2}{2}\right)\right|\leqslant\frac{\wp_2-\wp_1}{4(\nu_1+1)}\left[|\phi'(\wp_1)|+|\phi'(\wp_2)|\right],$$

which is established in the first step of proof of [29, Theorem 5].

Theorem 6 *Let $s\in[0,1]$, $\nu_1,\kappa_1>0$, $m\in\mathbb{N}$ and $\phi:[\wp_1,\wp_2]\to\mathbb{R}$ be a differentiable function on $(\wp_1,\wp_2)$ such that $\phi'\in\mathscr{L}[\wp_1,\wp_2]$. If $|\phi'|^{\rho_2}$ is m-polynomial exponentially s-type convex function on $[\wp_1,\wp_2]$ and $\eta\in\mathbb{R}$, then we have for $\frac{1}{\rho_1}+\frac{1}{\rho_2}=1$, $\rho_2>1$:*

$$\left|S_m^{\nu_1,\kappa_1}(\phi;\wp_1,\wp_2)\right|\leqslant\left(\frac{\wp_2-\wp_1}{4m}\right)\left(\frac{\kappa_1}{\rho_1\nu_1+\kappa_1}\right)^{\frac{1}{\rho_1}}\times\sum_{\jmath=0}^{m-1}\left\{\left(E_m(s)\frac{|\phi'(\boldsymbol{u}_{m,\jmath})|^{\rho_2}}{\mathrm{e}^{\eta\boldsymbol{u}_{m,\jmath}}}+F_m(s)\frac{|\phi'(\boldsymbol{u}_{m,\jmath+1})|^{\rho_2}}{\mathrm{e}^{\eta\boldsymbol{u}_{m,\jmath+1}}}\right)^{\frac{1}{\rho_2}}+\left(E_m(s)\frac{|\phi'(\boldsymbol{u}_{m,\jmath+1})|^{\rho_2}}{\mathrm{e}^{\eta\boldsymbol{u}_{m,\jmath+1}}}+F_m(s)\frac{|\phi'(\boldsymbol{u}_{m,\jmath})|^{\rho_2}}{\mathrm{e}^{\eta\boldsymbol{u}_{m,\jmath}}}\right)^{\frac{1}{\rho_2}}\right\}, \quad (25)$$

where

$$E_m(s)=\frac{1}{m}\sum_{\ell=1}^{m}\int_0^1\left[1-\left(s\left(1-\frac{\varrho}{2}\right)\right)^{\ell}\right]\mathrm{d}\varrho=1-\frac{2}{m}\sum_{\ell=1}^{m}\frac{s^{\ell}}{\ell+1}\left(1-\frac{1}{2^{\ell+1}}\right),$$

and

$$F_m(s)=\frac{1}{m}\sum_{\ell=1}^{m}\int_0^1\left[1-\left(s\frac{\varrho}{2}\right)^{\ell}\right]\mathrm{d}\varrho=1-\frac{1}{m}\sum_{\ell=1}^{m}\frac{s^{\ell}}{2^{\ell}(\ell+1)}.$$

Proof: By making use of Lemma 1, Hölder's inequality and properties of modulus, we can deduce

$$\begin{aligned}\left|S_m^{\nu_1,\kappa_1}(\phi;\wp_1,\wp_2)\right|&\leqslant\left(\frac{\wp_2-\wp_1}{4m}\right)\times\sum_{\jmath=0}^{m-1}\left\{\int_0^1\varrho^{\frac{\nu_1}{\kappa_1}}\left|\phi'\left(\frac{\varrho}{2}\mathbf{u}_{m,\jmath}+\frac{(2-\varrho)}{2}\mathbf{u}_{m,\jmath+1}\right)\right|\mathrm{d}\varrho+\int_0^1\varrho^{\frac{\nu_1}{\kappa_1}}\left|\phi'\left(\frac{(2-\varrho)}{2}\mathbf{u}_{m,\jmath}+\frac{\varrho}{2}\mathbf{u}_{m,\jmath+1}\right)\right|\mathrm{d}\varrho\right\}\\&\leqslant\left(\frac{\wp_2-\wp_1}{4m}\right)\left(\int_0^1\varrho^{\frac{\rho_1\nu_1}{\kappa_1}}d\varrho\right)^{\frac{1}{\rho_1}}\times\sum_{\jmath=0}^{m-1}\left\{\left(\int_0^1\left|\phi'\left(\frac{\varrho}{2}\mathbf{u}_{m,\jmath}+\frac{(2-\varrho)}{2}\mathbf{u}_{m,\jmath+1}\right)\right|^{\rho_2}\mathrm{d}\varrho\right)^{\frac{1}{\rho_2}}+\left(\int_0^1\left|\phi'\left(\frac{(2-\varrho)}{2}\mathbf{u}_{m,\jmath}+\frac{\varrho}{2}\mathbf{u}_{m,\jmath+1}\right)\right|^{\rho_2}\mathrm{d}\varrho\right)^{\frac{1}{\rho_2}}\right\}.\end{aligned}$$

Then, by making use of m-polynomial exponentially s-type convexity of $|\phi'|^{\rho_2}$, we get

