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Integral representations and properties of several finite sums containing central binomial coefficients

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ABSTRACT: For $m \geqslant 0$ and $n \in \mathbb{N}$, let $S^{\pm}_{m,n} = \sum_{k=1}^{n} \binom{2k}{k} \frac{1}{2^{2k}} \frac{(\pm 1)^k}{k+m}$. In the paper, the authors establish the integral representations, positivity, monotonic properties, convex properties, and limits of the sequences $S^{\pm}_{m,n}$ and $\sum_{k=0}^{n} \binom{2k}{k} \frac{(\pm 1)^k}{2^{2k}}$ for $n \in \mathbb{N}$. Several of these results recover the corresponding known conclusions.

KEYWORDS: integral representation, central binomial coefficient, finite sum, limit, sequence, positivity, monotonicity, convexity

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MOTIVATIONS

Lemma 3 in [1] can be rearranged as

$$\sum_{k=0}^{n} \binom{2k}{k} \frac{1}{2^{2k}} \frac{1}{k+1} = 2\left[1 - \frac{1}{2^{2(n+1)}} \binom{2(n+1)}{n+1}\right], \quad n \geqslant 0. \tag{1}$$

The first equality in [2, Theorem 23] can be reformulated as

$$\sum_{k=0}^{n} \binom{2k}{k} \frac{1}{2^{2k}} = \frac{2n+1}{2^{2n}} \binom{2n}{n}, \quad n \geqslant 0.$$

In [3], the infinite series

$$\sum_{k=1}^{\infty} \binom{2k}{k} \frac{1}{2^{2k}} \frac{(-1)^k}{k} = 2 \ln\left[2\left(\sqrt{2} - 1\right)\right] \tag{2}$$

was proved by three approaches.

Motivated by the above results, we would like to consider the sequences

$$a_n = \sum_{k=0}^{n} \binom{2k}{k} \frac{(-1)^k}{2^{2k}},$$

$$b_n = \sum_{k=1}^{n} \binom{2k}{k} \frac{1}{2^{2k}} \frac{(-1)^k}{k},$$

$$c_n = \sum_{k=0}^{n} \binom{2k}{k} \frac{1}{2^{2k}} \frac{(-1)^k}{k+1},$$

$$d_n = \sum_{k=1}^{n} \binom{2k}{k} \frac{1}{2^{2k}} \frac{1}{k}, \quad n \in \mathbb{N},$$

and, more generally, for $m \geqslant 0$ and $n \geqslant 1$,

$$S^{\pm}_{m,n} = \sum_{k=1}^{n} \binom{2k}{k} \frac{1}{2^{2k}} \frac{(\pm 1)^k}{k+m}.$$

What are the expressions of these finite sums without the sum mark?

At the website https://math.stackexchange.com/q/2746097, there existed some discussions on the above question for the sequence a_n.

A sequence of real numbers α_n for $n \geqslant 0$ is said to be convex if and only if $2\alpha_n \leqslant \alpha_{n-1} + \alpha_{n+1}$ for $n \geqslant 1$. A sequence of real numbers α_n for $n \geqslant 0$ is said to be concave if and only if $2\alpha_n \geqslant \alpha_{n-1} + \alpha_{n+1}$ for $n \geqslant 1$.

In this paper, we will give affirmative answers to the above question and present some properties, including both monotonicity and convexity, of the above sequences.

PROPERTIES OF THE SEQUENCE a_n

We now start out to present integral representations, positivity, monotonicity, convexity, and limit of the sequence a_n.

