RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2022.136

ScienceAsia 49 (2023): **200–204**

Several integral inequalities of the Hermite-Hadamard type for s-(β, F)-convex functions

Yan Wang[a], **Xi-Min Liu**[a], **Bai-Ni Guo**[b,*]

[a] School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, Liaoning, China
[b] School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo 454010, Henan, China

*Corresponding author, e-mail: bai.ni.guo@gmail.com

Received 12 Jul 2021, Accepted 25 Aug 2022
Available online 5 Dec 2022

ABSTRACT: In the paper, the authors introduce a new concept of s-(β, F)-convex functions and establish several integral inequalities of the Hermite-Hadamard type for s-(β, F)-convex functions.

KEYWORDS: integral inequality, Hermite-Hadamard type, convex function, s-(β, F)-convex function

MSC2020: 26A51 26D15 26D20 26E60 41A55

INTRODUCTION

In this paper, we denote a nonempty and open interval by $I \subseteq \mathbb{R}$. Let $H : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on I. Then the Hermite-Hadamard integral inequality reads that

$$H\left(\frac{r+t}{2}\right) \leqslant \frac{1}{t-r}\int_r^t H(x)\,\mathrm{d}x \leqslant \frac{H(r)+H(t)}{2}, \quad r,t \in I.$$

If the inequality

$$H(tu+(1-t)v) \leqslant tH(u)+(1-t)H(v)-ct(1-t)(u-v)^2$$

is valid for $u, v \in I$ and $t \in [0, 1]$, then the function $H : I \subseteq \mathbb{R} \to \mathbb{R}$ is called in [1] strongly convex with modulus $c > 0$. In [2, 3], the kind of strongly convex functions was generalized as follows.

Definition 1 ([2, 3]) Let $F : \mathbb{R} \to \mathbb{R}$ be a given function. A function $H : I \subseteq \mathbb{R} \to \mathbb{R}$ is called F-convex if the inequality

$$H(tu+(1-t)v) \leqslant tH(u)+(1-t)H(v)-t(1-t)F(u-v) \quad (1)$$

is valid for all $u, v \in I$ and $t \in [0, 1]$.

In [3], the following inequalities for strongly convex functions were established.

Theorem 1 ([3, Theorem 5]) *Let $F : \mathbb{R} \to \mathbb{R}$ be a given function which is integrable on each compact subinterval of $(-\alpha, \alpha)$, where $\alpha = (\sup I - \inf I)/2$. If an F-convex function $H : I \subseteq \mathbb{R} \to \mathbb{R}$ is one-sided differentiable and $H_- \leqslant H_+$, then*

$$H\left(\frac{r+t}{2}\right) + \frac{1}{t-r}\int_r^t F\left(u - \frac{r+t}{2}\right)\mathrm{d}u \leqslant \frac{1}{t-r}\int_r^t H(u)\,\mathrm{d}u \leqslant \frac{H(r)+H(t)}{2} - \frac{1}{6}F(r-t)$$

for all $r, t \in I$ with $r \neq t$.

Theorem 2 ([3, Theorem 6]) *Let $F : \mathbb{R} \to \mathbb{R}$ be a given function which is integrable on each compact subinterval of $(-\alpha, \alpha)$, where $\alpha = \sup I - \inf I$. If $H : I \subseteq \mathbb{R} \to \mathbb{R}$ is an F-convex function, then*

$$H\left(\frac{r+t}{2}\right) + \frac{1}{4(t-r)}\int_r^t F(2u-r-t)\,\mathrm{d}u \leqslant \frac{1}{t-r}\int_r^t H(u)\,\mathrm{d}u \leqslant \frac{H(r)+H(t)}{2} - \frac{1}{6}F(r-t)$$

for all $r, t \in I$ with $r \neq t$.

For more information on this topic, please refer to the papers [4–9] and references therein.

DEFINITION OF s-(β, F)-CONVEX FUNCTIONS

In [10, 11], the concept of s-convex functions was innovated as follows.

Definition 2 ([10, 11]) Let $s \in (0, 1]$. A function $H : I \subseteq \mathbb{R}_0 = [0, \infty) \to \mathbb{R}$ is said to be s-convex (in the second sense) if the inequality

$$H(tu+(1-t)v) \leqslant t^s H(u) + (1-t)^s H(v) \quad (2)$$

holds for all $u, v \in I$ and $t \in [0, 1]$.

