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Reverse order law for the Moore-Penrose invertible operators on Hilbert C^*-modules

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ABSTRACT: It is well known that an adjointable operator between Hilbert C^*-modules admits a Moore-Penrose inverse if and only if it has closed range. In this paper, we give certain necessary and sufficient conditions for the existence of the reverse order law for the Moore-Penrose inverse of closed range adjointable operators in Hilbert C^*-module settings. Some new related results are also derived, which can be used to establish connections with the reverse order law in Hilbert C^*-modules.

KEYWORDS: Hilbert C^*-module, reverse order law, adjointable operator, Moore-Penrose inverse

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INTRODUCTION

Let $\mathscr{S}$ be a semigroup containing unit. The order relation of elements $a, b \in \mathscr{S}$ is called the reverse order law for the ordinary inverse of $\mathscr{S}$ if a and b are invertible such that $(ab)^{-1} = b^{-1}a^{-1}$. The reverse order law is a nice property which makes it very useful in many fields of mathematics. From the operator theory, it can be seen that the reverse order law also applies to the invertible operators, but this case is generally not applicable to Moore-Penrose invertible operators [1]. Therefore, for the product of operators with Moore-Penrose inverses, it is one of the most important problems to find the conditions for the existence of the reverse order law.

There exist a lot of researchers investigated the above problem in various settings [2–4]. The reverse order law for Moore-Penrose invertible matrices was first proved by Greville [5]. Bouldin [6] devoted himself to the study of reverse order law for bounded linear operators on a complex Hilbert space, and some similar results were proved by Izumino, see [7]. Moreover, Cvetković-Ilić and Harte [8] extended the reverse order law for Moore-Penrose inverse to C^*-algebra elements. Another interesting result was contributed by Djordjević and Dinčić [9]. In recent years, Sharifi [10] and Bonakdar [11] studied the reverse order law for Moore-Penrose inverse of operators in Hilbert C^*-modules. Recent studies, especially [12–15], motivate us to study the problem in the framework of Hilbert C^*-modules.

In the present paper, space decomposition and operator matrix representation in Hilbert C^*-modules are used to continue and supplement this study. We give some equivalent conditions for the existence of the reverse order law in Hilbert C^*-module settings. Moreover, several related results are obtained, which can be used to establish connections with the existence of the reverse order law in Hilbert C^*-modules.

PRELIMINARIES

The theory of Hilbert C^*-modules generalizes the theory of Hilbert spaces, of one-sided norm-closed ideals of C^*-algebras, of (locally trivial) vector bundles over compact base spaces and of their noncommutative counterparts – the projective C^*-modules over unital C^*-algebras, among others (see [16, 17]). In the following, we recall some definitions and basic properties of operators in Hilbert C^*-modules.

Definition 1 Let $\mathscr{A}$ be a C^*-algebra. A pre-Hilbert $\mathscr{A}$-module is a complex linear space $\mathscr{H}$ which is a right $\mathscr{A}$-module with an $\mathscr{A}$-valued inner product $\langle\cdot,\cdot\rangle : \mathscr{H}\times\mathscr{H} \to \mathscr{A}$ satisfying the following properties:
(i) $\langle x, \alpha y+\beta z\rangle = \alpha\langle x, y\rangle+\beta\langle x, z\rangle$ whenever $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in \mathscr{H}$;
(ii) $\langle x, ya\rangle = \langle x, y\rangle a$ for all $a \in \mathscr{A}$, $x, y \in \mathscr{H}$;
(iii) $\langle y, x\rangle^* = \langle x, y\rangle$ for all $x, y \in \mathscr{H}$;
(iv) $\langle x, x\rangle \geqslant 0$ for all $x \in \mathscr{H}$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

For $x \in \mathscr{H}$, $\|x\| = \sqrt{\|\langle x, x\rangle\|_{\mathscr{A}}}$ defines a norm on $\mathscr{H}$. Throughout the present paper we suppose that $\mathscr{H}$ is complete with respect to that norm. So $\mathscr{H}$ becomes the structure of a Banach $\mathscr{A}$-module. In this case $\mathscr{H}$ is called a Hilbert $\mathscr{A}$-module.

Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert $\mathscr{A}$-modules. We define their direct sum $\mathscr{H}\oplus\mathscr{K}$ as the set of all ordered pairs $\{(h, k) : h \in \mathscr{H}, k \in \mathscr{K}\}$. The completion of $\mathscr{H}\oplus\mathscr{K}$, with respect to the $\mathscr{A}$-valued inner product for which

$$\langle (h_1, k_1), (h_2, k_2)\rangle = \langle h_1, h_2\rangle + \langle k_1, k_2\rangle, \quad h_1, h_2 \in \mathscr{H},\ k_1, k_2 \in \mathscr{K},$$

is also a Hilbert $\mathscr{A}$-module.