Theorem 1 *For $n \in \{0\} \cup \mathbb{N}$, we have the following conclusions:*

(i) *the sequence a_n has the integral representations*

$$a_n = \frac{2}{\pi} \int_0^{\pi/2} \frac{1 + (-1)^n \sin^{2n+2} x}{1 + \sin^2 x} \, \mathrm{d}x = \frac{2}{\pi} \int_0^{\infty} \frac{1}{2 + x^2} \left[1 + \frac{(-1)^n}{(1 + x^2)^{n+1}}\right] \mathrm{d}x; \tag{3}$$

(ii) *the sequence a_n is positive;*

(iii) *the sequence a_{2n} is decreasing and convex in $n \geqslant 0$;*

(iv) *the sequence a_{2n+1} is increasing and concave in $n \geqslant 0$;*

(v) *the limit*

$$\lim_{n \to \infty} a_n = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{(-1)^k}{2^{2k}} = \frac{\sqrt{2}}{2} \tag{4}$$

is valid;

(vi) *the definite integral*

$$\int_0^{\pi/2} \frac{1}{1+\sin^2 x}\,dx = \frac{\pi\sqrt{2}}{4} \tag{5}$$

is valid.

Proof: In [4, Corollary 3.2], in [5, Sec. 4.2], and in [6, Theorem 3.1], among other things, the integral representations

$$\binom{2n}{n} = \frac{2^{2n+1}}{\pi}\int_0^{\pi/2} \sin^{2n} x\,dx = \frac{2^{2n+1}}{\pi}\int_0^{\infty} \frac{1}{(1+x^2)^{n+1}}\,dx \tag{6}$$

were established. See also [7, p. 57] and [2, Theorem 7 in Sec. 2.4]. Then we have

$$a_n = \frac{2}{\pi}\int_0^{\pi/2}\left[\sum_{k=0}^{n}(-1)^k \sin^{2k} x\right]dx = \frac{2}{\pi}\int_0^{\pi/2} \frac{1+(-1)^n \sin^{2n+2} x}{1+\sin^2 x}\,dx$$

and

$$a_n = \frac{2}{\pi}\int_0^{\infty}\left[\sum_{k=0}^{n}\frac{(-1)^k}{(1+x^2)^{k+1}}\right]dx = \frac{2}{\pi}\int_0^{\infty}\frac{1}{2+x^2}\left[1+\frac{(-1)^n}{(1+x^2)^{n+1}}\right]dx.$$

The integral representations in (3) are thus proved.

From any one of the integral representations in (3), we can immediately derive the positivity of the sequence a_n, the decreasing property of the sequence a_{2n}, and the increasing property of the sequence a_{2n+1}.

The convexity of a_{2n} and concavity of a_{2n+1} follow from considering the convexity of the sequences $\sin^{2n+2} x$ and $\frac{1}{(1+x^2)^{n+1}}$ in n for fixed $t \in (0,1)$ in the integral representations in (3).

Setting $n \to \infty$ gives

$$\lim_{n\to\infty} a_n = \frac{2}{\pi}\int_0^{\pi/2}\frac{1}{1+\sin^2 x}\,dx = \frac{2}{\pi}\int_0^{\infty}\frac{1}{2+x^2}\,dx = \frac{\sqrt{2}}{2},$$

which are just the limit in (4) and the integral (5). The proof of Theorem 1 is complete. □

PROPERTIES OF THE SEQUENCE b_n

In this section, we now present integral representations, positivity, monotonicity, convexity, and limit of the sequence b_n.

Theorem 2 *For $n \in \mathbb{N}$, we have the following conclusions:*

(i) *the sequence b_n has the integral representations*

$$b_n = -\frac{2}{\pi}\int_0^{\pi/2}\int_0^{\sin^2 x}\frac{1-(-1)^n t^n}{1+t}\,dt\,dx = -\frac{2}{\pi}\int_0^{\infty}\frac{1}{1+x^2}\int_0^{1/(1+x^2)}\frac{1-(-1)^n t^n}{1+t}\,dt\,dx; \tag{7}$$

(ii) *the sequence b_n is negative;*

(iii) *the sequence b_{2n} is decreasing and convex in $n \geqslant 1$;*

(iv) *the sequence b_{2n-1} is increasing and concave in $n \geqslant 1$;*

(v) *the limit*

$$\lim_{n\to\infty} b_n = \sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{2^{2k}}\frac{(-1)^k}{k} = 2\ln\left[2\left(\sqrt{2}-1\right)\right] \tag{8}$$

is valid;

(vi) *the integrals*

$$\int_0^{\pi/2}\ln\left(1+\sin^2 x\right)dx = -\pi\ln\left[2\left(\sqrt{2}-1\right)\right] \tag{9}$$

and

$$\int_0^{\infty}\frac{1}{1+x^2}\ln\left(1+\frac{1}{1+x^2}\right)dx = -\pi\ln\left[2\left(\sqrt{2}-1\right)\right] \tag{10}$$

are valid.