In [12], the extended s-convex functions were defined as follows.

Definition 3 ([12]) For some $s \in [-1, 1]$, a function $H : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be extended s-convex if the inequality (2) is valid for all $u, v \in I$ and $t \in (0, 1)$.

We are now in a position to introduce the following new concept of convex functions, s-(β, F)-convex functions, which generalize the F-convex functions and the extended s-convex functions.

Definition 4 Let $s \in [-1,1]$, $0 < \beta \leqslant 1$, and $F : \mathbb{R} \to \mathbb{R}$ be a given function. A function $H : I \subseteq \mathbb{R} \to \mathbb{R}$ is called s-(β, F)-convex if

$$H(tu+(1-t)v) \leqslant t^s H(u)+(1-t)^s H(v)-t^\beta\big(1-t^\beta\big)F(u-v) \quad (3)$$

for all $u, v \in I$ and $t \in (0,1)$.

Remark 1 Let $F : \mathbb{R} \to \mathbb{R}$ and $H : I \subseteq \mathbb{R} \to \mathbb{R}$ be nonnegative functions and let $s \in [-1,1)$. If H is F-convex on I, then H is s-$(1,F)$-convex on I for $s \in [-1,1)$. If H is an s-(β,F)-convex function on I, then H is not necessarily an F-convex function on I.

Proposition 1 *For $s \in [-1,0]$ and $0 < \beta \leqslant 1$, let $H(u) = u \in \mathbb{R}^+ = (0,\infty)$ and $F(u) = |u|$ for $u \in \mathbb{R}$. Then $H(u) = u$ is an s-(β,F)-convex function on $\mathbb{R}^+$ for $s \in [-1,0]$, but $H(u) = u$ is not an F-convex function on $\mathbb{R}^+$.*

Proof: For $u, v \in \mathbb{R}^+$ with $u \leqslant v$ and $t \in (0,1)$, letting $w = \frac{u}{v} \in (0,1]$ and using (3) result in

$$\begin{aligned} t^s H(u) &+ (1-t)^s H(v) - t^\beta\big(1-t^\beta\big)F(u-v) \\ &- H(tx+(1-t)v) \\ &= v\big\{\big[t^s + t^\beta\big(1-t^\beta\big) - t\big]w \\ &+ \big[(1-t)^s - t^\beta\big(1-t^\beta\big) - (1-t)\big]\big\} \geqslant 0. \end{aligned}$$

For $u, v \in \mathbb{R}^+$ with $u < v$ and $t \in (0,1)$, by (1), we obtain

$$\begin{aligned} tH(u)+(1-t)H(v)-t(1-t)F(u-v)-H(tx+(1-t)v) \\ = -t(1-t)(v-u) < 0. \end{aligned}$$

Then $H(u) = u$ is an s-(β,F)-convex function on $\mathbb{R}^+$ for $s \in [-1,0]$, but $H(u) = u$ is not an F-convex function on $\mathbb{R}^+$. □

LEMMAS

In order to establish some integral inequalities of the Hermite-Hadamard type for s-(β,F)-convex functions, we recite the following two lemmas.

Lemma 1 ([13, Lemma 4]) *Let $I \subseteq \mathbb{R}$ and $H : I \to \mathbb{R}$ be a twice differentiable function on I°. If $H'' \in L_1([u,v])$ for $u, v \in I$ with $u < v$, then*

$$\frac{H(u)+H(v)}{2} - \frac{1}{v-u}\int_u^v H(w)\,\mathrm{d}w = \frac{(v-u)^2}{2}\int_0^1 t(1-t)H''(tu+(1-t)v)\,\mathrm{d}t.$$

Lemma 2 ([14, Lemma 2.1]) *Let $I \subseteq \mathbb{R}$ and $H : I \to \mathbb{R}$ be differentiable on I°. If $H' \in L_1([u,v])$ for $u, v \in I$ with $u < v$, then*

$$\begin{aligned} H\Big(\frac{u+v}{2}\Big) - \frac{1}{v-u}\int_u^v H(w)\,\mathrm{d}w &= (v-u)\Big[\int_0^{1/2} tH'(tu+(1-t)v)\,\mathrm{d}t \\ &+ \int_{1/2}^1 (t-1)H'(tu+(1-t)v)\,\mathrm{d}t\Big]. \end{aligned}$$

SEVERAL INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE

We now start out to establish several integral inequalities of the Hermite-Hadamard type for s-(β,F)-convex functions.