$$\begin{aligned}\left|S_m^{\nu_1,\kappa_1}(\phi;\wp_1,\wp_2)\right|&\leqslant\left(\frac{\wp_2-\wp_1}{4m}\right)\left(\frac{\kappa_1}{\rho_1\nu_1+\kappa_1}\right)^{\frac{1}{\rho_1}}\times\sum_{\jmath=0}^{m-1}\left\{\left(\int_0^1\left[\frac{1}{m}\sum_{\ell=1}^{m}\left[1-\left(s\left(1-\frac{\varrho}{2}\right)\right)^{\ell}\right]\frac{|\phi'(\mathbf{u}_{m,\jmath})|^{\rho_2}}{\mathrm{e}^{\eta\mathbf{u}_{m,\jmath}}}+\frac{1}{m}\sum_{\ell=1}^{m}\left[1-\left(s\frac{\varrho}{2}\right)^{\ell}\right]\frac{|\phi'(\mathbf{u}_{m,\jmath+1})|^{\rho_2}}{\mathrm{e}^{\eta\mathbf{u}_{m,\jmath+1}}}\right]\mathrm{d}\varrho\right)^{\frac{1}{\rho_2}}\right.\\&\left.\quad+\left(\int_0^1\left[\frac{1}{m}\sum_{\ell=1}^{m}\left[1-\left(s\left(1-\frac{\varrho}{2}\right)\right)^{\ell}\right]\frac{|\phi'(\mathbf{u}_{m,\jmath+1})|^{\rho_2}}{\mathrm{e}^{\eta\mathbf{u}_{m,\jmath+1}}}+\frac{1}{m}\sum_{\ell=1}^{m}\left[1-\left(s\frac{\varrho}{2}\right)^{\ell}\right]\frac{|\phi'(\mathbf{u}_{m,\jmath})|^{\rho_2}}{\mathrm{e}^{\eta\mathbf{u}_{m,\jmath}}}\right]\mathrm{d}\varrho\right)^{\frac{1}{\rho_2}}\right\}\\&=\left(\frac{\wp_2-\wp_1}{4m}\right)\left(\frac{\kappa_1}{\rho_1\nu_1+\kappa_1}\right)^{\frac{1}{\rho_1}}\times\sum_{\jmath=0}^{m-1}\left\{\left(E_m(s)\frac{|\phi'(\mathbf{u}_{m,\jmath})|^{\rho_2}}{\mathrm{e}^{\eta\mathbf{u}_{m,\jmath}}}+F_m(s)\frac{|\phi'(\mathbf{u}_{m,\jmath+1})|^{\rho_2}}{\mathrm{e}^{\eta\mathbf{u}_{m,\jmath+1}}}\right)^{\frac{1}{\rho_2}}+\left(E_m(s)\frac{|\phi'(\mathbf{u}_{m,\jmath+1})|^{\rho_2}}{\mathrm{e}^{\eta\mathbf{u}_{m,\jmath+1}}}+F_m(s)\frac{|\phi'(\mathbf{u}_{m,\jmath})|^{\rho_2}}{\mathrm{e}^{\eta\mathbf{u}_{m,\jmath}}}\right)^{\frac{1}{\rho_2}}\right\}.\end{aligned}$$

This ends our proof. □

Corollary 10 *Theorem 6 with $\eta=0$ leads to*

$$\left|S_m^{\nu_1,\kappa_1}(\phi;\wp_1,\wp_2)\right|\leqslant\left(\frac{\wp_2-\wp_1}{4m}\right)\left(\frac{\kappa_1}{\rho_1\nu_1+\kappa_1}\right)^{\frac{1}{\rho_1}}\times\sum_{\jmath=0}^{m-1}\left\{\left(E_m(s)|\phi'(\boldsymbol{u}_{m,\jmath})|^{\rho_2}+F_m(s)|\phi'(\boldsymbol{u}_{m,\jmath+1})|^{\rho_2}\right)^{\frac{1}{\rho_2}}+\left(E_m(s)|\phi'(\boldsymbol{u}_{m,\jmath+1})|^{\rho_2}+F_m(s)|\phi'(\boldsymbol{u}_{m,\jmath})|^{\rho_2}\right)^{\frac{1}{\rho_2}}\right\}. \quad (26)$$

Corollary 11 *Theorem 6 with $s=0$ leads to*

$$\left|S_m^{\nu_1,\kappa_1}(\phi;\wp_1,\wp_2)\right| \leqslant \left(\frac{\wp_2-\wp_1}{2m}\right)\left(\frac{\kappa_1}{\rho_1\nu_1+\kappa_1}\right)^{\frac{1}{\rho_1}} \times \sum_{\jmath=0}^{m-1}\left(\frac{|\phi'(\boldsymbol{u}_{m,\jmath})|^{\rho_2}}{e^{\eta \boldsymbol{u}_{m,\jmath}}}+\frac{|\phi'(\boldsymbol{u}_{m,\jmath+1})|^{\rho_2}}{e^{\eta \boldsymbol{u}_{m,\jmath+1}}}\right)^{\frac{1}{\rho_2}}. \quad (27)$$

Corollary 12 *Theorem 6 with $s=1$ leads to*

$$\left|S_m^{\nu_1,\kappa_1}(\phi;\wp_1,\wp_2)\right| \leqslant \left(\frac{\wp_2-\wp_1}{4m}\right)\left(\frac{\kappa_1}{\rho_1\nu_1+\kappa_1}\right)^{\frac{1}{\rho_1}} \times \sum_{\jmath=0}^{m-1}\left\{\left(E_m\frac{|\phi'(\boldsymbol{u}_{m,\jmath})|^{\rho_2}}{e^{\eta \boldsymbol{u}_{m,\jmath}}}+F_m\frac{|\phi'(\boldsymbol{u}_{m,\jmath+1})|^{\rho_2}}{e^{\eta \boldsymbol{u}_{m,\jmath+1}}}\right)^{\frac{1}{\rho_2}} + \left(E_m\frac{|\phi'(\boldsymbol{u}_{m,\jmath+1})|^{\rho_2}}{e^{\eta \boldsymbol{u}_{m,\jmath+1}}}+F_m\frac{|\phi'(\boldsymbol{u}_{m,\jmath})|^{\rho_2}}{e^{\eta \boldsymbol{u}_{m,\jmath}}}\right)^{\frac{1}{\rho_2}}\right\}, \quad (28)$$

where

$$E_m = \frac{1}{m}\sum_{\ell=1}^{m}\int_0^1\left[1-\left(1-\frac{\varrho}{2}\right)^{\ell}\right]d\varrho = 1-\frac{2}{m}\sum_{\ell=1}^{m}\frac{1}{\ell+1}\left(1-\frac{1}{2^{\ell+1}}\right),$$

and

$$F_m = \frac{1}{m}\sum_{\ell=1}^{m}\int_0^1\left[1-\left(\frac{\varrho}{2}\right)^{\ell}\right]d\varrho = 1-\frac{1}{m}\sum_{\ell=1}^{m}\frac{1}{2^{\ell}(\ell+1)}.$$

