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Meromorphic solutions of the seventh-order KdV equation by using an extended complex method and Painlevé analysis

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ABSTRACT: Using the traveling wave transformation, the seventh-order KdV equation reduces to a sixth-order complex differential equation (CDE), and we first prove that all meromorphic solutions of the CDE belong to the class W via Nevanlinna's value distribution theory. Then abundant new meromorphic solutions of the sixth-order CDE have been established in the finite complex plane with the aid of an extended complex method and Painlevé analysis, which contains Weierstrass elliptic function solutions and exponential function solutions, some of them are whole new solutions comparing to the opening literature. We give the computer simulations of some elliptic and exponential solutions. At last, we investigate the meromorphic solutions of the nonlinear dispersive Kawahara equation as an application.

KEYWORDS: Nevanlinna's value distribution theory, complex differential equations, elliptic function solutions

MSC2020: 35Q53 30D30 35C05

INTRODUCTION AND MAIN RESULTS

A function $w(z)$ is called a meromorphic function if $w(z)$ is analytic in the complex plane $\mathbb{C}$ excepts for poles. $\wp(z, g_2, g_3)$ is the Weierstrass elliptic function with invariants g_2 and g_3. It is required that the reader is familiar with the standard notations and basic results of the Nevanlinna's value distribution, for example,

$$T(r,f),\ m(r,f),\ N(r,f),\ \overline{N}(r,f),\ \ldots.$$

We denote by $S(r,f)$ a quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$, outside of a set of finite measure. For more details on Nevanlinna's value distribution theory, see [1,2]. Eremenko [3] defined that a meromorphic function $f(z)$ belongs to the class W if $f(z)$ is an elliptic function, or a rational function of $\mathrm{e}^{\alpha z}(\alpha \in \mathbb{C})$, or a rational function of z.

Baldwin et al [4] used symbolic computation to find hyperbolic and elliptic solutions to some KdV-like equations. Mancas and Hereman [5] applied an elliptic function method, Ma [6] applied a trial function method, and Wazwaz [7] applied a sech-csch method to the following nonlinear dispersive seventh-order KdV equation

$$u_t + 6uu_x + u_{3x} - u_{5x} + \alpha u_{7x} = 0, \qquad (1)$$

where $u_{nx} = \frac{\partial^n u}{\partial x^n}$, α is a non-zero constant, u_{3x} and u_{5x} are dispersion terms.

The classical KdV equation $u_t + 6uu_x + u_{3x} = 0$ describes the shallow-water waves and ion-acoustic waves in plasmas [5]. Eq. (1) was first given by Pomeau et al [8] to studying the structural stability of the KdV equation under singular perturbation. An extended tanh-coth method was used to obtain periodic and soliton wave solutions on a modified KdV equation with higher-order nonlinearity [9]. Eq. (1) has been widely used in the applied sciences and engineering, including shallow-water waves, electrical pulses in transmission lines, waves in plasmas etc [5]. The related seventh-order KdV-like equations, such as the Kaup-Kupershmidt equation, the seventh-order Lax equation and the seventh-order Sawada-Kotera-Ito equation are also attract much attention, see [10–12] for details.

Eq. (1) can be written in the Hamiltonian form [13] as

$$u_t = \frac{\partial}{\partial x}\Big(\frac{\delta H}{\delta u}\Big), \qquad (2)$$

where $\frac{\delta}{\delta u}$ is the variational derivative and

$$H = -u^3 + \frac{1}{2}u_x^2 + \frac{1}{2}u_{2x}^2 + \frac{\alpha}{2}u_{3x}^2 \qquad (3)$$

is the Hamiltonian. Then the energy integral $\int_{-\infty}^{\infty} H\,\mathrm{d}x$ does not change with t, hence the Hamiltonian describes the conserved energy density [13].