Theorem 3 *Suppose $s \in [-1,1]$, $\beta \in (0,1]$, and $u, v \in \mathbb{R}$ with $u < v$. Let $F : [u-v, v-u] \to \mathbb{R}$ be a given function. If $H : [u,v] \to \mathbb{R}$ is s-(β,F)-convex with $H \in L_1([u,v])$ and $F \in L_1([u-v, v-u])$, then*

$$2^{s-1}H\Big(\frac{u+v}{2}\Big) + \frac{2^{s-1}(2^\beta-1)}{(v-u)2^{2\beta+1}}\int_{u-v}^{v-u} F(x)\,\mathrm{d}x \leqslant \frac{1}{v-u}\int_u^v H(x)\,\mathrm{d}x.$$

Proof: Using the s-(β,F)-convexity of H, for all $t \in [0,1]$, we obtain

$$\begin{aligned} H\Big(\frac{u+v}{2}\Big) &= H\Big(\frac{tu+(1-t)v+(1-t)u+tv}{2}\Big) \\ &\leqslant \frac{1}{2^s}\big[H(tu+(1-t)v)+H((1-t)u+tv)\big] \\ &\quad - \frac{2^\beta-1}{2^{2\beta}}F((1-2t)(u-v)). \end{aligned}$$

Integrating on both sides with respect to $t \in [0,1]$ and changing variables lead to

$$\begin{aligned} H\Big(\frac{u+v}{2}\Big) &\leqslant \frac{1}{2^s}\int_0^1 [H(tu+(1-t)v)+H((1-t)u+tv)]\,\mathrm{d}t \\ &\quad - \frac{2^\beta-1}{2^{2\beta}}\int_0^1 F((1-2t)(u-v))\,\mathrm{d}t \\ &= \frac{1}{2^{s-1}(v-u)}\int_u^v H(x)\,\mathrm{d}x - \frac{2^\beta-1}{2^{2\beta}}\int_0^1 F((1-2t)(u-v))\,\mathrm{d}t \\ &= \frac{1}{2^{s-1}(v-u)}\int_u^v H(x)\,\mathrm{d}x - \frac{2^\beta-1}{(v-u)2^{2\beta+1}}\int_{u-v}^{v-u} F(x)\,\mathrm{d}x. \end{aligned}$$

Theorem 3 is thus proved. □

Theorem 4 *Suppose $s\in(-1,1]$, $\beta\in(0,1]$, and $u,v\in\mathbb{R}$ with $u<v$. Let $F:[u-v,0)\to\mathbb{R}$ and $H:[u,v]\to\mathbb{R}$. If H is s-(β,F)-convex on $[u,v]$ and $H\in L_1([u,v])$, then*

$$\frac{1}{v-u}\int_u^v H(x)\,\mathrm{d}x \leqslant \frac{H(u)+H(v)}{s+1}-\frac{\beta}{(\beta+1)(2\beta+1)}F(u-v).$$

Proof: Changing the variable $x=tu+(1-t)v$ for $t\in(0,1)$ and using the s-(β,F)-convexity of H yield

$$\begin{aligned}\frac{1}{v-u}\int_u^v H(x)\,\mathrm{d}x &= \int_0^1 H(tu+(1-t)v)\,\mathrm{d}t\\ &\leqslant \int_0^1 \Big[t^s H(u)+(1-t)^s H(v)-t^\beta\big(1-t^\beta\big)F(u-v)\Big]\mathrm{d}t\\ &= \frac{H(u)+H(v)}{s+1}-\frac{\beta}{(\beta+1)(2\beta+1)}F(u-v).\end{aligned}$$

Theorem 4 is thus proved. □

Theorem 5 *Suppose $s\in[-1,1]$ and $\beta\in(0,1]$, and $u,v\in\mathbb{R}$ with $u<v$. Let function $F:[u-v,0)\to\mathbb{R}$ and a twice differentiable function $H:[u,v]\to\mathbb{R}$. If $|H''|^q$ is s-(β,F)-convex on $[u,v]$ for $q\geqslant 1$ and $H''\in L_1([u,v])$, then*

$$\left|\frac{H(u)+H(v)}{2}-\frac{1}{v-u}\int_u^v H(x)\,\mathrm{d}x\right| \leqslant \frac{(v-u)^2}{2\times 6^{1-1/q}}\left[\frac{|H''(u)|^q+|H''(v)|^q}{(s+2)(s+3)} - \frac{\beta(3\beta+5)F(u-v)}{2(\beta+1)(\beta+2)(\beta+3)(2\beta+3)}\right]^{1/q}. \quad (4)$$