Corollary 13 *Theorem 6 with $m=\kappa_1=1$ leads to*

$$\left|\frac{2^{\nu_1-1}\Gamma(\nu_1+1)}{(\wp_2-\wp_1)^{\nu_1}}\left\{\mathscr{I}^{\nu_1}_{(\frac{\wp_1+\wp_2}{2})^+}\phi(\wp_2)+\mathscr{I}^{\nu_1}_{(\frac{\wp_1+\wp_2}{2})^-}\phi(\wp_1)\right\} - \phi\left(\frac{\wp_1+\wp_2}{2}\right)\right| \leqslant \left(\frac{\wp_2-\wp_1}{4}\right)\left(\frac{1}{\rho_1\nu_1+1}\right)^{\frac{1}{\rho_1}}\left(\frac{1}{4}\right)^{\frac{1}{\rho_2}} \times\left\{\left((4-3s)\frac{|\phi'(\wp_1)|^{\rho_2}}{e^{\eta\wp_1}}+(4-s)\frac{|\phi'(\wp_2)|^{\rho_2}}{e^{\eta\wp_2}}\right)^{\frac{1}{\rho_2}} + \left((4-3s)\frac{|\phi'(\wp_2)|^{\rho_2}}{e^{\eta\wp_2}}+(4-s)\frac{|\phi'(\wp_1)|^{\rho_2}}{e^{\eta\wp_1}}\right)^{\frac{1}{\rho_2}}\right\}. \quad (29)$$

Moreover, if $\eta=0$ and $s=1$, we get

$$\left|\frac{2^{\nu_1-1}\Gamma(\nu_1+1)}{(\wp_2-\wp_1)^{\nu_1}}\left\{\mathscr{I}^{\nu_1}_{(\frac{\wp_1+\wp_2}{2})^+}\phi(\wp_2)+\mathscr{I}^{\nu_1}_{(\frac{\wp_1+\wp_2}{2})^-}\phi(\wp_1)\right\} - \phi\left(\frac{\wp_1+\wp_2}{2}\right)\right| \leqslant \left(\frac{\wp_2-\wp_1}{4}\right)\left(\frac{1}{\rho_1\nu_1+1}\right)^{\frac{1}{\rho_1}} \times\left\{\left(\frac{|\phi'(\wp_1)|^{\rho_2}+3|\phi'(\wp_2)|^{\rho_2}}{4}\right)^{\frac{1}{\rho_2}} + \left(\frac{3|\phi'(\wp_1)|^{\rho_2}+|\phi'(\wp_2)|^{\rho_2}}{4}\right)^{\frac{1}{\rho_2}}\right\},$$

which is established in [29, Theorem 6].

Theorem 7 *Let $s\in[0,1]$, $\nu_1,\kappa_1>0$, $m\in\mathbb{N}$ and $\phi:[\wp_1,\wp_2]\to\mathbb{R}$ be a differentiable function on $(\wp_1,\wp_2)$ such that $\phi'\in\mathscr{L}[\wp_1,\wp_2]$. If $|\phi'|^{\rho_2}$ is m-polynomial exponentially s-type convex function on $[\wp_1,\wp_2]$ and $\eta\in\mathbb{R}$, then we have for $\rho_2>1$:*

$$\left|S_m^{\nu_1,\kappa_1}(\phi;\wp_1,\wp_2)\right| \leqslant \left(\frac{\wp_2-\wp_1}{4m}\right)\left(\frac{\kappa_1}{\nu_1+\kappa_1}\right)^{1-\frac{1}{\rho_2}} \times\sum_{\jmath=0}^{m-1}\left\{\left(C_m^{\nu_1,\kappa_1}(s)\frac{|\phi'(\boldsymbol{u}_{m,\jmath})|^{\rho_2}}{e^{\eta \boldsymbol{u}_{m,\jmath}}}+D_m^{\nu_1,\kappa_1}(s)\frac{|\phi'(\boldsymbol{u}_{m,\jmath+1})|^{\rho_2}}{e^{\eta \boldsymbol{u}_{m,\jmath+1}}}\right)^{\frac{1}{\rho_2}} +\left(C_m^{\nu_1,\kappa_1}(s)\frac{|\phi'(\boldsymbol{u}_{m,\jmath+1})|^{\rho_2}}{e^{\eta \boldsymbol{u}_{m,\jmath+1}}}+D_m^{\nu_1,\kappa_1}(s)\frac{|\phi'(\boldsymbol{u}_{m,\jmath})|^{\rho_2}}{e^{\eta \boldsymbol{u}_{m,\jmath}}}\right)^{\frac{1}{\rho_2}}\right\}, \quad (30)$$

where $C_m^{\nu_1,\kappa_1}(s)$ and $D_m^{\nu_1,\kappa_1}(s)$ are as given in Theorem 5.

Proof: By making use of Lemma 1, the power mean inequality and properties of modulus, we have

$$\left|S_m^{\nu_1,\kappa_1}(\phi;\wp_1,\wp_2)\right| \leqslant \left(\frac{\wp_2-\wp_1}{4m}\right) \times\sum_{\jmath=0}^{m-1}\left\{\int_0^1 \varrho^{\frac{\nu_1}{\kappa_1}}\left|\phi'\left(\frac{\varrho}{2}\mathbf{u}_{m,\jmath}+\frac{(2-\varrho)}{2}\mathbf{u}_{m,\jmath+1}\right)\right|d\varrho + \int_0^1 \varrho^{\frac{\nu_1}{\kappa_1}}\left|\phi'\left(\frac{(2-\varrho)}{2}\mathbf{u}_{m,\jmath}+\frac{\varrho}{2}\mathbf{u}_{m,\jmath+1}\right)\right|d\varrho\right\}$$
$$\leqslant \left(\frac{\wp_2-\wp_1}{4m}\right)\left(\int_0^1\varrho^{\frac{\nu_1}{\kappa_1}}d\varrho\right)^{1-\frac{1}{\rho_2}} \times\sum_{\jmath=0}^{m-1}\left\{\left(\int_0^1 \varrho^{\frac{\nu_1}{\kappa_1}}\left|\phi'\left(\frac{\varrho}{2}\mathbf{u}_{m,\jmath}+\frac{(2-\varrho)}{2}\mathbf{u}_{m,\jmath+1}\right)\right|^{\rho_2}d\varrho\right)^{\frac{1}{\rho_2}} + \left(\int_0^1 \varrho^{\frac{\nu_1}{\kappa_1}}\left|\phi'\left(\frac{(2-\varrho)}{2}\mathbf{u}_{m,\jmath}+\frac{\varrho}{2}\mathbf{u}_{m,\jmath+1}\right)\right|^{\rho_2}d\varrho\right)^{\frac{1}{\rho_2}}\right\}.$$