Proof: Using integral representations in (6) and the integral representation

$$\sum_{k=1}^{n}(-1)^k\frac{x^k}{k} = -\int_0^x \frac{1-(-1)^n t^n}{1+t}\,dt, \quad n \in \mathbb{N}, \tag{11}$$

we obtain

$$b_n = \frac{2}{\pi}\int_0^{\pi/2}\left[\sum_{k=1}^{n}(-1)^k\frac{\sin^{2k} x}{k}\right]dx = -\frac{2}{\pi}\int_0^{\pi/2}\int_0^{\sin^2 x}\frac{1-(-1)^n t^n}{1+t}\,dt\,dx$$

and

$$b_n = \frac{2}{\pi}\int_0^{\infty}\left[\sum_{k=1}^{n}\frac{1}{(1+x^2)^{k+1}}\frac{(-1)^k}{k}\right]dx = -\frac{2}{\pi}\int_0^{\infty}\frac{1}{1+x^2}\int_0^{1/(1+x^2)}\frac{1-(-1)^n t^n}{1+t}\,dt\,dx.$$

The integral representations in (7) follow.

From any one of the integral representations in (7), we can readily deduce the negativity of the sequence b_n, the decreasing property of the sequence b_{2n}, and the increasing property of the sequence b_{2n-1}.

The convexity of b_{2n} and concavity of b_{2n-1} follow from considering the convexity of the sequence t^n in n for fixed $t \in (0,1)$ in the integral representations in (7).

The limit in (8) is just the one in (2).

Taking $n \to \infty$ in (7) and combining it with (8) lead to

$$\lim_{n\to\infty} b_n = -\frac{2}{\pi}\int_0^{\pi/2}\int_0^{\sin^2 x}\frac{1}{1+t}\,\mathrm{d}t\,\mathrm{d}x = -\frac{2}{\pi}\int_0^{\pi/2}\ln\left(1+\sin^2 x\right)\mathrm{d}x = 2\ln\left[2\left(\sqrt{2}-1\right)\right]$$

and

$$\lim_{n\to\infty} b_n = -\frac{2}{\pi}\int_0^{\infty}\frac{1}{1+x^2}\int_0^{1/(1+x^2)}\frac{1}{1+t}\,\mathrm{d}t\,\mathrm{d}x = -\frac{2}{\pi}\int_0^{\infty}\frac{1}{1+x^2}\ln\left(1+\frac{1}{1+x^2}\right)\mathrm{d}x = 2\ln\left[2\left(\sqrt{2}-1\right)\right].$$

The integrals in (9) and (10) are thus verified. The proof of Theorem 2 is complete. □

PROPERTIES OF THE SEQUENCE c_n

Now we present integral representations, positivity, monotonicity, convexity, and limit of the sequence c_n.

Theorem 3 *For $n \in \mathbb{N}$, we have the following conclusions:*

(i) *the sequence c_n has the integral representations*

$$c_n = \frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\sin^2 x}\int_0^{\sin^2 x}\frac{1-(-t)^{n+1}}{1+t}\,\mathrm{d}t\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\infty}\int_0^{1/(1+x^2)}\frac{1-(-t)^{n+1}}{1+t}\,\mathrm{d}t\,\mathrm{d}x; \quad (12)$$

(ii) *the sequence c_n is positive;*

(iii) *the sequence c_{2n} is decreasing and convex in $n \geqslant 1$;*

(iv) *the sequence c_{2n-1} is increasing and concave in $n \geqslant 1$;*

(v) *the limit*

$$\lim_{n\to\infty} c_n = \sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{2^{2k}}\frac{(-1)^k}{k+1} = 2\left(\sqrt{2}-1\right) \quad (13)$$

is valid;

(vi) *the improper definite integrals*

$$\int_0^{\pi/2}\frac{\ln\left(1+\sin^2 x\right)}{\sin^2 x}\,\mathrm{d}x = \frac{1}{2}\int_0^1\frac{\ln(1+t)}{t^{3/2}\sqrt{1-t}}\,\mathrm{d}t = \left(\sqrt{2}-1\right)\pi \quad (14)$$

are valid.