Proof: By Lemma 1 and the Hölder integral inequality (see the monograph [15, Theorem 7, p. 54]), we have

$$\left|\frac{H(u)+H(v)}{2}-\frac{1}{v-u}\int_u^v H(x)\,\mathrm{d}x\right| \leqslant \frac{(v-u)^2}{2}\left[\int_0^1 t(1-t)\,\mathrm{d}t\right]^{1-1/q} \times\left(\int_0^1 t(1-t)|H''(tu+(1-t)v)|^q\,\mathrm{d}t\right)^{1/q}. \quad (5)$$

Using the s-(β,F)-convexity of $|H''|^q$ gives

$$\begin{aligned}\int_0^1 t(1-t)|H''(tu+(1-t)v)|^q\,\mathrm{d}t &\leqslant \int_0^1 t(1-t)\Big[t^s|H''(u)|^q+(1-t)^s|H''(v)|^q - t^\beta\big(1-t^\beta\big)F(u-v)\Big]\mathrm{d}t\\ &= \frac{|H''(u)|^q+|H''(v)|^q}{(s+2)(s+3)} - \frac{\beta(3\beta+5)}{2(\beta+1)(\beta+2)(\beta+3)(2\beta+3)}F(u-v). \quad (6)\end{aligned}$$

By inequalities (5) and (6), we deduce (4). The proof of Theorem 5 is thus complete. □

Theorem 6 *Suppose $s\in[-1,1]$, $\beta\in(0,1]$, and $u,v\in\mathbb{R}$ with $u<v$. Let $F:[u-v,0)\to\mathbb{R}$ and let $H:[u,v]\to\mathbb{R}$ be a twice differentiable function. If $|H''|^q$ is s-(β,F)-convex on $[u,v]$ for $q>1$ and $q\geqslant\ell\geqslant 0$ and if $H''\in L_1([u,v])$, then*

$$\begin{aligned}\left|\frac{H(u)+H(v)}{2}-\frac{1}{v-u}\int_u^v H(x)\,\mathrm{d}x\right| &\leqslant \frac{(v-u)^2}{2}\left[B\left(\frac{2q-\ell-1}{q-1},\frac{2q-\ell-1}{q-1}\right)\right]^{1-1/q}\\ &\times\Big(B(s+\ell+1,\ell+1)[|H''(u)|^q+|H''(v)|^q]\\ &-\big[B(\beta+\ell+1,\ell+1)-B(2\beta+\ell+1,\ell+1)\big]F(u-v)\Big)^{1/q}, \quad (7)\end{aligned}$$

where

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t,\quad \operatorname{Re}(\alpha),\operatorname{Re}(\beta)>0$$

denotes the classical Beta function.

Proof: By Lemma 1 and the Hölder integral inequality, it follows that

$$\begin{aligned}\left|\frac{H(u)+H(v)}{2}-\frac{1}{v-u}\int_u^v H(x)\,\mathrm{d}x\right| &\leqslant \frac{(v-u)^2}{2}\left(\int_0^1 [t(1-t)]^{\frac{q-\ell}{q-1}}\,\mathrm{d}t\right)^{1-1/q}\\ &\times\left(\int_0^1 [t(1-t)]^\ell |H''(tu+(1-t)v)|^q\,\mathrm{d}t\right)^{1/q}, \quad (8)\end{aligned}$$

where

$$\int_0^1 [t(1-t)]^{\frac{q-\ell}{q-1}}\,\mathrm{d}t = B\left(\frac{2q-\ell-1}{q-1},\frac{2q-\ell-1}{q-1}\right) \quad (9)$$

and, by utilizing the s-(β,F)-convexity of $|H''|^q$,

$$\begin{aligned}&\int_0^1 [t(1-t)]^\ell |H''(tu+(1-t)v)|^q\,\mathrm{d}t\\ &\leqslant \int_0^1 [t(1-t)]^\ell \Big[t^s|H''(u)|^q+(1-t)^s|H''(v)|^q\\ &\quad -t^\beta\big(1-t^\beta\big)F(u-v)\Big]\mathrm{d}t\\ &= B(s+\ell+1,\ell+1)[|H''(u)|^q+|H''(v)|^q]\\ &\quad -[B(\beta+\ell+1,\ell+1)-B(2\beta+\ell+1,\ell+1)]F(u-v). \qquad (10)\end{aligned}$$