Then, by making use of m-polynomial exponentially s-type convexity of $|\phi'|^{\rho_2}$, we have

$$\left|S_m^{\nu_1,\kappa_1}(\phi;\wp_1,\wp_2)\right| \leqslant \left(\frac{\wp_2-\wp_1}{4m}\right)\left(\frac{\kappa_1}{\nu_1+\kappa_1}\right)^{1-\frac{1}{\rho_2}} \times\sum_{\jmath=0}^{m-1}\left\{\left(\int_0^1\varrho^{\frac{\nu_1}{\kappa_1}}\left[\frac{1}{m}\sum_{\ell=1}^{m}\left[1-\left(s\left(1-\frac{\varrho}{2}\right)\right)^{\ell}\right]\frac{|\phi'(\mathbf{u}_{m,\jmath})|^{\rho_2}}{e^{\eta\mathbf{u}_{m,\jmath}}} + \frac{1}{m}\sum_{\ell=1}^{m}\left[1-\left(s\frac{\varrho}{2}\right)^{\ell}\right]\frac{|\phi'(\mathbf{u}_{m,\jmath+1})|^{\rho_2}}{e^{\eta\mathbf{u}_{m,\jmath+1}}}\right]d\varrho\right)^{\frac{1}{\rho_2}} + \left(\int_0^1\varrho^{\frac{\nu_1}{\kappa_1}}\left[\frac{1}{m}\sum_{\ell=1}^{m}\left[1-\left(s\left(1-\frac{\varrho}{2}\right)\right)^{\ell}\right]\frac{|\phi'(\mathbf{u}_{m,\jmath+1})|^{\rho_2}}{e^{\eta\mathbf{u}_{m,\jmath+1}}} + \frac{1}{m}\sum_{\ell=1}^{m}\left[1-\left(s\frac{\varrho}{2}\right)^{\ell}\right]\frac{|\phi'(\mathbf{u}_{m,\jmath})|^{\rho_2}}{e^{\eta\mathbf{u}_{m,\jmath}}}\right]d\varrho\right)^{\frac{1}{\rho_2}}\right\}$$

$$=\left(\frac{\wp_2-\wp_1}{4m}\right)\left(\frac{\kappa_1}{\nu_1+\kappa_1}\right)^{1-\frac{1}{\rho_2}}$$
$$\times\sum_{\jmath=0}^{m-1}\left\{\left(C_m^{\nu_1,\kappa_1}(s)\frac{|\phi'(\mathbf{u}_{m,\jmath})|^{\rho_2}}{e^{\eta\mathbf{u}_{m,\jmath}}}+D_m^{\nu_1,\kappa_1}(s)\frac{|\phi'(\mathbf{u}_{m,\jmath+1})|^{\rho_2}}{e^{\eta\mathbf{u}_{m,\jmath+1}}}\right)^{\frac{1}{\rho_2}}\right.$$
$$\left.+\left(C_m^{\nu_1,\kappa_1}(s)\frac{|\phi'(\mathbf{u}_{m,\jmath+1})|^{\rho_2}}{e^{\eta\mathbf{u}_{m,\jmath+1}}}+D_m^{\nu_1,\kappa_1}(s)\frac{|\phi'(\mathbf{u}_{m,\jmath})|^{\rho_2}}{e^{\eta\mathbf{u}_{m,\jmath}}}\right)^{\frac{1}{\rho_2}}\right\},$$

which completes the proof. □

Corollary 14 *Theorem 7 with $\eta=0$ leads to*

$$\left|S_m^{\nu_1,\kappa_1}(\phi;\wp_1,\wp_2)\right|\leqslant\left(\frac{\wp_2-\wp_1}{4m}\right)\left(\frac{\kappa_1}{\nu_1+\kappa_1}\right)^{1-\frac{1}{\rho_2}}$$
$$\times\sum_{\jmath=0}^{m-1}\left\{\left(C_m^{\nu_1,\kappa_1}(s)|\phi'(\boldsymbol{u}_{m,\jmath})|^{\rho_2}+D_m^{\nu_1,\kappa_1}(s)|\phi'(\boldsymbol{u}_{m,\jmath+1})|^{\rho_2}\right)^{\frac{1}{\rho_2}}\right.$$
$$\left.+\left(C_m^{\nu_1,\kappa_1}(s)|\phi'(\boldsymbol{u}_{m,\jmath+1})|^{\rho_2}+D_m^{\nu_1,\kappa_1}(s)|\phi'(\boldsymbol{u}_{m,\jmath})|^{\rho_2}\right)^{\frac{1}{\rho_2}}\right\}.\quad(31)$$

Corollary 15 *Theorem 7 with $s=0$ leads to*

$$\left|S_m^{\nu_1,\kappa_1}(\phi;\wp_1,\wp_2)\right|\leqslant\left(\frac{\wp_2-\wp_1}{2m}\right)\left(\frac{\kappa_1}{\nu_1+\kappa_1}\right)\times\sum_{\jmath=0}^{m-1}\left(\frac{|\phi'(\boldsymbol{u}_{m,\jmath})|^{\rho_2}}{e^{\eta\boldsymbol{u}_{m,\jmath}}}+\frac{|\phi'(\boldsymbol{u}_{m,\jmath+1})|^{\rho_2}}{e^{\eta\boldsymbol{u}_{m,\jmath+1}}}\right)^{\frac{1}{\rho_2}}.\quad(32)$$