Proof: Making use of integral representations in (6) and (11), we obtain

$$c_n = -\frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\sin^2 x}\left[\sum_{k=0}^{n}(-1)^{k+1}\frac{\sin^{2(k+1)} x}{k+1}\right]\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\sin^2 x}\int_0^{\sin^2 x}\frac{1-(-1)^{n+1}t^{n+1}}{1+t}\,\mathrm{d}t\,\mathrm{d}x$$

and

$$c_n = \frac{2}{\pi}\int_0^{\infty}\left[\sum_{k=0}^{n}\frac{1}{(1+x^2)^{k+1}}\frac{(-1)^k}{k+1}\right]\mathrm{d}x = \frac{2}{\pi}\int_0^{\infty}\int_0^{1/(1+x^2)}\frac{1-(-1)^{n+1}t^{n+1}}{1+t}\,\mathrm{d}t\,\mathrm{d}x.$$

The integral representations in (12) are thus deduced.

From any one of the integral representations in (12), we can deduce the positivity of the sequence c_n, the decreasing property of the sequence c_{2n}, and the increasing property of the sequence c_{2n-1} straightforwardly.

The convexity of c_{2n} and concavity of c_{2n-1} follow from considering the convexity of the sequence t^{n+1} in n for fixed $t \in (0,1)$ in the integral representations in (12).

Letting $n \to \infty$ in (12) gives

$$\lim_{n\to\infty} c_n = \frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\sin^2 x}\int_0^{\sin^2 x}\frac{1}{1+t}\,\mathrm{d}t\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi/2}\frac{\ln\left(1+\sin^2 x\right)}{\sin^2 x}\,\mathrm{d}x = \frac{2}{\pi}\int_0^1\frac{\ln\left(1+t^2\right)}{t^2\sqrt{1-t^2}}\,\mathrm{d}t = \frac{1}{\pi}\int_0^1\frac{\ln(1+t)}{t^{3/2}\sqrt{1-t}}\,\mathrm{d}t$$

and

$$\lim_{n\to\infty} c_n = \frac{2}{\pi}\int_0^{\infty}\int_0^{1/(1+x^2)}\frac{1}{1+t}\,\mathrm{d}t\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\infty}\ln\frac{2+x^2}{1+x^2}\,\mathrm{d}x = 2\left(\sqrt{2}-1\right),$$

where we used the formula

$$\int_0^{\infty}\ln\frac{a^2+x^2}{b^2+x^2}\,\mathrm{d}x = (a-b)\pi, \quad a,b>0$$

in the handbook [8, formula 4.222 on p. 533]. The limit in (13) and the definite integrals in (14) are thus verified. The proof of Theorem 3 is complete. □

PROPERTIES OF THE SEQUENCE d_n

In this section, we present integral representations, concavity, and a limit of the sequence d_n.

It is trivial that the sequence d_n for $n \geqslant 0$ is positive and increasing.

Theorem 4 *For $n \in \mathbb{N}$, we have the following conclusions:*

(i) *the sequence d_n has the integral representations*

$$d_n = \frac{2}{\pi}\int_0^{\pi/2}\int_0^{\sin^2 x}\frac{1-t^n}{1-t}\,\mathrm{d}t\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\infty}\frac{1}{1+x^2}\int_0^{1/(1+x^2)}\frac{1-t^n}{1-t}\,\mathrm{d}t\,\mathrm{d}x; \quad (15)$$

(ii) *the sequence d_n is concave in n;*

(iii) *the limit*

$$\lim_{n\to\infty} d_n = \sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{2^{2k}}\frac{1}{k} = 2\ln 2 \quad (16)$$

is valid;

(iv) *the improper integral*

$$\int_0^{\infty}\frac{1}{1+x^2}\ln\left(1-\frac{1}{1+x^2}\right)\mathrm{d}x = -\pi\ln 2 \quad (17)$$

is valid.