In the special case where $\mathscr{A}$ is the field $\mathbb{C}$ of complex numbers, the above definition reproduces the definition of Hilbert spaces. However, by no means all theorems of Hilbert space theory can be simply generalized to the situation of Hilbert C^*-modules. To appreciate this, consider the C^*-algebra $\mathscr{A}$ of all bounded linear operators on the separable Hilbert space H together with its two-sided norm-closed ideal $\mathscr{L}$ of all compact operators on H. The C^*-algebra $\mathscr{A}$ equipped with the $\mathscr{A}$-valued inner product defined by the formula $\langle a, b\rangle = a^* b$ becomes a Hilbert $\mathscr{A}$-module over itself. The restriction of this $\mathscr{A}$-valued inner product to the ideal $\mathscr{L}$ turns $\mathscr{L}$ into a Hilbert $\mathscr{A}$-module, too. So we can form the new Hilbert $\mathscr{A}$-module $\mathscr{H} = \mathscr{A} \oplus \mathscr{L}$ as defined in the previous paragraph. We now consider some properties of $\mathscr{H}$. First of all, the analogue of the Riesz representation theorem for bounded $\mathscr{A}$-linear mappings $f : \mathscr{H} \to \mathscr{A}$ is not valid for $\mathscr{H}$. For example, the mapping $f((a,l)) = a + l$, $a \in \mathscr{A}$, $l \in \mathscr{L}$, cannot be realized by applying the $\mathscr{A}$-valued inner product to $\mathscr{H}$ with one fixed entry of $\mathscr{H}$ in its second place. Secondly, the bounded $\mathscr{A}$-linear operator T on $\mathscr{H}$ defined by the rule $T((a,l)) = (l, 0_A)$, $a \in \mathscr{A}$, $l \in \mathscr{L}$, does not have an adjoint operator T^* in the usual sense. Furthermore, the Hilbert $\mathscr{A}$-submodule $\mathscr{L}$ of the Hilbert $\mathscr{A}$-module $\mathscr{A}$ is not a direct summand, neither an orthogonal nor a topological one.

Hence the reader should be aware that every formally generalized formulation of Hilbert space theorems has to be checked for any larger class of Hilbert C^*-modules carefully and in each case separately. There are some further surprising situations in Hilbert C^*-module theory which cannot happen in Hilbert space theory. Due to their minor importance for our considerations we refer the interested reader to the standard reference sources on Hilbert C^*-modules [16–18].

Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert $\mathscr{A}$-modules. A mapping $T : \mathscr{H} \to \mathscr{K}$ is said to be adjointable if there exists a mapping $T^* : \mathscr{K} \to \mathscr{H}$ such that $\langle Tx, y\rangle = \langle x, T^* y\rangle$ for each $x \in \mathscr{H}$, $y \in \mathscr{K}$. The operator T is called selfadjoint if $T = T^*$. We also reserve the notation $\mathrm{Hom}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K})$ for the set of all adjointable operators from $\mathscr{H}$ to $\mathscr{K}$ and we denote $\mathrm{Hom}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{H})$ by $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$. It is easy to see that every element of $\mathrm{Hom}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K})$ is a bounded mapping.

Definition 2 Let $T \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K})$. Then an operator $X \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{K}, \mathscr{H})$ is called Moore-Penrose inverse of T if
(1) $TXT = T$,
(2) $XTX = X$,
(3) $(TX)^* = TX$,
(4) $(XT)^* = XT$.

Let $T\{1,2,3\}$, $T\{1,2,4\}$ and $T\{1,2,3,4\}$ be the set of operators $X \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{K}, \mathscr{H})$ which satisfy above equations {(1), (2), (3)}, {(1), (2), (4)} and {(1), (2), (3), (4)}, respectively. Obviously, the operator X is Moore-Penrose inverse of T if and only if $X \in T\{1,2,3,4\}$. In symbols, this is denoted by $T^\dagger$. The equations (1) to (4) imply that $T^\dagger$ is unique and $T^\dagger T$ and $TT^\dagger$ are orthogonal projections. It has been proven that an adjointable operator between two Hilbert C^*-modules admits a Moore-Penrose inverse if and only if it has closed range (see [19]).

Let us recall here some basic properties concerning the Moore-Penrose inverse of adjointable operators in Hilbert C^*-modules from [11, 19, 20]. It is useful in obtaining our results in this paper. For information about theory and applications of Moore-Penrose inverse we refer to the book [21]. In what follows, the symbols $\mathrm{ran}(\cdot)$ and $\ker(\cdot)$ refer, respectively, to the range and kernel of an operator.

Proposition 1 *Let $T \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K})$ admitting the Moore-Penrose inverse $T^\dagger$. Then*
(i) $\mathrm{ran}(T) = \mathrm{ran}(TT^\dagger)$ and $\mathrm{ran}(T^\dagger) = \mathrm{ran}(T^\dagger T)$,
(ii) $\ker(T) = \ker(T^\dagger T)$ and $\ker(T^\dagger) = \ker(TT^\dagger)$,
(iii) $\mathrm{ran}(T^\dagger) = \mathrm{ran}(T^*)$ and $\ker(T^\dagger) = \ker(T^*)$,
(iv) $T^\dagger = (T^*T)^\dagger T^* = T^*(TT^*)^\dagger$ and $(T^*T)^\dagger = T^\dagger T^{*\dagger}$,
(v) $T^* = T^\dagger TT^* = T^*TT^\dagger$ and $T = TT^*T^{*\dagger} = T^{*\dagger}T^*T$.