Corollary 16 *Theorem 7 with $s=1$ leads to*

$$\left|S_m^{\nu_1,\kappa_1}(\phi;\wp_1,\wp_2)\right|\leqslant\left(\frac{\wp_2-\wp_1}{4m}\right)\left(\frac{\kappa_1}{\nu_1+\kappa_1}\right)^{1-\frac{1}{\rho_2}}$$
$$\times\sum_{\jmath=0}^{m-1}\left\{\left(C_m^{\nu_1,\kappa_1}\frac{|\phi'(\boldsymbol{u}_{m,\jmath})|^{\rho_2}}{e^{\eta\boldsymbol{u}_{m,\jmath}}}+D_m^{\nu_1,\kappa_1}\frac{|\phi'(\boldsymbol{u}_{m,\jmath+1})|^{\rho_2}}{e^{\eta\boldsymbol{u}_{m,\jmath+1}}}\right)^{\frac{1}{\rho_2}}\right.$$
$$\left.+\left(C_m^{\nu_1,\kappa_1}\frac{|\phi'(\boldsymbol{u}_{m,\jmath+1})|^{\rho_2}}{e^{\eta\boldsymbol{u}_{m,\jmath+1}}}+D_m^{\nu_1,\kappa_1}\frac{|\phi'(\boldsymbol{u}_{m,\jmath})|^{\rho_2}}{e^{\eta\boldsymbol{u}_{m,\jmath}}}\right)^{\frac{1}{\rho_2}}\right\},\quad(33)$$

where $C_m^{\nu_1,\kappa_1}$ and $D_m^{\nu_1,\kappa_1}$ are as given in Corollary 8.

Corollary 17 *Theorem 7 with $m=\kappa_1=1$ leads to*

$$\left|\frac{2^{\nu_1-1}\Gamma(\nu_1+1)}{(\wp_2-\wp_1)^{\nu_1}}\left\{\mathscr{I}^{\nu_1}_{\left(\frac{\wp_1+\wp_2}{2}\right)^+}\phi(\wp_2)+\mathscr{I}^{\nu_1}_{\left(\frac{\wp_1+\wp_2}{2}\right)^-}\phi(\wp_1)\right\}-\phi\left(\frac{\wp_1+\wp_2}{2}\right)\right|\leqslant\left(\frac{\wp_2-\wp_1}{4}\right)\left(\frac{1}{\nu_1+1}\right)^{1-\frac{1}{\rho_2}}$$
$$\times\left\{\left(\left[\frac{1-s}{\nu_1+1}+\frac{1}{2(\nu_1+2)}\right]\frac{|\phi'(\wp_1)|^{\rho_2}}{e^{\eta\wp_1}}+\left[\frac{1}{\nu_1+1}-\frac{1}{2(\nu_1+2)}\right]\frac{|\phi'(\wp_2)|^{\rho_2}}{e^{\eta\wp_2}}\right)^{\frac{1}{\rho_2}}\right.$$
$$\left.+\left(\left[\frac{1-s}{\nu_1+1}+\frac{1}{2(\nu_1+2)}\right]\frac{|\phi'(\wp_2)|^{\rho_2}}{e^{\eta\wp_2}}+\left[\frac{1}{\nu_1+1}-\frac{1}{2(\nu_1+2)}\right]\frac{|\phi'(\wp_1)|^{\rho_2}}{e^{\eta\wp_1}}\right)^{\frac{1}{\rho_2}}\right\}.\quad(34)$$

Moreover, if $\eta=0$ and $s=1$, we get

$$\left|\frac{2^{\nu_1-1}\Gamma(\nu_1+1)}{(\wp_2-\wp_1)^{\nu_1}}\left\{\mathscr{I}^{\nu_1}_{\left(\frac{\wp_1+\wp_2}{2}\right)^+}\phi(\wp_2)+\mathscr{I}^{\nu_1}_{\left(\frac{\wp_1+\wp_2}{2}\right)^-}\phi(\wp_1)\right\}-\phi\left(\frac{\wp_1+\wp_2}{2}\right)\right|\leqslant\left(\frac{\wp_2-\wp_1}{4(\nu_1+1)}\right)$$
$$\times\left\{\left(\frac{\nu_1+1}{2(\nu_1+2)}|\phi'(\wp_1)|^{\rho_2}+\frac{\nu_1+3}{2(\nu_1+2)}|\phi'(\wp_2)|^{\rho_2}\right)^{\frac{1}{\rho_2}}+\left(\frac{\nu_1+3}{2(\nu_1+2)}|\phi'(\wp_1)|^{\rho_2}+\frac{\nu_1+1}{2(\nu_1+2)}|\phi'(\wp_2)|^{\rho_2}\right)^{\frac{1}{\rho_2}}\right\},$$

which is established in [29, Theorem 5].

EXAMPLES AND APPLICATIONS

Bessel functions

Consider the function $\mathfrak{B}_\tau:(0,\infty)\to[1,\infty)$ given by

$$\mathfrak{B}_\tau(x)=2^\tau\Gamma(\tau+1)x^{-\tau}\mathscr{N}_\tau(x),$$

where $\mathscr{N}_\tau$ is the modified Bessel function of the first kind defined by (see [30, Eq. (2), pp 77]):

$$\mathscr{N}_\tau(x)=\sum_{m=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{\tau+2m}}{m!\Gamma(\tau+1+m)},\quad x\in\mathbb{R}.$$

Following [30], we have

$$\mathfrak{B}'_\tau(x)=\frac{x}{2(\tau+1)}\mathfrak{B}_{\tau+1}(x),\quad(35)$$

$$\mathfrak{B}''_\tau(x)=\frac{x^2\mathfrak{B}_{\tau+2}(x)}{4(\tau+1)(\tau+2)}+\frac{\mathfrak{B}_{\tau+1}(x)}{2(\tau+1)}.\quad(36)$$

Assume that all assumptions of the used corollaries in the following examples are satisfied.