Proof: Making use of integral representations in (6), we acquire

$$d_n = \frac{2}{\pi}\int_0^{\pi/2}\left(\sum_{k=1}^{n}\frac{\sin^{2k}x}{k}\right)\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi/2}\int_0^{\sin^2 x}\frac{1-t^n}{1-t}\,\mathrm{d}t\,\mathrm{d}x,$$

where we employed

$$\sum_{k=1}^{n}\frac{x^k}{k} = \int_0^x \frac{1-t^n}{1-t}\,\mathrm{d}t, \quad n\in\mathbb{N},$$

and

$$d_n = \frac{2}{\pi}\int_0^{\infty}\left[\sum_{k=1}^{n}\frac{1}{(1+x^2)^{k+1}}\frac{1}{k}\right]\mathrm{d}x = \frac{2}{\pi}\int_0^{\infty}\frac{1}{1+x^2}\int_0^{1/(1+x^2)}\frac{1-t^n}{1-t}\,\mathrm{d}t\,\mathrm{d}x.$$

The integral representations in (15) follow.

Since the sequence t^n for any fixed $t\in(0,1)$ is convex in n, from the integral representations in (15), we conclude that the sequence d_n is concave in $n \geqslant 1$.

Taking $n\to\infty$ in (15) results in

$$\lim_{n\to\infty} d_n = \frac{2}{\pi}\int_0^{\pi/2}\int_0^{\sin^2 x}\frac{1}{1-t}\,\mathrm{d}t\,\mathrm{d}x = -\frac{2}{\pi}\int_0^{\pi/2}\ln\left(1-\sin^2 x\right)\mathrm{d}x = -\frac{4}{\pi}\int_0^{\pi/2}\ln\cos x\,\mathrm{d}x = 2\ln 2$$

and

$$\lim_{n\to\infty} d_n = \frac{2}{\pi}\int_0^{\infty}\frac{1}{1+x^2}\int_0^{1/(1+x^2)}\frac{1}{1-t}\,\mathrm{d}t\,\mathrm{d}x = -\frac{2}{\pi}\int_0^{\infty}\frac{1}{1+x^2}\ln\left(1-\frac{1}{1+x^2}\right)\mathrm{d}x,$$

where the integral

$$\int_0^{\pi/2}\ln\cos x\,\mathrm{d}x = -\frac{\ln 2}{2}\pi$$

can be found in the handbook [8, formula 4.224 on p. 534]. The limit (16) and the improper integral (17) are thus proved. The proof of Theorem 4 is complete. □

PROPERTIES OF THE SEQUENCES $S^{\pm}_{m,n}$

In this section, we present integral representations, positivity, monotonicity, and limit of the sequences $S^{\pm}_{m,n}$.

It is trivial that the sequence $S^{+}_{m,n}$ is positive, increasing in $n \geqslant 1$, and decreasing in $m \geqslant 0$.

Theorem 5 *For $n\in\mathbb{N}$ and $m\geqslant 0$, we have the following conclusions:*

(i) *the sequence $S^{+}_{m,n}$ has the integral representations*

$$S^{+}_{m,n} = \frac{2}{\pi}\int_0^{\pi/2}\int_0^{\sin^2 x}\left(\frac{t}{\sin^2 x}\right)^m\frac{1-t^n}{1-t}\,\mathrm{d}t\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\infty}\frac{1}{1+x^2}\int_0^{1/(1+x^2)}\left[\left(1+x^2\right)t\right]^m\frac{1-t^n}{1-t}\,\mathrm{d}t\,\mathrm{d}x; \quad (18)$$

(ii) *the sequence $S^{+}_{m,n}$ is convex in m for fixed $n\in\mathbb{N}$;*

(iii) *the sequence $S^{+}_{m,n}$ is concave in n for fixed $m\geqslant 0$;*

(iv) *the limits*

$$\lim_{n\to\infty} S^{+}_{m,n} = \frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\sin^{2m}x}\int_0^{\sin^2 x}\frac{t^m}{1-t}\,\mathrm{d}t\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\infty}\left(1+x^2\right)^{m-1}\int_0^{1/(1+x^2)}\frac{t^m}{1-t}\,\mathrm{d}t\,\mathrm{d}x \quad (19)$$

are valid for $m\geqslant 0$.