The pivotal tools in our investigation are the notions of space decomposition and operator matrix representation. Let $\mathscr{M}$ be a closed submodule of a Hilbert $\mathscr{A}$-module $\mathscr{H}$ and $\mathscr{M}^\perp := \{x \in \mathscr{H} : \langle x, y\rangle = 0,\ y \in \mathscr{M}\}$ be orthogonal complement of $\mathscr{M}$ in $\mathscr{H}$. We say $\mathscr{M}$ is orthogonally complemented if $\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^\perp$. Bearing in mind that a closed submodule of a Hilbert C^*-module need not be orthogonally complemented. Fortunately, we have the following well known result which enables us to conclude that certain submodules are orthogonally complemented. Suppose $T \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K})$, the operator T has closed range if and only if T^* has closed range. In this case, $\mathscr{H} = \ker(T) \oplus \mathrm{ran}(T^*)$ and $\mathscr{K} = \ker(T^*) \oplus \mathrm{ran}(T)$ (see [16, Theorem 3.2]).

The matrix form of an operator $T \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K})$ is induced by some natural decompositions of Hilbert C^*-modules. If $\mathscr{H} = \mathscr{Y} \oplus \mathscr{Y}^\perp$, $\mathscr{K} = \mathscr{Z} \oplus \mathscr{Z}^\perp$, then T can be written as the following 2×2 matrix

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix},$$

where $T_1 \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{Y}, \mathscr{Z})$, $T_2 \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{Y}^\perp, \mathscr{Z})$, $T_3 \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{Y}, \mathscr{Z}^\perp)$, $T_4 \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{Y}^\perp, \mathscr{Z}^\perp)$.

MAIN RESULTS

We begin with some technical lemmas, which will be used repeatedly in this paper.

Lemma 1 ([11]) *Suppose that $T \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K})$ has closed range. Then T has the following matrix decomposition with respect to the orthogonal decompositions of* *submodules* $\mathscr{H} = \ker(T) \oplus \mathrm{ran}(T^*)$ *and* $\mathscr{K} = \ker(T^*) \oplus \mathrm{ran}(T)$*:*

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathrm{ran}(T^*) \\ \ker(T) \end{pmatrix} \to \begin{pmatrix} \mathrm{ran}(T) \\ \ker(T^*) \end{pmatrix},$$

where T_1 *is invertible. Moreover*

$$T^\dagger = \begin{pmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathrm{ran}(T) \\ \ker(T^*) \end{pmatrix} \to \begin{pmatrix} \mathrm{ran}(T^*) \\ \ker(T) \end{pmatrix}.$$

Lemma 2 ([11]) *Suppose that* $T \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K})$ *has closed range. Let* $\mathscr{H}_1, \mathscr{H}_2$ *be closed submodules of* $\mathscr{H}$ *and* $\mathscr{K}_1, \mathscr{K}_2$ *be closed submodules of* $\mathscr{K}$ *such that* $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ *and* $\mathscr{K} = \mathscr{K}_1 \oplus \mathscr{K}_2$*. Then the operator* T *has the following matrix representations with respect to the orthogonal sums of submodules* $\mathscr{H} = \ker(T) \oplus \mathrm{ran}(T^*)$ *and* $\mathscr{K} = \ker(T^*) \oplus \mathrm{ran}(T)$*:*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_2 \end{pmatrix} \to \begin{pmatrix} \mathrm{ran}(T) \\ \ker(T^*) \end{pmatrix}.$$

Moreover

$$T^\dagger = \begin{pmatrix} T_1^* D^{-1} & 0 \\ T_2^* D^{-1} & 0 \end{pmatrix}$$

where $D = T_1 T_1^* + T_2 T_2^* \in \mathrm{End}^*_{\mathscr{A}}(\mathrm{ran}(T))$ *is positive and invertible.*

Lemma 3 ([15]) *Suppose that* $T \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K})$ *has closed range. and* $U \in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$ *is an orthogonal projection commuting with* $T^\dagger T$*. Then* TUT^* *has closed range.*

Lemma 4 ([22]) *Let* $T \in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$ *be an idempotent and contraction operator (*$\|T\| \leqslant 1$*). Then* T *is a projection.*

Armed with these lemmas, we can now state and prove some equivalent conditions for the existence of the reverse order law in Hilbert C^*-module settings. We first give some necessary and sufficient conditions for $S^\dagger T^\dagger \in (TS)\{1, 2, 3\}$.