Substituting inequalities (9) and (10) into the inequality (8) yields (7). The proof of Theorem 6 is thus complete. □

Corollary 1 *Suppose $s\in(-1,1]$, $\beta\in(0,1]$, and $u,v\in\mathbb{R}$ with $u<v$. Let $F:[u-v,0)\to\mathbb{R}$ and let $H:[u,v]\to\mathbb{R}$ be a twice differentiable function. If $|H''|^q$ is s-(β,F)-convex on $[u,v]$ for $q>1$ and $H''\in L_1([u,v])$, then*

$$\begin{aligned}&\left|\frac{H(u)+H(v)}{2}-\frac{1}{v-u}\int_u^v H(x)\,\mathrm{d}x\right|\\ &\quad\leqslant \frac{(v-u)^2}{2}B^{1-1/q}\left(\frac{2q-1}{q-1},\frac{2q-1}{q-1}\right)\\ &\quad\quad\times\left[\frac{|H''(u)|^q+|H''(v)|^q}{s+1}-\frac{\beta F(u-v)}{(\beta+1)(2\beta+1)}\right]^{1/q}.\end{aligned}$$

Proof: This is the special case $\ell=0$ in Theorem 6. The proof of Corollary 1 is thus complete. □

Theorem 7 *Suppose $s\in(-1,1]$, $\beta\in(0,1]$, and $u,v\in\mathbb{R}$ with $u<v$. Let $F:[u-v,0)\to\mathbb{R}$ and let $H:[u,v]\to\mathbb{R}$ be a differentiable function. If $|H'|^q$ is s-(β,F)-convex on $[u,v]$ for $q\geqslant 1$ and $H'\in L_1([u,v])$, then*

$$\begin{aligned}&\left|H\left(\frac{u+v}{2}\right)-\frac{1}{v-u}\int_u^v H(x)\,\mathrm{d}x\right|\\ &\leqslant \frac{v-u}{8^{1-1/q}}\Bigg[\Bigg(\frac{(s+1)|H'(u)|^q+(2^{s+2}-s-3)|H'(v)|^q}{2^{s+2}(s+1)(s+2)}\\ &\quad-\frac{2^{\beta+1}(\beta+1)-(\beta+2)}{2^{2\beta+3}(\beta+1)(\beta+2)}F(u-v)\Bigg)^{1/q}\\ &\quad+\Bigg(\frac{(2^{s+2}-s-3)|H'(u)|^q+(s+1)|H'(v)|^q}{2^{s+2}(s+1)(s+2)}\\ &-\frac{3\cdot4^{\beta+1}\beta+(2\beta+3)(\beta+2)-2^{\beta+1}(\beta+3)(2\beta+1)}{2^{2\beta+3}(\beta+1)(\beta+2)(2\beta+1)}F(u-v)\Bigg)^{1/q}\Bigg]. \qquad (11)\end{aligned}$$

Proof: Using Lemma 2 and the Hölder integral inequality, we obtain

$$\begin{aligned}&\left|H\left(\frac{u+v}{2}\right)-\frac{1}{v-u}\int_u^v H(x)\,\mathrm{d}x\right|\\ &\leqslant (v-u)\Bigg[\left(\int_0^{1/2} t\,\mathrm{d}t\right)^{1-1/q}\left(\int_0^{1/2} t|H'(tu+(1-t)v)|^q\,\mathrm{d}t\right)^{1/q}\\ &+\left(\int_{1/2}^1(1-t)\,\mathrm{d}t\right)^{1-1/q}\left(\int_{1/2}^1(1-t)|H'(tu+(1-t)v)|^q\mathrm{d}t\right)^{1/q}\Bigg]. \qquad (12)\end{aligned}$$