Proof: By means of integral representations in (6) and (11), we acquire

$$S^{+}_{m,n} = \frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\sin^{2m}x}\left[\sum_{k=1}^{n+m}\frac{\sin^{2k}x}{k} - \sum_{k=1}^{m}\frac{\sin^{2k}x}{k}\right]\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi/2}\int_0^{\sin^2 x}\left(\frac{t}{\sin^2 x}\right)^m\frac{1-t^n}{1-t}\,\mathrm{d}t\,\mathrm{d}x$$

and

$$S^+_{m,n}=\frac{2}{\pi}\int_0^\infty(1+x^2)^{m-1}\left[\sum_{k=1}^{n+m}\frac{1}{(1+x^2)^k}\frac{1}{k}-\sum_{k=1}^{m}\frac{1}{(1+x^2)^k}\frac{1}{k}\right]\mathrm{d}x$$
$$=\frac{2}{\pi}\int_0^\infty\frac{1}{1+x^2}\int_0^{1/(1+x^2)}\left[(1+x^2)t\right]^m\frac{1-t^n}{1-t}\,\mathrm{d}t\,\mathrm{d}x.$$

The integral representations in (18) are thus deduced.

The convexity of $S^+_{m,n}$ in m is deduced from considering the convexity of $(t/\sin^2 x)^m$ and $[(1+x^2)t]^m$ in m in the integral representations in (18). The concavity of the sequence $S^+_{m,n}$ in n is deduced from the convexity of t^n in n in the integral representations in (18).

Letting $n\to\infty$ in (18) gives the limits in (19). The proof of Theorem 5 is complete. □

Theorem 6 *For $n\in\mathbb{N}$ and $m\geqslant 0$, we have the following conclusions:*

(i) *the sequence $S^-_{m,n}$ has the integral representations*

$$S^-_{m,n}=-\frac{2}{\pi}\int_0^{\pi/2}\int_0^{\sin^2 x}\left(\frac{t}{\sin^2 x}\right)^m\frac{1-(-1)^n t^n}{1+t}\,\mathrm{d}t\,\mathrm{d}x$$
$$=-\frac{2}{\pi}\int_0^\infty\frac{1}{1+x^2}\int_0^{1/(1+x^2)}\left[(1+x^2)t\right]^m\frac{1-(-1)^n t^n}{1+t}\,\mathrm{d}t\,\mathrm{d}x; \quad (20)$$

(ii) *the sequence $S^-_{m,n}$ is negative for all $n\geqslant 1$ and $m\geqslant 0$;*

(iii) *the sequence $S^-_{m,n}$ is increasing and concave in $m\geqslant 0$ for fixed $n\geqslant 1$;*

(iv) *the sequence $S^-_{m,2n}$ is decreasing and convex in $n\geqslant 1$ for fixed $m\geqslant 0$;*

(v) *the sequence $S^-_{m,2n-1}$ is increasing and concave in $n\geqslant 1$ for fixed $m\geqslant 0$;*

(vi) *the limits*

$$\lim_{n\to\infty}S^-_{m,n}=-\frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\sin^{2m}x}\int_0^{\sin^2 x}\frac{t^m}{1+t}\,\mathrm{d}t\,\mathrm{d}x$$
$$=-\frac{2}{\pi}\int_0^\infty(1+x^2)^{m-1}\int_0^{1/(1+x^2)}\frac{t^m}{1+t}\,\mathrm{d}t\,\mathrm{d}x \quad (21)$$

are valid for $m\geqslant 0$.