Theorem 1 *Let* $\mathscr{H}, \mathscr{K}$ *and* $\mathscr{G}$ *be Hilbert* $\mathscr{A}$*-modules. Suppose that* $S \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K})$*,* $T \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{K}, \mathscr{G})$*,* $TS \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{G})$ *have closed ranges. Then the following statements are equivalent:*
(i) $(TSS^\dagger)^\dagger = SS^\dagger T^\dagger$*,*
(ii) $S^\dagger (TSS^\dagger)^\dagger = S^\dagger T^\dagger$*,*
(iii) $(TSS^\dagger)(TSS^\dagger)^\dagger = TSS^\dagger T^\dagger$*,*
(iv) $S^\dagger T^\dagger \in (TS)\{1, 2, 3\}$.

Proof: By Lemma 1, the operator S and its Moore-Penrose inverse $S^\dagger$ have the following matrix forms:

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathrm{ran}(S^*) \\ \ker(S) \end{pmatrix} \to \begin{pmatrix} \mathrm{ran}(S) \\ \ker(S^*) \end{pmatrix},$$

$$S^\dagger = \begin{pmatrix} S_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathrm{ran}(S) \\ \ker(S^*) \end{pmatrix} \to \begin{pmatrix} \mathrm{ran}(S^*) \\ \ker(S) \end{pmatrix}.$$

From Lemma 2, it follows that the operator T and its Moore-Penrose inverse $T^\dagger$ have the following matrix forms:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathrm{ran}(S) \\ \ker(S^*) \end{pmatrix} \to \begin{pmatrix} \mathrm{ran}(T) \\ \ker(T^*) \end{pmatrix},$$

$$T^\dagger = \begin{pmatrix} T_1^* D^{-1} & 0 \\ T_2^* D^{-1} & 0 \end{pmatrix},$$

where $D = T_1 T_1^* + T_2 T_2^* \in \mathrm{End}^*_{\mathscr{A}}(\mathrm{ran}(T))$ is positive and invertible. Then we have the following products

$$TS = \begin{pmatrix} T_1 S_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (TS)^\dagger = \begin{pmatrix} (T_1 S_1)^\dagger & 0 \\ 0 & 0 \end{pmatrix},$$
$$S^\dagger T^\dagger = \begin{pmatrix} S_1^{-1} T_1^* D^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Notice that $\mathrm{ran}(TSS^\dagger) = \mathrm{ran}(TS)$ is closed, so there exists $(TSS^\dagger)^\dagger$. Before proceeding, we find the equivalent expressions for our statements in terms of T_1, T_2 and S_1.
(i) $(TSS^\dagger)^\dagger = SS^\dagger T^\dagger \iff T_1^\dagger = T_1^* D^{-1}$.
(ii) $S^\dagger (TSS^\dagger)^\dagger = S^\dagger T^\dagger \iff T_1^\dagger = T_1^* D^{-1}$, so (i) $\iff$(ii).
(iii) $(TSS^\dagger)(TSS^\dagger)^\dagger = TSS^\dagger T^\dagger \iff T_1 T_1^\dagger = T_1 T_1^* D^{-1}$.
(iv) $S^\dagger T^\dagger \in (TS)\{1,2,3\} \iff \begin{cases} T_1 = T_1 T_1^* D^{-1} T_1, \\ T_1 T_1^* D^{-1} = D^{-1} T_1 T_1^*. \end{cases}$

Based on these equivalent expressions, we now prove the declared equivalent statements. We proceed with the following steps: (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iii): This is obvious.

(iii) $\Rightarrow$ (iv): Since $T_1 T_1^\dagger = T_1 T_1^* D^{-1}$ is self-adjoint and $T_1 T_1^\dagger D = T_1 T_1^*$, we get that $T_1 T_1^* D^{-1} = D^{-1} T_1 T_1^*$ and $T_1 T_1^\dagger T_2 T_2^* = 0$. Hence

$$\mathrm{ran}(T_2 T_2^*) \subset \ker(T_1 T_1^\dagger) = \ker(T_1^*).$$

It follows that $T_1^* T_2 T_2^* = 0$ and $T_2 T_2^* T_1 = 0$. Now,

$$D T_1 = (T_1 T_1^* + T_2 T_2^*) T_1 = T_1 T_1^* T_1.$$

Thus $T_1 = D^{-1} T_1 T_1^* T_1 = T_1 T_1^* D^{-1} T_1$.

(iv) $\Rightarrow$ (i): Assume that $T_1 = T_1 T_1^* D^{-1} T_1$ and $T_1 T_1^* D^{-1} = D^{-1} T_1 T_1^*$. We now check that $T_1^* D^{-1}$ satisfies all four Moore-Penrose inverse equations for operator T_1,

$$\begin{cases} T_1 T_1^* D^{-1} T_1 = T_1, \\ T_1^* D^{-1} T_1 T_1^* D^{-1} = T_1^* D^{-1}, \\ (T_1 T_1^* D^{-1})^* = D^{-1} T_1 T_1^* = T_1 T_1^* D^{-1}, \\ (T_1^* D^{-1} T_1)^* = T_1^* D^{-1} T_1. \end{cases}$$

Due to the uniqueness property of the Moore-Penrose inverse as mentioned in preliminaries, we obtain that $T_1^\dagger = T_1^* D^{-1}$, as claimed. □

In what follows, we present some necessary and sufficient conditions for $S^\dagger T^\dagger \in (TS)\{1, 2, 4\}$.