Example 1 Let $0<\wp_1<\wp_2$ and $\tau>-1$. Then, by applying Corollary 9 with $\nu_1=1$ and the identities (35) and (36), we obtain

$$\left|\frac{\mathfrak{B}_\tau(\wp_2)-\mathfrak{B}_\tau(\wp_1)}{\wp_2-\wp_1}-\frac{(\wp_1+\wp_2)}{4(\tau+1)}\mathfrak{B}_{\tau+1}\left(\frac{\wp_1+\wp_2}{2}\right)\right|$$
$$\leqslant\frac{(2-s)(\wp_2-\wp_1)}{8}\left[\frac{1}{e^{\eta\wp_1}}\left(\frac{\wp_1^2\mathfrak{B}_{\tau+2}(\wp_1)}{4(\tau+1)(\tau+2)}+\frac{\mathfrak{B}_{\tau+1}(\wp_1)}{2(\tau+1)}\right)+\frac{1}{e^{\eta\wp_2}}\left(\frac{\wp_2^2\mathfrak{B}_{\tau+2}(\wp_2)}{4(\tau+1)(\tau+2)}+\frac{\mathfrak{B}_{\tau+1}(\wp_2)}{2(\tau+1)}\right)\right].$$

Example 2 Let $0<\wp_1<\wp_2$ and $\tau>-1$. Then, by applying Corollary 13 with $\nu_1=1$ and the identities

(35) and (36), we obtain

$$\left|\frac{\mathfrak{B}_\tau(\wp_2)-\mathfrak{B}_\tau(\wp_1)}{\wp_2-\wp_1}-\frac{(\wp_1+\wp_2)}{4(\tau+1)}\mathfrak{B}_{\tau+1}\left(\frac{\wp_1+\wp_2}{2}\right)\right|$$
$$\leqslant(\wp_2-\wp_1)\left(\frac{1}{4}\right)^{1+\frac{1}{\rho_2}}\left(\frac{1}{\rho_1+1}\right)^{\frac{1}{\rho_1}}$$
$$\times\left\{\left[\frac{(4-3s)}{e^{\eta\wp_1}}\left(\frac{\wp_1^2\mathfrak{B}_{\tau+2}(\wp_1)}{4(\tau+1)(\tau+2)}+\frac{\mathfrak{B}_{\tau+1}(\wp_1)}{2(\tau+1)}\right)^{\rho_2}\right.\right.$$
$$\left.+\frac{(4-s)}{e^{\eta\wp_2}}\left(\frac{\wp_2^2\mathfrak{B}_{\tau+2}(\wp_2)}{4(\tau+1)(\tau+2)}+\frac{\mathfrak{B}_{\tau+1}(\wp_2)}{2(\tau+1)}\right)^{\rho_2}\right]^{\frac{1}{\rho_2}}$$
$$+\left[\frac{(4-s)}{e^{\eta\wp_1}}\left(\frac{\wp_1^2\mathfrak{B}_{\tau+2}(\wp_1)}{4(\tau+1)(\tau+2)}+\frac{\mathfrak{B}_{\tau+1}(\wp_1)}{2(\tau+1)}\right)^{\rho_2}\right.$$
$$\left.\left.+\frac{(4-3s)}{e^{\eta\wp_2}}\left(\frac{\wp_2^2\mathfrak{B}_{\tau+2}(\wp_2)}{4(\tau+1)(\tau+2)}+\frac{\mathfrak{B}_{\tau+1}(\wp_2)}{2(\tau+1)}\right)^{\rho_2}\right]^{\frac{1}{\rho_2}}\right\}.$$

Example 3 Let $0<\wp_1<\wp_2$ and $\tau>-1$. Then, by applying Corollary 17 with $\nu_1=1$ and the identities (35) and (36), we obtain

$$\left|\frac{\mathfrak{B}_\tau(\wp_2)-\mathfrak{B}_\tau(\wp_1)}{\wp_2-\wp_1}-\frac{(\wp_1+\wp_2)}{4(\tau+1)}\mathfrak{B}_{\tau+1}\left(\frac{\wp_1+\wp_2}{2}\right)\right|$$
$$\leqslant(\wp_2-\wp_1)\left(\frac{1}{2}\right)^{3-\frac{1}{\rho_2}}$$
$$\times\left\{\left[\frac{(4-3s)}{6\,e^{\eta\wp_1}}\left(\frac{\wp_1^2\mathfrak{B}_{\tau+2}(\wp_1)}{4(\tau+1)(\tau+2)}+\frac{\mathfrak{B}_{\tau+1}(\wp_1)}{2(\tau+1)}\right)^{\rho_2}\right.\right.$$
$$\left.+\frac{1}{3\,e^{\eta\wp_2}}\left(\frac{\wp_2^2\mathfrak{B}_{\tau+2}(\wp_2)}{4(\tau+1)(\tau+2)}+\frac{\mathfrak{B}_{\tau+1}(\wp_2)}{2(\tau+1)}\right)^{\rho_2}\right]^{\frac{1}{\rho_2}}$$
$$+\left[\frac{1}{3\,e^{\eta\wp_1}}\left(\frac{\wp_1^2\mathfrak{B}_{\tau+2}(\wp_1)}{4(\tau+1)(\tau+2)}+\frac{\mathfrak{B}_{\tau+1}(\wp_1)}{2(\tau+1)}\right)^{\rho_2}\right.$$
$$\left.\left.+\frac{(4-3s)}{6\,e^{\eta\wp_2}}\left(\frac{\wp_2^2\mathfrak{B}_{\tau+2}(\wp_2)}{4(\tau+1)(\tau+2)}+\frac{\mathfrak{B}_{\tau+1}(\wp_2)}{2(\tau+1)}\right)^{\rho_2}\right]^{\frac{1}{\rho_2}}\right\}.$$

Special means

Let $\wp_1,\wp_2\in\mathbb{R}$ with $\wp_1\neq\wp_2$.

- The arithmetic mean:

$$\mathscr{A}(\wp_1,\wp_2)=\frac{\wp_1+\wp_2}{2}.$$

- The generalized logarithmic mean:

$$\mathscr{L}_r(\wp_1,\wp_2)=\left[\frac{\wp_2^{r+1}-\wp_1^{r+1}}{(r+1)(\wp_2-\wp_1)}\right]^{1/r},\quad r\in\mathbb{R}\backslash\{-1,0\}.$$