By traveling wave transformation $u(x,t) = w(z)$, $z = x + \lambda t$ onto Eq. (1), we yield

$$\lambda w + 3w^2 + w'' - w^{(4)} + \alpha w^{(6)} + \mu = 0, \qquad (4)$$

where the superscript (k) denotes the k-th derivative with respect to z, μ is a constant, and we should assume that $\alpha \neq 0$. In the case of $\alpha = 0$, Eq. (4) becomes a fourth-order CDE:

$$\lambda w + 3w^2 + w'' - w^{(4)} + \mu = 0. \qquad (5)$$

If we multiply w' to Eq. (4) and integrate once, we get

$$\frac{\lambda}{2}w^2+w^3+\frac{1}{2}w'^2-w'w'''+\frac{1}{2}w''^2 + \alpha(w'w^{(5)}-w''w^{(4)}+\frac{1}{2}w'''^2)+\mu w+C=0, \quad (6)$$

where C is a constant. Demina and Kudryashov [14, 15], Yuan et al [16, 17] studied the existence and representations of meromorphic solutions for some nonlinear complex differential equations based on the Laurent series expansion method or the complex method. Most recently, with the aid of the complex method, Dang investigated the sixth-order thin-film equation [18] with an arbitrary degree n and the $(2+1)$-dimensional and the $(3+1)$-dimensional Boiti-Leon-Manna-Pempinelli equations and the $(2+1)$-dimension Kundu-Mukherjee-Naskar equation [19]. Ng and Wu [20] investigated the second-order nonlinear Loewy factorizable algebraic ordinary differential equations (ODEs) and showed that the conjecture proposed by Hayman in 1996 holds for some certain second-order ODEs. Although many methods for constructing solutions of ordinary differential equations have made enormous progress [21], higher-order nonlinear ordinary differential equations are rarely investigated, especially analytical solutions. This work is an attempt to find exact meromorphic solutions in the finite complex plane for complex differential equation (CDE) (4). In fact, we derive the following results.

Theorem 1 *If Eq. (6) has a meromorphic solution w, then w belongs to the class W.*

Theorem 2 *Eq. (4) has the following elliptic solutions:*

$$w(z)=-924\alpha\wp''''(z-z_0,g_2,g_3)+\frac{462}{5}\wp''(z-z_0,g_2,g_3) - \frac{115500\alpha-11319}{16375\alpha}\wp(z-z_0,g_2,g_3), \quad (7)$$

where $z_0\in\mathbb{C}$ is arbitrary, and

$$g_2=\frac{3890700000\,\alpha^3 g_3-818750\,\alpha^2\lambda-85125\alpha+16366}{45391500\alpha^2},$$

$$g_3=\frac{1}{\alpha^3}\left(\frac{\alpha^2\lambda}{4752}+\frac{\alpha}{69168}+\frac{2429}{389070000}\pm\frac{\sqrt{m}}{15562800000}\right),$$

$$m=991047750000\alpha^2\lambda-5775000000\alpha^2+80264415000\alpha +4399258710;$$

$$18019050000\sqrt{m}\alpha^3\lambda+991047750000000\alpha^4\lambda -68481399525000000\alpha^3\lambda+1268211000\sqrt{m}\alpha^2+71567265000000\alpha^3 +928027842\sqrt{m}\alpha-525296825592000\alpha^2-65882439471156\alpha=0;$$

$$-413086721250000000\alpha^4\lambda^2-3091125125000000\mu\alpha^4 +20593200000\sqrt{m}\alpha^2\lambda-3741098000000000\alpha^3\lambda +266250000\sqrt{m}\alpha^2+12546875000000\alpha^3-5034411023500000\alpha^2\lambda +708925000\sqrt{m}\alpha-75573885000000\alpha^2+662733661\sqrt{m} -313056476146500\alpha-50385107208926=0.$$

Eq. (4) has the following exponential function solutions:

$$w(z)=\frac{5544000}{591361}\frac{(\mathrm{e}^{\frac{5}{\sqrt{1538}}(z-z_0)})^6}{(\mathrm{e}^{\frac{10}{\sqrt{1538}}(z-z_0)}+1)^6}-\frac{\lambda}{6}-\frac{30000}{591361}, \quad (8)$$

where $\alpha=\frac{769}{2500}$, $\mu=\frac{1}{12}\lambda^2-\frac{2700000000}{349707832321}$, and $z_0\in\mathbb{C}$ is arbitrary.