Proof: By the aid of integral representations in (6) and (11), we arrive at

$$S^-_{m,n}=\frac{2}{\pi}\int_0^{\pi/2}\frac{(-1)^m}{\sin^{2m}x}\left[\sum_{k=1}^{n+m}\frac{(-1)^k\sin^{2k}x}{k}-\sum_{k=1}^{m}\frac{(-1)^k\sin^{2k}x}{k}\right]\mathrm{d}x$$
$$=-\frac{2}{\pi}\int_0^{\pi/2}\int_0^{\sin^2 x}\left(\frac{t}{\sin^2 x}\right)^m\frac{1-(-1)^n t^n}{1+t}\,\mathrm{d}t\,\mathrm{d}x$$

and

$$S^-_{m,n}=\frac{2}{\pi}\int_0^\infty(-1)^m(1+x^2)^{m-1}\left[\sum_{k=1}^{n+m}\frac{(-1)^k}{(1+x^2)^k}\frac{1}{k}-\sum_{k=1}^{m}\frac{(-1)^k}{(1+x^2)^k}\frac{1}{k}\right]\mathrm{d}x$$
$$=-\frac{2}{\pi}\int_0^\infty\frac{1}{1+x^2}\int_0^{1/(1+x^2)}\left[(1+x^2)t\right]^m\frac{1-(-1)^n t^n}{1+t}\,\mathrm{d}t\,\mathrm{d}x.$$

The integral representations in (20) are thus deduced.

Considering the convexity of $(t/\sin^2 x)^m$, $[(1+x^2)t]^m$, and t^n in m and n respectively, from the integral representations in (20), we can derive all the negativity, increasing and decreasing properties, convex and concave properties of the sequences $S^-_{m,n}$, $S^-_{m,2n}$, and $S^-_{m,2n-1}$ respectively.

Letting $n\to\infty$ in (20) gives the limits in (21). The proof of Theorem 6 is complete. □

REMARKS

After establishing our main results, we now give several remarks.

Remark 1 If letting $m=0$ in (19), then we recover the limit (16) in Theorem 4.

When taking $m=1$ in (19), we derive

$$\sum_{k=0}^\infty\binom{2k}{k}\frac{1}{2^{2k}}\frac{1}{k+1}=1,$$

which is a limit of the finite sum in (1).

When $2\leqslant m\leqslant 16$, the limits in (19), that is, the sums of the infinite series

$$\sum_{k=1}^\infty\binom{2k}{k}\frac{1}{2^{2k}}\frac{1}{k+m},$$

are

$$\frac{5}{6},\ \frac{11}{15},\ \frac{93}{140},\ \frac{193}{315},\ \frac{793}{1386},\ \frac{1619}{3003},\ \frac{26333}{51480},\ \frac{53381}{109395},\ \frac{43191}{92378},\ \frac{436109}{969969},\ \frac{1172755}{2704156},\ \frac{7088533}{16900975},\ \frac{28539857}{70204050},\ \frac{57414019}{145422675},\ \frac{1846943453}{4808643120},$$

respectively. These special values are computed by the famous software Wolfram Mathematica 12.0.

Remark 2 If letting $m=0$ in (21), then we recover the sum (2).

When taking $m=1$ in (21), we recover (13) in Theorem 3.

When $2\leqslant m\leqslant 16$, the limits in (21), that is, the sequence in m of the infinite series

$$\sum_{k=1}^\infty\binom{2k}{k}\frac{1}{2^{2k}}\frac{(-1)^k}{k+m},$$

are

$$-\frac{4\sqrt{2}-5}{6},\quad -\frac{7(3-2\sqrt{2})}{15},\quad -\frac{3(24\sqrt{2}-31)}{140},$$
$$-\frac{319-214\sqrt{2}}{315},\quad -\frac{604\sqrt{2}-793}{1386},\quad -\frac{2477-1670\sqrt{2}}{3003},$$
$$-\frac{17(1168\sqrt{2}-1549)}{51480},\quad -\frac{77691-52582\sqrt{2}}{109395},$$
$$-\frac{161708\sqrt{2}-215955}{461890},\quad -\frac{23(26629-18078\sqrt{2})}{969969},$$
$$-\frac{5(174728\sqrt{2}-234551)}{2704156},\quad -\frac{9688683-6593918\sqrt{2}}{16900975},$$
$$-\frac{21175372\sqrt{2}-28539857}{70204050},\quad -\frac{31(2477539-1689674\sqrt{2})}{145422675},$$
$$-\frac{1365921568\sqrt{2}-1846943453}{4808643120}$$

respectively. These special values are computed by the famous software Wolfram Mathematica 12.0.