From the s-(β,F)-convexity of $|H'|^q$, we deduce

$$\begin{aligned}&\int_0^{1/2} t|H'(tu+(1-t)v)|^q\,\mathrm{d}t\\ &\leqslant \int_0^{1/2} t\Big[t^s|H'(u)|^q+(1-t)^s|H'(v)|^q-t^\beta\big(1-t^\beta\big)F(u-v)\Big]\mathrm{d}t\\ &= \frac{(s+1)|H'(u)|^q+(2^{s+2}-s-3)|H'(v)|^q}{2^{s+2}(s+1)(s+2)}\\ &\quad\quad -\frac{2^{\beta+1}(\beta+1)-(\beta+2)}{2^{2\beta+3}(\beta+1)(\beta+2)}F(u-v) \qquad (13)\end{aligned}$$

and

$$\begin{aligned}&\int_{1/2}^1 (1-t)|H'(tu+(1-t)v)|^q\,\mathrm{d}t\\ &\leqslant \int_{1/2}^1 (1-t)\Big[t^s|H'(u)|^q+(1-t)^s|H'(v)|^q-t^\beta\big(1-t^\beta\big)F(u-v)\Big]\mathrm{d}t\\ &= \frac{(2^{s+2}-s-3)|H'(u)|^q+(s+1)|H'(v)|^q}{2^{s+2}(s+1)(s+2)}\\ &\quad -\frac{3\cdot4^{\beta+1}\beta+(2\beta+3)(\beta+2)-2^{\beta+1}(\beta+3)(2\beta+1)}{2^{2\beta+3}(\beta+1)(\beta+2)(2\beta+1)}F(u-v). \qquad (14)\end{aligned}$$

Substituting inequalities (13) and (14) into the inequality (12) yields (11). The proof of Theorem 7 is thus complete. □

Theorem 8 *Suppose $s\in(-1,1]$, $0<\beta\leqslant 1$, and $u,v\in\mathbb{R}$ with $u<v$. Let $F:[u-v,0)\to\mathbb{R}$ and a differentiable function $H:[u,v]\to\mathbb{R}$. If $|H'|^q$ is s-(β,F)-convex on $[u,v]$ for $q>1$ and $H'\in L_1([u,v])$, then*

$$\begin{aligned}&\left|H\left(\frac{u+v}{2}\right)-\frac{1}{v-u}\int_u^v H(x)\,\mathrm{d}x\right|\leqslant\frac{v-u}{4}\left[\frac{q-1}{2^{(2q-1)/(q-1)}(2q-1)}\right]^{1-\frac{1}{q}}\times\\ &\Bigg[\left(\frac{|H'(u)|^q+(2^{s+1}-1)|H'(v)|^q}{2^{s+1}(s+1)}-\frac{2^\beta(2\beta+1)-\beta-1}{2^{2\beta+1}(\beta+1)(2\beta+1)}F(u-v)\right)^{\frac{1}{q}}+\\ &\left(\frac{(2^{s+1}-1)|H'(u)|^q+|H'(v)|^q}{2^{s+1}(s+1)}-\frac{2^{2\beta+1}\beta-2^\beta(2\beta+1)+\beta+1}{2^{2\beta+1}(\beta+1)(2\beta+1)}F(u-v)\right)^{\frac{1}{q}}\Bigg].\end{aligned}$$

Proof: By Lemma 2 and the Hölder integral inequality, it follows that

$$\left|H\left(\frac{u+v}{2}\right)-\frac{1}{v-u}\int_u^v H(x)\,\mathrm{d}x\right|$$
$$\leqslant (v-u)\left[\left(\int_0^{1/2} t^{q/(q-1)}\,\mathrm{d}t\right)^{1-1/q}\left(\int_0^{1/2}|H'(tu+(1-t)v)|^q\,\mathrm{d}t\right)^{1/q}\right.$$
$$\left.+\left(\int_{1/2}^{1}(1-t)^{q/(q-1)}\,\mathrm{d}t\right)^{1-1/q}\left(\int_{1/2}^{1}|H'(tu+(1-t)v)|^q\,\mathrm{d}t\right)^{1/q}\right].$$

It is easy to see that

$$\int_0^{1/2} t^{q/(q-1)}\,\mathrm{d}t=\int_{1/2}^{1}(1-t)^{q/(q-1)}\,\mathrm{d}t=\frac{q-1}{2^{(2q-1)/(q-1)}(2q-1)}.$$