Theorem 2 *Let $\mathscr{H}, \mathscr{K}$ and $\mathscr{G}$ be Hilbert $\mathscr{A}$-modules. Suppose that $S \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K})$, $T \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{K}, \mathscr{G})$, $TS \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{G})$ have closed ranges. Then the following statements are equivalent:*
(i) $(T^\dagger TS)^\dagger = S^\dagger T^\dagger T$,
(ii) $(T^\dagger TS)^\dagger T^\dagger = S^\dagger T^\dagger$,
(iii) $(T^\dagger TS)^\dagger (T^\dagger TS) = S^\dagger T^\dagger TS$,
(iv) $S^\dagger T^\dagger \in (TS)\{1,2,4\}$.

Proof: Let us follow the strategy used in the proof of above theorem. We keep the matrix forms of T and S as in previous theorem. We see that

$$\mathrm{ran}((T^\dagger TS)^*) = \mathrm{ran}(S^* T^\dagger T) = \mathrm{ran}(S^* T^*) = \mathrm{ran}((TS)^*)$$

is closed, so there exists $(T^\dagger TS)^\dagger$. Notice that

$$T^\dagger TS = \begin{pmatrix} T_1^* D^{-1} T_1 S_1 & 0 \\ T_2^* D^{-1} T_1 S_1 & 0 \end{pmatrix},$$
$$S^\dagger T^\dagger T = \begin{pmatrix} S_1^{-1} T_1^* D^{-1} T_1 & S_1^{-1} T_1^* D^{-1} T_2 \\ 0 & 0 \end{pmatrix}.$$

Using the formula $T^\dagger = (T^*T)^\dagger T^*$ (see (iv) of Proposition 1), we obtain that

$$(T^\dagger TS)^\dagger = \begin{pmatrix} (S_1^* T_1^* D^{-1} T_1 S_1)^\dagger S_1^* T_1^* D^{-1} T_1 & (S_1^* T_1^* D^{-1} T_1 S_1)^\dagger S_1^* T_1^* D^{-1} T_2 \\ 0 & 0 \end{pmatrix}.$$

Similarly to Theorem 1, we find the equivalent expressions for our statements in terms of T_1, T_2 and S_1.
(i) $(T^\dagger TS)^\dagger = S^\dagger T^\dagger T \iff$
$$\begin{cases} (S_1^* T_1^* D^{-1} T_1 S_1)^\dagger S_1^* T_1^* D^{-1} T_1 = S_1^{-1} T_1^* D^{-1} T_1, \\ (S_1^* T_1^* D^{-1} T_1 S_1)^\dagger S_1^* T_1^* D^{-1} T_2 = S_1^{-1} T_1^* D^{-1} T_2. \end{cases}$$
(ii) $(T^\dagger TS)^\dagger T^\dagger = S^\dagger T^\dagger \iff S_1 (S_1^* T_1^* D^{-1} T_1 S_1)^\dagger S_1^* T_1^* = T_1^*$.
(iii) $(T^\dagger TS)^\dagger (T^\dagger TS) = S^\dagger T^\dagger TS \iff$
$$(S_1^* T_1^* D^{-1} T_1 S_1)^\dagger S_1^* T_1^* D^{-1} T_1 S_1 = S_1^{-1} T_1^* D^{-1} T_1 S_1.$$
(iv) $S^\dagger T^\dagger \in (TS)\{1,2,4\} \iff$
$$\begin{cases} T_1 = T_1 T_1^* D^{-1} T_1, \\ S_1 S_1^* T_1^* D^{-1} T_1 = T_1^* D^{-1} T_1 S_1 S_1^*. \end{cases}$$
According to these equivalent expressions, we now prove the claimed equivalent statements. We proceed with the following steps: (a) (i) $\iff$ (ii); (b) (ii) $\iff$ (iv); (c) (iii) $\iff$ (iv).

Step (a):

(i) $\Rightarrow$ (ii): If $(T^\dagger TS)^\dagger = S^\dagger T^\dagger T$, then by equivalent expression we can get

$$\begin{cases} (S_1^* T_1^* D^{-1} T_1 S_1)^\dagger S_1^* T_1^* D^{-1} T_1 T_1^* = S_1^{-1} T_1^* D^{-1} T_1 T_1^*, \\ (S_1^* T_1^* D^{-1} T_1 S_1)^\dagger S_1^* T_1^* D^{-1} T_2 T_2^* = S_1^{-1} T_1^* D^{-1} T_2 T_2^*. \end{cases}$$

By summing the above two equalities we obtain (ii).

(ii) $\Rightarrow$ (i): This is obvious.