PRELIMINARIES

Let ω_1, ω_2 be two fixed complex numbers such that $\mathrm{Im}(\omega_1/\omega_2)>0$, $L=L[2\omega_1,\ 2\omega_2]$ be discrete subset $L[2\omega_1,2\omega_2]=\{\omega\,|\,\omega=2n\omega_1+2m\omega_2,\ n,m\in\mathbb{Z}\}$, which is isomorphic to $\mathbb{Z}\times\mathbb{Z}$. The discriminant $\Delta=\Delta(c_1,c_2):=c_1^3-27c_2^2$ and

$$s_n=s_n(L):=\sum_{\omega\in L\setminus\{0\}}\frac{1}{\omega^n}, \quad n\geqslant 3,\ n\in\mathbb{N}.$$

Weierstrass elliptic function [22] $\wp(z):=\wp(z,g_2,g_3)$ is a meromorphic function with two periods $2\omega_1$, $2\omega_2$ and satisfies

$$(\wp'(z))^2=4\wp(z)^3-g_2\wp(z)-g_3, \quad (9)$$

where the invariants $g_2=60s_4$, $g_3=140s_6$ and discriminant $\Delta(g_2,g_3)\neq 0$.

Furthermore, $\wp'(-z)=-\wp'(z)$, $2\wp''(z)=12\wp^2(z)-g_2$, $\wp'''(z)=12\wp(z)\wp'(z)$, $\ldots$, any k-th derivatives of $\wp$ can be deduced by these identities, and $\wp$ has the Laurent series expansion $\wp(z)=\frac{1}{z^2}+\frac{g_2z^2}{20}+\frac{g_3z^4}{28}+O(|z|^6)$, and the addition formula

$$\wp(z-z_0)=-\wp(z)-\wp(z_0)+\frac{1}{4}\left[\frac{\wp'(z)+\wp'(z_0)}{\wp(z)-\wp(z_0)}\right]^2. \quad (10)$$

If modify Eq. (9) to the following equation

$$(\wp'(z))^2=4(\wp(z)-\mathrm{e}_1)(\wp(z)-\mathrm{e}_2)(\wp(z)-\mathrm{e}_3),$$

we have $\mathrm{e}_1=\wp(\omega_1)$, $\mathrm{e}_2=\wp(\omega_2)$, $\mathrm{e}_3=\wp(\omega_1+\omega_2)$.

Reversely, given two complex numbers g_2 and g_3 such that $\Delta(g_2,g_3)\neq 0$, then there exists a double periodic $2\omega_1$, $2\omega_2$ Weierstrass elliptic function $\wp(z)$ such that the above equation hold.

Given an algebraic ODE

$$P(w,w',\ldots,w^{(m)})=bw^n, \quad (11)$$

P is a polynomial in $w(z)$ and its derivatives with constant coefficients.

We assume that the Laurent series expansion of meromorphic solutions of Eq. (11) are the form of

$$w(z)=\sum_{k=-q}^{\infty}c_k(z-z_0)^k\ (q>0). \quad (12)$$

Definition 1 If there are exactly p distinct formal meromorphic Laurent series

$$w(z)=\sum_{k=-q}^{\infty} c_k z^k \tag{13}$$

satisfy Eq. (11), we say Eq. (11) satisfy $\langle p,q\rangle$ condition [17]. If only determine p distinct principle parts $\sum_{k=-q}^{-1} c_k z^k$, we say Eq. (11) satisfy weak $\langle p,q\rangle$ condition.

Lemma 1 ([2]) *A meromorphic function f is a rational function if and only if $T(r,f)=O(\log r)$.*

Lemma 2 (Clunie's lemma [1]) *Let f be a transcendental meromorphic solution of equation*

$$f^n P(z,f,f',\dots)=Q(z,f,f',\dots),$$

where n is a non-zero positive integer, P and Q are polynomials in f and its derivatives with meromorphic coefficients $\{a_\lambda|\lambda\in I\}$, such that for each $\lambda\in I$, $m(r,a_\lambda)=S(r,f)$, where I is a index set. If the total degree of Q as a polynomial in $f,f',f'',\dots$ is at most n, then

$$m(r,P(z,f,f',\dots)=S(r,f).$$

Lemma 3 ([23, 24]) *Giving the following k-th-order Briot-Bouquet equation*

$$P(f^{(k)},f)=0, \tag{14}$$

where P is a polynomial with constant coefficients. If f is a meromorphic solution of Eq. (14) and f has at least a pole, then $f\in W$.