Using the s-(β,F)-convexity of $|H'|^q$, we acquire

$$\int_0^{1/2}|H'(tu+(1-t)v)|^q\,\mathrm{d}t$$
$$\leqslant\int_0^{1/2}\left[t^s|H'(u)|^q+(1-t)^s|H'(v)|^q-t^\beta\left(1-t^\beta\right)F(u-v)\right]\mathrm{d}t$$
$$=\frac{|H'(u)|^q+(2^{s+1}-1)|H'(v)|^q}{2^{s+1}(s+1)}-\frac{2^\beta(2\beta+1)-\beta-1}{2^{2\beta+1}(\beta+1)(2\beta+1)}F(u-v)$$

and

$$\int_{1/2}^{1}|H'(tu+(1-t)v)|^q\,\mathrm{d}t$$
$$\leqslant\int_{1/2}^{1}\left[t^s|H'(u)|^q+(1-t)^s|H'(v)|^q-t^\beta\left(1-t^\beta\right)F(u-v)\right]\mathrm{d}t$$
$$=\frac{(2^{s+1}-1)|H'(u)|^q+|H'(v)|^q}{2^{s+1}(s+1)}$$
$$-\frac{2^{2\beta+1}\beta-2^\beta(2\beta+1)+\beta+1}{2^{2\beta+1}(\beta+1)(2\beta+1)}F(u-v).$$

The proof of Theorem 8 is thus complete. □

Acknowledgements: This work was partially supported by the National Natural Science Foundation of China (Grant no. 11901322), the Natural Science Foundation of Inner Mongolia (Grant no. 2019MS01007), and the Foundation of the Research Program of Science and Technology at Universities of Inner Mongolia Autonomous Region (Grant no. NJZY20119), China.

REFERENCES

1. Polyak BT (1966) Existence theorems and convergence of minimizing sequences in extremum problems with restrictions. *Soviet Math Dokl* **7**, 72–75.
2. Adamek M (2016) On a problem connected with strongly convex functions. *Math Inequal Appl* **19**, 1287–1293.
3. Adamek M (2020) On Hermite-Hadamard type inequalities for F-convex functions. *J Math Inequal* **14**, 867–874.
4. Li L, Gu XM, Zhao J (2018) The weighted arithmetic mean-geometric mean inequality is equivalent to the Hölder inequality. *Symmetry* **10**, 380.
5. Shuang Y, Qi F (2021) Integral inequalities of Hermite-Hadamard type for GA-F-convex functions. *AIMS Math* **6**, 9582–9589.
6. Wu Y, Qi F (2022) Discussions on two integral inequalities of Hermite-Hadamard type for convex functions. *J Comput Appl Math* **406**, 114049.
7. Xi BY, Gao DD, Qi F (2020) Integral inequalities of Hermite-Hadamard type for (α,s)-convex and (α,s,m)-convex functions. *Italian J Pure Appl Math* **44**, 499–510.
8. Xi BY, Qi F, Zhang TY (2015) Some inequalities of Hermite-Hadamard type for m-harmonic-arithmetically convex functions. *ScienceAsia* **41**, 357–361.
9. Zhang YC, Du TS, Pan J (2017) On new inequalities of Fejér-Hermite-Hadamard type for differentiable (α,m)-preinvex mappings. *ScienceAsia* **43**, 258–266.
10. Breckner WW (1978) Stetigkeitsaussagen Hür eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen. *Publ Inst Math (Beograd) (NS)* **23**, 13–20. [in German]
11. Hudzik H, Maligranda L (1994) Some remarks on s-convex functions. *Aequationes Math* **48**, 100–111.
12. Xi BY, Qi F (2015) Inequalities of Hermite-Hadamard type for extended s-convex functions and applications to means. *J Nonlinear Convex Anal* **16**, 873–890.
13. Dragomir SS, Pearce CEM (2000) *Selected Topics on Hermite-Hadamard Type Inequalities and Applications*, *RGMIA Monographs*, Victoria University.
14. Kirmaci US (2004) Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula. *Appl Math Comput* **147**, 137–146.
15. Mitrinović DS (1970) *Analytic Inequalities*, In cooperation with Vasić PM, Die Grundlehren der mathematischen Wissenschaften, Springer-Verlag, New York-Berlin.