Step (b):

(ii) $\Rightarrow$ (iv): If we multiply the left side of $S_1 (S_1^* T_1^* D^{-1} T_1 S_1)^\dagger S_1^* T_1^* = T_1^*$ by $S_1^* T_1^* D^{-1} T_1$ and the right side by $D^{-1} T_1 S_1$, then we get

$$S_1^* T_1^* D^{-1} T_1 S_1 (S_1^* T_1^* D^{-1} T_1 S_1)^\dagger S_1^* T_1^* D^{-1} T_1 S_1 = S_1^* T_1^* D^{-1} T_1 T_1^* D^{-1} T_1 S_1,$$

and therefore $T_1^* D^{-1} T_1 = T_1^* D^{-1} T_1 T_1^* D^{-1} T_1$. Now, $T_1^* D^{-1} T_1$ is an orthogonal projection onto a subspace of $\mathrm{ran}(T_1^*)$, so

$$T_1 = T_1 T_1^* D^{-1} T_1.$$

Since $(S_1^* T_1^* D^{-1} T_1 S_1)^\dagger S_1^* T_1^* D^{-1} T_1 = S_1^{-1} T_1^* D^{-1} T_1$ is self-adjoint, we obtain that

$$S_1 S_1^* T_1^* D^{-1} T_1 = T_1^* D^{-1} T_1 S_1 S_1^*.$$

(iv) $\Rightarrow$ (ii): Using the formula $T^\dagger = (T^*T)^\dagger T^*$, we have

$$(D^{-\frac{1}{2}} T_1 S_1)^\dagger = (S_1^* T_1^* D^{-1} T_1 S_1)^\dagger S_1^* T_1^* D^{-\frac{1}{2}},$$

which means that

$$S_1 (S_1^* T_1^* D^{-1} T_1 S_1)^\dagger S_1^* T_1^* = S_1 (D^{-\frac{1}{2}} T_1 S_1)^\dagger D^{\frac{1}{2}}.$$

In following we will show $(D^{-\frac{1}{2}} T_1 S_1)^\dagger = S_1^{-1} T_1^* D^{-\frac{1}{2}}$, by proving that $S_1^{-1} T_1^* D^{-\frac{1}{2}}$ satisfies four Moore-Penrose equations for operator $D^{-\frac{1}{2}} T_1 S_1$. It is easy to see that

$$\begin{aligned} D^{-\frac{1}{2}} T_1 S_1 S_1^{-1} T_1^* D^{-\frac{1}{2}} D^{-\frac{1}{2}} T_1 S_1 &= D^{-\frac{1}{2}} T_1 T_1^* D^{-1} T_1 S_1 \\ &= D^{-\frac{1}{2}} T_1 S_1, \\ S_1^{-1} T_1^* D^{-\frac{1}{2}} D^{-\frac{1}{2}} T_1 S_1 S_1^{-1} T_1^* D^{-\frac{1}{2}} &= S_1^{-1} T_1^* D^{-1} T_1 T_1^* D^{-\frac{1}{2}} \\ &= S_1^{-1} T_1^* D^{-\frac{1}{2}}, \end{aligned}$$
$$D^{-\frac{1}{2}} T_1 S_1 S_1^{-1} T_1^* D^{-\frac{1}{2}} = D^{-\frac{1}{2}} T_1 T_1^* D^{-\frac{1}{2}} \text{ is self-adjoint,}$$
$$S_1^{-1} T_1^* D^{-\frac{1}{2}} D^{-\frac{1}{2}} T_1 S_1 = S_1^{-1} T_1^* D^{-1} T_1 S_1 \text{ is self-adjoint.}$$

This implies that $S_1^{-1} T_1^* D^{-\frac{1}{2}} \in (D^{-\frac{1}{2}} T_1 S_1)\{1,2,3,4\}$, as desired.

Step (c):

(iii)$\Rightarrow$ (iv): If we multiply the left side of $(S_1^* T_1^* D^{-1} T_1 S_1)^\dagger S_1^* T_1^* D^{-1} T_1 S_1 = S_1^{-1} T_1^* D^{-1} T_1 S_1$ by $S_1^* T_1^* D^{-1} T_1 S_1$, we get that

$$T_1^* D^{-1} T_1 = T_1^* D^{-1} T_1 T_1^* D^{-1} T_1.$$

Now, $T_1^* D^{-1} T_1$ is an orthogonal projection onto a subspace of $\mathrm{ran}(T_1^*)$, so $T_1 = T_1 T_1^* D^{-1} T_1$. On the other hand, since

$$(S_1^* T_1^* D^{-1} T_1 S_1)^\dagger S_1^* T_1^* D^{-1} T_1 S_1 = S_1^{-1} T_1^* D^{-1} T_1 S_1$$

is self-adjoint, we obtain that

$$S_1 S_1^* T_1^* D^{-1} T_1 = T_1^* D^{-1} T_1 S_1 S_1^*.$$

(iv) ⇒ (iii): As discussed in (iv) ⇒ (ii), we know that

$$S^\dagger T^\dagger \in (TS)\{1,2,4\} \Rightarrow S_1(S_1^* T_1^* D^{-1} T_1 S_1)^\dagger S_1^* T_1^* = T_1^*.$$

Hence, $(S_1^* T_1^* D^{-1} T_1 S_1)^\dagger S_1^* T_1^* D^{-1} T_1 S_1 = S_1^{-1} T_1^* D^{-1} T_1 S_1$. □

We are now in the position to give some sufficient and necessary conditions for $(TS)^\dagger = S^\dagger T^\dagger$.