Proposition 2 *Let $s\in(0,1]$, $m\in\mathbb{N}$ and $0<\wp_1<\wp_2$. Then, we have for $\eta\leqslant0$:*

$$\left|m\mathscr{L}_s^s(\wp_1,\wp_2)-\sum_{\jmath=0}^{m-1}\frac{1}{m^s}\mathscr{A}^s\left((2(m-\jmath)-1)\wp_1,(2\jmath+1)\wp_2\right)\right|$$
$$\leqslant\frac{2^{s-3}(\wp_2-\wp_1)s}{m}\left[C_m(s)+D_m(s)\right]$$
$$\times\sum_{\jmath=0}^{m-1}\left[\frac{1}{e^{\eta((m-\jmath)\wp_1+\jmath\wp_2)}}\mathscr{A}^{s-1}\left((m-\jmath)\wp_1,\jmath\wp_2\right)\right.$$
$$\left.+\frac{1}{e^{\eta((m-\jmath-1)\wp_1+(\jmath+1)\wp_2)}}\mathscr{A}^{s-1}\left((m-\jmath-1)\wp_1,(\jmath+1)\wp_2\right)\right],\quad(37)$$

where

$$C_m(s)=\frac{1}{2}-\frac{4}{m}\sum_{\ell=1}^{m}s^\ell\left[\frac{1}{\ell+1}\left(1-\frac{1}{2^{\ell+1}}\right)\right.$$
$$\left.-\frac{1}{\ell+2}\left(1-\frac{1}{2^{\ell+2}}\right)\right];$$
$$D_m(s)=\frac{1}{2}-\frac{1}{m}\sum_{\ell=1}^{m}\frac{s^\ell}{2^\ell(\ell+2)}.$$

Proof: By making use of Proposition 1 and Theorem 5 with $\phi(x)=x^s$, $x\in[\wp_1,\wp_2]$, $s\in(0,1]$ and $\nu_1=\kappa_1=1$, we can obtain the desired result (37). □

Proposition 3 *Let $s\in(0,1]$, $m\in\mathbb{N}$ and $0<\wp_1<\wp_2$. Then, we have for $\eta\leqslant0$:*

$$\left|m\mathscr{L}_s^s(\wp_1,\wp_2)-\sum_{\jmath=0}^{m-1}\frac{1}{m^s}\mathscr{A}^s\left((2(m-\jmath)-1)\wp_1,(2\jmath+1)\wp_2\right)\right|$$
$$\leqslant\frac{2^{s-3}(\wp_2-\wp_1)s}{m}\left(\frac{1}{\rho_1+1}\right)^{\frac{1}{\rho_1}}$$
$$\times\sum_{\jmath=0}^{m-1}\left\{\left[\frac{E_m(s)}{e^{\eta((m-\jmath)\wp_1+\jmath\wp_2)}}\mathscr{A}^{\rho_2(s-1)}\left((m-\jmath)\wp_1,\jmath\wp_2\right)\right.\right.$$
$$\left.+\frac{F_m(s)}{e^{\eta((m-\jmath-1)\wp_1+(\jmath+1)\wp_2)}}\mathscr{A}^{\rho_2(s-1)}\left((m-\jmath-1)\wp_1,(\jmath+1)\wp_2\right)\right]^{\frac{1}{\rho_2}}$$
$$+\left[\frac{F_m(s)}{e^{\eta((m-\jmath)\wp_1+\jmath\wp_2)}}\mathscr{A}^{\rho_2(s-1)}\left((m-\jmath)\wp_1,\jmath\wp_2\right)\right.$$
$$+\frac{E_m(s)}{e^{\eta((m-\jmath-1)\wp_1+(\jmath+1)\wp_2)}}$$
$$\left.\left.\times\mathscr{A}^{\rho_2(s-1)}\left((m-\jmath-1)\wp_1,(\jmath+1)\wp_2\right)\right]^{\frac{1}{\rho_2}}\right\},\quad(38)$$

where $\frac{1}{\rho_1}+\frac{1}{\rho_2}=1$, $\rho_2>1$, and $E_m(s)$ and $F_m(s)$ are as given in Theorem 6.

Proof: By making use of Proposition 1 and Theorem 6 with $\phi(x)=x^s$, $x\in[\wp_1,\wp_2]$, $s\in(0,1]$ and $\nu_1=\kappa_1=1$, we can obtain the desired result (38). □

Proposition 4 *Let $s \in (0,1]$, $m \in \mathbb{N}$ and $0 < \wp_1 < \wp_2$. Then, we have for $\eta \leqslant 0$:*

$$\left| m\mathscr{L}_s^s(\wp_1,\wp_2) - \sum_{\jmath=0}^{m-1} \frac{1}{m^s} \mathscr{A}^s\left((2(m-\jmath)-1)\wp_1,(2\jmath+1)\wp_2\right) \right|$$
$$\leqslant \frac{2^{s-3}(\wp_2-\wp_1)s}{m}\left(\frac{1}{2}\right)^{3-\frac{1}{\rho_2}}$$
$$\times \sum_{\jmath=0}^{m-1}\left\{\left[\frac{C_m(s)}{e^{\eta((m-\jmath)\wp_1+\jmath\wp_2)}}\mathscr{A}^{\rho_2(s-1)}((m-\jmath)\wp_1,\jmath\wp_2)\right.\right.$$
$$\left.+\frac{D_m(s)}{e^{\eta((m-\jmath-1)\wp_1+(\jmath+1)\wp_2)}}\mathscr{A}^{\rho_2(s-1)}((m-\jmath-1)\wp_1,(\jmath+1)\wp_2)\right]^{\frac{1}{\rho_2}}$$
$$+\left[\frac{D_m(s)}{e^{\eta((m-\jmath)\wp_1+\jmath\wp_2)}}\mathscr{A}^{\rho_2(s-1)}((m-\jmath)\wp_1,\jmath\wp_2)\right.$$
$$+\frac{C_m(s)}{e^{\eta((m-\jmath-1)\wp_1+(\jmath+1)\wp_2)}}$$
$$\left.\left.\times\mathscr{A}^{\rho_2(s-1)}((m-\jmath-1)\wp_1,(\jmath+1)\wp_2)\right]^{\frac{1}{\rho_2}}\right\}, \quad (39)$$

where $\rho_2 > 1$, and $C_m(s)$ and $D_m(s)$ are as given in Proposition 2.

Proof: By making use of Proposition 1 and Theorem 7 with $\phi(x) = x^s$, $x \in [\wp_1,\wp_2], s \in (0,1]$ and $\nu_1 = \kappa_1 = 1$, we can obtain the desired result (39). □

CONCLUSION

In this paper we introduced a new class of convex functions namely the m-polynomial exponentially s-type convex functions. Also, we studied some of their algebraic properties. We have established new inequality of H-H type for the new defined convex function. In addition, we proved a new midpoint identity and some related integral inequalities for the new defined convex function. In Corollaries 6–17, we discussed many special cases which are obtained from the main results and we pointed out those are exist in the literature. Finally, we presented some applications of Bessel functions and special means by applying our results on specific values and functions. In any case, we hope that these results continue to sharpen our understanding of the nature of fractional calculus and their applications in different fields.