Remark 3 An anonymous referee pointed out that
(i) for $m \in \mathbb{C}\setminus\{0\}$ and $m+k \neq 0$,

$$\sum_{k=1}^{\infty}\binom{2k}{k}\frac{1}{2^{2k}(k+m)} = \frac{\sqrt{\pi}\,\Gamma(m+2)}{m(m+1)\Gamma\left(m+\frac{1}{2}\right)} - \frac{1}{m},$$

where

$$\Gamma(z) = \lim_{m\to\infty}\frac{m!m^z}{\prod_{j=0}^{m}(z+j)}, \quad z \in \mathbb{C}\setminus\{0,-1,-2,\dots\}$$

denotes the classical Euler's gamma function, see the paper [9] or Chapter 3 in the monograph [10]; in particular, for $m \in \mathbb{N}$,

$$\sum_{k=1}^{\infty}\binom{2k}{k}\frac{1}{2^{2k}(k+m)} = \frac{2^m(m-1)!}{(2m-1)!!} - \frac{1}{m};$$

(ii) for $n \in \mathbb{N}$ and $m \in \mathbb{C}\setminus\{0\}$ with $m+k \neq 0$,

$$\sum_{k=1}^{n}\binom{2k}{k}\frac{1}{2^{2k}(k+m)} = \frac{\sqrt{\pi}\,\Gamma(m)}{\Gamma\left(m+\frac{1}{2}\right)} - \frac{1}{m} - \frac{(2n+1)!!}{2^{n+1}(m+n+1)(n+1)!} \times {}_3F_2\left(1, n+\frac{3}{2}, m+n+1; n+2, m+n+2; 1\right),$$

where the generalized hypergeometric series

$${}_pF_q(\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;z) = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}$$

is defined in [8] for complex numbers $\alpha_i \in \mathbb{C}$ and $\beta_i \in \mathbb{C}\setminus\{0,-1,-2,\dots\}$, for positive integers $p,q \in \mathbb{N}$, and in terms of the rising factorial

$$(\alpha)_n = \prod_{\ell=0}^{n-1}(\alpha+\ell) = \begin{cases}\alpha(\alpha+1)\cdots(\alpha+n-1), & n \geqslant 1;\\ 1, & n=0.\end{cases}$$

The anonymous referee also pointed out that similar expressions can be developed for the alternating cases.

CONCLUSION

For $m \geqslant 0$ and $n \in \mathbb{N}$, let

$$S_{m,n}^{\pm} = \sum_{k=1}^{n}\binom{2k}{k}\frac{1}{2^{2k}}\frac{(\pm 1)^k}{k+m}.$$

In this paper, we established the integral representations, positivity, monotonic properties, convex properties, and limits of the sequences $S_{m,n}^{\pm}$ for $m = 0, 1$ and the sequences $\sum_{k=0}^{n}\binom{2k}{k}\frac{(\pm 1)^k}{2^{2k}}$ for $n \in \mathbb{N}$. Several of these results recover corresponding known conclusions.

The Catalan numbers, denoted by C_n for $n \geqslant 0$, constitute a sequence of integers. This sequence is an important object in combinatorial number theory. For more information on the Catalan numbers C_n, please refer to the papers [2, 4–6] and a number of literature therein.

The central binomial coefficients $\binom{2n}{n}$ and the Catalan numbers C_n have the relation $C_n = \frac{1}{n+1}\binom{2n}{n}$ for $n \geqslant 0$. Therefore, all of the above sequences containing central binomial coefficients $\binom{2n}{n}$ can be reformulated in terms of the Catalan numbers C_n.

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