Lemma 4 ([16, 25]) *Suppose that an equation*

$$P(w,w',\cdots,w^{(m)})=bw^n \tag{15}$$

satisfies the $\langle p,q\rangle$ condition, where $p,l,m,n\in\mathbb{N}$, $\deg P(w,w',\dots,w^{(m)})<n$. Then all meromorphic solutions w belong to the class W. Furthermore, each elliptic solution with a pole at $z=0$ can be written as

$$w(z)=\sum_{i=1}^{l-1}\sum_{j=2}^{q}\frac{(-1)^j c_{-ij}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\left(\frac{1}{4}\left[\frac{\wp'(z)+B_i}{\wp(z)-A_i}\right]^2-\wp(z)\right)$$
$$+\sum_{i=1}^{l-1}\frac{c_{-i1}}{2}\frac{\wp'(z)+B_i}{\wp(z)-A_i}+\sum_{j=2}^{q}\frac{(-1)^j c_{-lj}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\wp(z)+c_0, \tag{16}$$

where c_{-ij} are given by (13), $B_i^2=4A_i^3-g_2A_i-g_3$, $\sum_{i=1}^{l} c_{-i1}=0$, and $c_0\in\mathbb{C}$.

Each rational function solution $w:=R(z)$ is of the form

$$R(z)=\sum_{i=1}^{l}\sum_{j=1}^{q}\frac{c_{ij}}{(z-z_i)^j}+c_0, \tag{17}$$

with $l(\leqslant p)$ distinct poles of multiplicity q.

Each simply periodic solution is a rational function $R(\xi)$ of $\xi=e^{\alpha z}(\alpha\in\mathbb{C})$. $R(\xi)$ is of the form

$$R(\xi)=\sum_{i=1}^{l}\sum_{j=1}^{q}\frac{c_{ij}}{(\xi-\xi_i)^j}+c_0, \tag{18}$$

where $R(\xi)$ has $l(\leqslant p)$ distinct poles with multiplicity q.

Remark 1 ([25]) Let $p,l,m,n\in\mathbb{N}$, $\deg P(w,w',\dots,w^{(m)})<n$, and Eq. (11) satisfies the weak $\langle p,q\rangle$ condition, then we can build meromorphic solutions by utilizing the undetermined forms of solutions (16)–(18).

Remark 2 Starting from the Laurent series, Demina and Kudryashov [14, 15] first obtained systematically the forms of elliptic, simply periodic and rational meromorphic solutions to ordinary differential equations in 2010.

Definition 2 ([25]) Let w be a meromorphic solution of a m-th-order algebraic differential equation $E(z,w)=0$. We call the involved term of $E(z,w)=0$ which determining the multiplicity q in w as the dominant term. The dominant part of $E(z,w)=0$ is consists of all dominant terms, and is denoted by $\widehat{E}=\widehat{E}(z,w)$. The multiplicity of a pole of each term in $\widehat{E}(z,w(z))$ is the same integer denoted by $D(q)$. The multiplicity of pole of each monomial $M_r[z]$ in $E(z,w)-\widehat{E}(z,w)$ is denoted by $D_r(q)$.

Definition 3 ([26]) For any meromorphic function v, the derivative operator of dominant part $\widehat{E}(z,w(z))$ with respect to w is defined by

$$\widehat{E}'(z,w)v:=\lim_{\lambda\to 0}\frac{\widehat{E}(z,w+\lambda v)-\widehat{E}(z,w)}{\lambda}. \tag{19}$$

The root of the following equation

$$P(i)=\lim_{\chi\to 0}\chi^{-i+D(q)}\widehat{E}'(\chi,c_{-q}\chi^{-q})\chi^{i-q}=0 \tag{20}$$

is called the Fuchs index of the equation $E(z,w)=0$.

By the former discussion, the extended complex method can be described concerning Eq. (4) as follows:

Step 1. Substituting the transform $T:u(x,y,t)\to w(z)$, $(x,y,t)\to z$ into the given Eq. (1), and obtaining the nonlinear ODE (4).

Step 2. Proving that all meromorphic solutions of Eq. (6) belong to the class W by using Nevanlinna's value distribution theory.