For future developments, we will derive several new interesting inequalities via Hölder-İşcan, Chebyshev, Markov, Young and Minkowski inequalities using fractional calculus for m-polynomial exponentially s-type convex functions. Moreover, interested reader can consider the mathematical equivalence (see e.g., [31]) among these proposed results.

Acknowledgements: This research was supported by Taif University Researchers Supporting Project Number (TURSP-2020/217), Taif University, Taif, Saudi Arabia, and the author B. Abdalla would like to thank Prince Sultan University for paying the APC and for the support through the TAS research lab.

REFERENCES

1. Hadamard J (1893) Étude sur les propriétés des fonctions entières en particulier d'une fonction considérée par Riemann. *J Math Pures Appl* **58**, 171–215.
2. Zhang TY, Ji AP, Qi F (2013) Some inequalities of Hermite-Hadamard type for *GA*-convex functions with applications to means. *Le Mat* **68**, 229–239.
3. Mohammed PO, Vivas-Cortez M, Abdeljawad T, Rangel-Oliveros Y (2020) Integral inequalities of Hermite-Hadamard type for quasi-convex functions with applications. *AIMS Math* **5**, 7316–7331.
4. Zhang TY, Ji AP, Qi F (2012) On integral inequalities of Hermite-Hadamard type for s-geometrically convex functions. *Abst Appl Anal* **2012**, 560586.
5. Shi DP, Xi BY, Qi, F (2014) Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals of (α,m)-convex functions. *Fract Differ Calc* **4**, 31–43.
6. Dragomir SS, Pearce CEM (2000) *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, Footscray, Australia.
7. Dragomir SS, Pečarić J, Persson LE (1995) Some inequalities of Hadamard type. *Soochow J Math* **21**, 335–341.
8. Awan MU, Noor MA, Noor KI (2018) Hermite-Hadamard inequalities for exponentially convex functions. *Appl Math Inf Sci* **12**, 405–409.
9. Toply T, Kadakal M, İşcan İ (2020) On n-polynomial convexity and some related inequalities. *AIMS Math* **5**, 1304–1318.
10. Rashid S, İşcan İ, Baleanu D, Chu YM (2020) Generalization of new fractional inequalities via n-polynomials s-type convexity with applications. *Adv Differ Equ* **2020**, 264.
11. Mubeen S, Habibullah GM (2012) k-Fractional integrals and application. *Int J Contemp Math Sci* **7**, 89–94.
12. Han J, Mohammed PO, Zeng H (2020) Generalized fractional integral inequalities of Hermite-Hadamard-type for a convex function. *Open Math* **18**, 794–80.
13. Fernandez A, Mohammed P (2020) Hermite-Hadamard inequalities in fractional calculus defined using Mittag-Leffler kernels. *Math Meth Appl Sci* **44**, 8414–8431.
14. Baleanu D, Mohammed PO, Zeng S (2020) Inequalities of trapezoidal type involving generalized fractional integrals. *Alex Eng J* **59**, 2975–2984.
15. Mohammed PO, Abdeljawad T, Zeng S, Kashuri A (2020) Fractional Hermite-Hadamard integral inequalities for a new class of convex functions. *Symmetry* **12**, 1485.
16. Akhtar N, Awan MU, Javed MZ, Rassias MT, Mihai MV, Noor MA, Noor KI (2021) Ostrowski type inequalities involving harmonically convex functions and applications. *Symmetry* **13**, 201.
17. Noor MA, Noor KI, Rassias MT (2020) New trends in general variational inequalities. *Acta Appl Math* **170**, 981–1064.
18. Noor MA, Noor KI, Rassias MT (2021) Characterizations of higher order strongly generalized convex functions. In: Rassias TM (ed) *Nonlinear Analysis, Differential Equations, and Applications*, Springer Optimization and Its Applications, vol 173, Springer, Cham., pp 341–364.

19. Mohammed PO, Brevik I (2020) A new version of the Hermite-Hadamard inequality for Riemann-Liouville fractional integrals. *Symmetry* **12**, 610.
20. Li JF, Rashid S, Liu JB, Akdemir AO, Safda F (2020) Inequalities involving conformable approach for exponentially convex functions and their applications. *J Funct Spaces* **2020**, 6517068.
21. Mohammed PO, Sarikaya MZ (2020) On generalized fractional integral inequalities for twice differentiable convex functions. *J Comput Appl Math* **372**, 112740.
22. Ferreira RAC (2021) Generalized discrete operators. *J Frac Calc & Nonlinear Sys* **2**, 18–23.
23. Mohammed PO (2021) Hermite-Hadamard inequalities for Riemann-Liouville fractional integrals of a convex function with respect to a monotone function. *Math Meth Appl Sci* **44**, 2314–2324.
24. Mohammed PO, Sarikaya MZ, Baleanu D (2020) On the generalized Hermite-Hadamard inequalities via the tempered fractional integrals. *Symmetry* **12**, 595.
25. Kashuri A, Liko R (2019) On some k-fractional integral inequalities of Hermite-Hadamard type for twice differentiable generalized beta (r,g)-preinvex functions. *J Appl Anal* **25**, 59–72.
26. Set E, Choi J, Gözpinar A (2017) Hermite-Hadamard type inequalities for the generalized k-fractional integral operators. *J Inequal Appl* **2017**, 206.
27. Du TS, Wang H, Latif MA, Zhang Y (2019) Estimation type results associated to k-fractional integral inequalities with applications. *J King Saud Univ Sci* **31**, 1083–1088.
28. Du TS, Awan MU, Kashuri A, Zhao SS (2021) Some k-fractional extensions of the trapezium inequalities through generalized relative semi-(m,h)-preinvexity. *Appl Anal* **100**, 642–662.
29. Sarikaya MZ, Yildirim H (2017) On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals. *Miskolc Math Notes* **17**, 1049–1059.
30. Watson GN (1944) *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, UK.
31. Li Y, Gu XM, Zhao J (2018) The weighted arithmetic mean-geometric mean inequality is equivalent to the Hölder inequality. *Symmetry* **10**, 380.