Corollary 1 *Let $\mathscr{H}, \mathscr{K}$ and $\mathscr{G}$ be Hilbert $\mathscr{A}$-modules. Suppose that $S \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K})$, $T \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{K}, \mathscr{G})$, $TS \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{G})$ have closed ranges. Then the following statements are equivalent:*
(i) $(TS)^\dagger = S^\dagger T^\dagger$;
(ii) $(TSS^\dagger)^\dagger = SS^\dagger T^\dagger$ *and* $(T^\dagger TS)^\dagger = S^\dagger T^\dagger T$;
(iii) $S^\dagger (TSS^\dagger)^\dagger = S^\dagger T^\dagger$ *and* $(T^\dagger TS)^\dagger T^\dagger = S^\dagger T^\dagger$;
(iv) $(TSS^\dagger)(TSS^\dagger)^\dagger = TSS^\dagger T^\dagger$ *and* $(T^\dagger TS)^\dagger (T^\dagger TS) = S^\dagger T^\dagger TS$.

Proof: For proof argument we refer to the Theorems 1 and 2. □

In what follows, we find the inverses of two special operators by using Moore-Penrose inverses of operators in Hilbert C^*-modules. Some related results have been derived that can be used to establish connections with the reverse order law.

Theorem 3 *Let $\mathscr{H}, \mathscr{K}$ and $\mathscr{G}$ be Hilbert $\mathscr{A}$-modules. Suppose that $S \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K})$, $T \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{K}, \mathscr{G})$, $TS \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{G})$ have closed ranges. If $SS^\dagger$ commutes with $T^\dagger T$, then the following statements hold.*
(i) $\mathrm{ran}(TSS^\dagger)$ *is closed;*
(ii) $I - TSS^\dagger T^\dagger + TSS^\dagger T^*$ *is invertible with inverse* $I - TSS^\dagger T^\dagger + (TSS^\dagger T^*)^\dagger$;
(iii) $(I - TSS^\dagger T^\dagger + TSS^\dagger T^*)(TSS^\dagger)(TSS^\dagger)^\dagger = TSS^\dagger T^*$.

Proof: The statement (i) follows from Theorem 1. For the proofs of (ii) and (iii), we keep the matrix forms of T and S as in previous theorems. Notice that

$$TSS^\dagger T^* = \begin{pmatrix} T_1 T_1^* & 0 \\ 0 & 0 \end{pmatrix}, \ (TSS^\dagger)(TSS^\dagger)^\dagger = \begin{pmatrix} T_1 T_1^\dagger & 0 \\ 0 & 0 \end{pmatrix},$$

$$I - TSS^\dagger T^\dagger + TSS^\dagger T^* = \begin{pmatrix} I_1 - T_1 T_1^* D^{-1} + T_1 T_1^* & 0 \\ 0 & I_2 \end{pmatrix}.$$

It follows that

$$(I - TSS^\dagger T^\dagger + TSS^\dagger T^*)(TSS^\dagger)(TSS^\dagger)^\dagger = \begin{pmatrix} (I_1 - T_1 T_1^* D^{-1} + T_1 T_1^*)(T_1 T_1^\dagger) & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $SS^\dagger$ commutes with $T^\dagger T$, we have $T_1 T_1^* D^{-1} T_1 = T_1$. Moreover, $T_1^\dagger = T_1^* D^{-1}$. So by Proposition 1 we get

$$\begin{aligned}
(I_1 - T_1 T_1^* D^{-1} + T_1 T_1^*)(T_1 T_1^\dagger) &= T_1 T_1^\dagger - T_1 T_1^* D^{-1} T_1 T_1^\dagger + T_1 T_1^* T_1 T_1^\dagger \\
&= T_1 T_1^\dagger - T_1 T_1^\dagger + T_1 T_1^* \\
&= T_1 T_1^*,
\end{aligned}$$

therefore, the equation of (iii) holds.

In the following, we will show that $I - TSS^\dagger T^\dagger + TSS^\dagger T^*$ is invertible. By Lemma 3 we know that $TSS^\dagger T^*$ has closed range, hence there exists $(TSS^\dagger T^*)^\dagger$. Put

$$U = I - TSS^\dagger T^\dagger + TSS^\dagger T^* = \begin{pmatrix} I_1 - T_1 T_1^* D^{-1} + T_1 T_1^* & 0 \\ 0 & I_2 \end{pmatrix},$$

$$V = I - TSS^\dagger T^\dagger + (TSS^\dagger T^*)^\dagger = \begin{pmatrix} I_1 - T_1 T_1^* D^{-1} + (T_1 T_1^*)^\dagger & 0 \\ 0 & I_2 \end{pmatrix}.$$