Step 3. Substituting (13) into Eq. (4) to determine that the weak $\langle p,q\rangle$ condition holds by using the Painlevé analysis.

Step 4. By indeterminant relations (16)–(18), building the elliptic, rational and simply periodic solutions $w(z)$ of Eq. (4) with pole at $z=0$, respectively.

Step 5. By Lemma 4 mainly, obtaining all meromorphic solutions $w(z-z_0)$.

Step 6. Substituting the inverse transform T^{-1} into these meromorphic solutions $w(z-z_0)$, then we get all exact solutions $u(x,t)$ of the original given Eq. (1).

PROOF OF Theorem 1

If w is a rational solution of Eq. (6), then $w \in W$, Theorem 1 holds. Next, we assume w be a transcendental solution.

Case 1. If w has finite many poles. Rewrite Eq. (6) into the following form

$$-w^3 = \frac{\lambda}{2}w^2 + \frac{1}{2}w'^2 - w'w''' + \frac{1}{2}w''^2 + \alpha(w'w^{(5)} - w''w^{(4)} + \frac{1}{2}w'''^2) + \mu w + C. \quad (21)$$

According to Clunie's lemma, we have $n=2$, $P=w$, $Q = \frac{\lambda}{2}w^2 + \frac{1}{2}w'^2 - w'w''' + \frac{1}{2}w''^2 + \alpha(w'w^{(5)} - w''w^{(4)} + \frac{1}{2}w'''^2) + \mu w + C$. Therefore, $\deg Q = 2$, then $m(r,w) = S(r,w)$. Hence, $T(r,w) - N(r,w) = S(r,w)$. From the assumption, $N(r,w) = O(\log r) = o(T(r,w))$. Therefore, $(1-o(1))T(r,w) = S(r,w)$, by Lemma 1, w must be a rational function, which is a contradiction.

Case 2. If w has infinite many poles. We assume that $z_1, z_2, \dots, z_p, \dots$ are distinct poles of w on the complex plane $\mathbb{C}$, then $w(z+z_1), w(z+z_2), \dots, w(z+z_p), \dots$ are pole distinct meromorphic solutions of Eq. (6). But there exists at most one Laurent series at the pole of z_0 (see Proof of Theorem 2), some of $w(z+z_j-z_0)$ must the same. Therefore, there exists some $i \neq j$, such that $w(z+z_i-z_0) = w(z+z_j-z_0)$, then we have $w(z) = w(z-z_i+z_j)$, therefore, w is periodic. Without lost generality, there exist $l \leqslant |i-j|$ distinct poles such that all poles of w can be described as complex number sets $z_1+\Gamma, \dots, z_l+\Gamma$, which Γ is a non-trivial discrete subgroup on $(\mathbb{C},+)$. Then Γ isomorphic to $\mathbb{Z}$ or $\mathbb{Z}\times\mathbb{Z}$. Therefore, If Γ somorphic to $\mathbb{Z}\times\mathbb{Z}$, then w is an elliptic function with at most $l \leqslant p$ distinct poles with multiplicity q in each periodic parallelogram. Otherwise, If Γ isomorphic to $\mathbb{Z}$, then $\mathbb{C}/\Gamma = \mathbb{C} - \{0\}$, hence, by using (18), w is a simply periodic meromorphic function which can be expressed by $R(e^{\alpha z})$. We must mention here that the major idea of this proof is original from [25, 27]. Thus the proof of Theorem 1 is complete.

PROOF OF Theorem 2

Assume that a meromorphic solution $w(z)$ satisfies Eq. (4), and if $w(z)$ has a movable pole at $z=0$, then in a neighbourhood of $z=z_0$, the Laurent series of w is of the form of $\sum_{k=-q}^{\infty} c_k(z-z_0)^k$ $(q>0, c_{-q}\neq 0)$. Substituting this Laurent series into Eq. (4), we have $p=1, q=6$, and

$$w(z) = -110880\alpha(z-z_0)^{-6} + \frac{2772}{5}(z-z_0)^{-4} + \left(-\frac{924}{131} + \frac{11319}{16375\alpha}\right)(z-z_0)^{-2} + \cdots. \quad (22)$$