Since

$$\begin{aligned}
&(I_1 - T_1 T_1^* D^{-1} + T_1 T_1^*)(I_1 - T_1 T_1^* D^{-1} + (T_1 T_1^*)^\dagger) \\
&= I_1 - T_1 T_1^* D^{-1} + (T_1 T_1^*)^\dagger - T_1 T_1^* D^{-1} + T_1 T_1^* D^{-1} T_1 T_1^* D^{-1} \\
&\quad - T_1 T_1^* D^{-1} (T_1 T_1^*)^\dagger + T_1 T_1^* - T_1 T_1^* T_1 T_1^* D^{-1} + T_1 T_1^* (T_1 T_1^*)^\dagger \\
&= I_1 + (T_1 T_1^*)^\dagger - T_1 T_1^\dagger - T_1 T_1^\dagger (T_1^*)^\dagger T_1^\dagger + T_1 T_1^* - T_1 T_1^* T_1 T_1^\dagger \\
&\quad + T_1 T_1^* (T_1^*)^\dagger T_1^\dagger \\
&= I_1 + (T_1 T_1^*)^\dagger - T_1 T_1^\dagger - (T_1^*)^\dagger T_1^\dagger + T_1 T_1^* - T_1 T_1^* + T_1 T_1^\dagger \\
&= I_1,
\end{aligned}$$

we have

$$UV = \begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix} = I.$$

Similarly we can prove that $VU = I$. Hence $I - TSS^\dagger T^\dagger + TSS^\dagger T^*$ is invertible with the desired inverse, so the statement (ii) is proved. □

Using the same ideas as in above theorem we can get the following result.

Theorem 4 *Let $\mathscr{H}, \mathscr{K}$ and $\mathscr{G}$ be Hilbert $\mathscr{A}$-modules. Suppose that $S \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K})$, $T \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{K}, \mathscr{G})$, $TS \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{G})$ have closed ranges. If $SS^\dagger$ commutes with $T^\dagger T$, then the following statements hold.*
(i) $\mathrm{ran}(T^\dagger TS)$ *is closed;*
(ii) $I - S^\dagger T^\dagger TS + S^* T^\dagger TS$ *is invertible with inverse* $I - S^\dagger T^\dagger TS + (S^* T^\dagger TS)^\dagger$;
(iii) $(T^\dagger TS)^\dagger (T^\dagger TS)(I - S^\dagger T^\dagger TS + S^* T^\dagger TS) = S^* T^\dagger TS$.

Proof: Replace in Theorem 3, T and S by S^* and T^*, respectively, and take the adjoints. □

We end the paper with the following useful corollary which offers another necessary and sufficient condition for the existence of the reverse order law $(TS)^\dagger = S^\dagger T^\dagger$ in Hilbert C^*-modules.

Corollary 2 *Let $\mathscr{H}, \mathscr{K}$ and $\mathscr{G}$ be Hilbert $\mathscr{A}$-modules. Suppose that $S \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K})$, $T \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{K}, \mathscr{G})$, $TS \in \mathrm{Hom}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{G})$ have closed ranges. Then $(TS)^\dagger = S^\dagger T^\dagger$ if and only if $SS^\dagger$ commutes with $T^\dagger T$, $TSS^\dagger T^\dagger = TSS^\dagger T^*$ and $S^\dagger T^\dagger TS = S^* T^\dagger TS$.*

Proof: Suppose that $(TS)^\dagger = S^\dagger T^\dagger$. We first show that $T^\dagger T$ and $SS^\dagger$ are commutative. Since

$$TS = TS(TS)^\dagger TS = TSS^\dagger T^\dagger TS,$$

we have

$$T^{\dagger}TSS^{\dagger} = T^{\dagger}TSS^{\dagger}T^{\dagger}TSS^{\dagger} = (T^{\dagger}TSS^{\dagger})^2.$$

Besides, clearly $\|T^{\dagger}TSS^{\dagger}\| \leqslant 1$. Hence by Lemma 4, $T^{\dagger}TSS^{\dagger}$ is a projection, so that $T^{\dagger}T$ and $SS^{\dagger}$ are commutative. Now by (i) $\Longleftrightarrow$ (iv) of Corollary 1 we see that

$$(TSS^{\dagger})(TSS^{\dagger})^{\dagger} = TSS^{\dagger}T^{\dagger},\ (T^{\dagger}TS)^{\dagger}(T^{\dagger}TS) = S^{\dagger}T^{\dagger}TS.$$

Now, exploiting Theorem 3 and Theorem 4 simultaneously, we conclude that

$$TSS^{\dagger}T^{\dagger} = TSS^{\dagger}T^{*}, \quad S^{\dagger}T^{\dagger}TS = S^{*}T^{\dagger}TS.$$

The converse conclusion is also a simple calculation. Using Theorem 3 and Theorem 4, we see that

$$(TSS^{\dagger})(TSS^{\dagger})^{\dagger} = TSS^{\dagger}T^{*} = TSS^{\dagger}T^{\dagger},$$
$$(T^{\dagger}TS)^{\dagger}(T^{\dagger}TS) = S^{*}T^{\dagger}TS = S^{\dagger}T^{\dagger}TS.$$

From equivalent statement (iv) of Corollary 1, we get $(TS)^{\dagger} = S^{\dagger}T^{\dagger}$. □

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