According to Eq. (6), we know that $\widehat{E} = \widehat{E}(z,w) = w^3 + \alpha(w'w^{(5)} - w''w^{(4)} + \frac{1}{2}w'''^2)$, hence,

$$\begin{aligned}
&\widehat{E}'(z,w)v \\
&= \lim_{\lambda\to 0}\frac{1}{\lambda}\Big\{(w+\lambda v)^3 + \alpha\big[(w+\lambda v)'(w+\lambda v)^{(5)} \\
&\quad -(w+\lambda v)''(w+\lambda v)^{(4)} + \frac{1}{2}(w+\lambda v)'''^2\big] - \widehat{E}(z,w)\Big\} \\
&= 3w^2v + \alpha\big[w'v^{(5)} + w^{(5)}v' - (w''v^{(4)} + w^{(4)}v'') + w'''v'''\big] \\
&= \Big\{3w^2 + \alpha\big[w'\frac{\partial^5}{\partial z^5} + w^{(5)}\frac{\partial}{\partial z} \\
&\quad -(w''\frac{\partial^4}{\partial z^4} + w^{(4)}\frac{\partial^2}{\partial z^2}) + w'''\frac{\partial^3}{\partial z^3}\big]\Big\}v. \quad (23)
\end{aligned}$$

Hence, the Fuchs index equation of Eq. (6) reads

$$P(i) = \lim_{\chi\to 0}\chi^{-i+D(q)}\widehat{E}'(\chi, c_{-q}\chi^{-q})\chi^{i-q} = 0. \quad (24)$$

Putting (12) into Eq. (6), we have $c_{-6} = -110880\alpha$. Setting $w = c_{-6}\chi^{-6}, v = \chi^{i-6}$, we have

$$\begin{aligned}
&P(i) = \lim_{\chi\to 0}\chi^{-i+D(6)}\widehat{E}'(\chi, c_{-6}\chi^{-6})\chi^{i-6} \\
&= \lim_{\chi\to 0}\chi^{-i+18}\Big\{3w^2 + \alpha\big[w'\frac{\partial^5}{\partial z^5} + w^{(5)}\frac{\partial}{\partial z} \\
&\quad -(w''\frac{\partial^4}{\partial z^4} + w^{(4)}\frac{\partial^2}{\partial z^2}) + w'''\frac{\partial^3}{\partial z^3}\big]\Big\}\chi^{i-6} \\
&= 3c_{-6}^2 + \alpha\big[-6c_{-6}(i-6)(i-7)(i-8)(i-9)(i-10) \\
&\quad + c_{-6}(-6)(-7)(-8)(-9)(-10)(i-6) \\
&\quad - c_{-6}(-6)(-7)(i-6)(i-7)(i-8)(i-9) \\
&\quad - c_{-6}(-6)(-7)(-8)(-9)(i-6)(i-7) \\
&\quad + c_{-6}(-6)(-7)(-8)(i-6)(i-7)(i-8)\big] \\
&= -110880\alpha\big[-151200 - 6(i-6)(i-7)(i-8)(i-9)(i-10) \\
&\quad - 30240i - 42(i-6)(i-7)(i-8)(i-9) \\
&\quad - 3024(i-6)(i-7) - 336(i-6)(i-7)(i-8)\big] = 0. \quad (25)
\end{aligned}$$

Further, the roots of the Fuchs index equation $P(i)=0$ are -1, $\frac{17}{2} - \frac{1}{2}\sqrt{-163-4i\sqrt{5711}}$, $\frac{17}{2} - \frac{1}{2}\sqrt{-163+4i\sqrt{5711}}$, $\frac{17}{2} + \frac{1}{2}\sqrt{-163-4i\sqrt{5711}}$, $\frac{17}{2} + \frac{1}{2}\sqrt{-163+4i\sqrt{5711}}$, so Eq. (6) does not have any non-negative integer Fuchs index. This means that all other coefficients in the Laurent series (12) are uniquely determined [26] by the leading coefficient c_{-6} and are independent of z_0, thus we only have one distinct Laurent series. Therefore, there exists one meromorphic solution with a pole at z_0 satisfies Eq. (6). Furthermore, Eq. (4) satisfies the weak $\langle p,q\rangle = \langle 1,6\rangle$

condition, and then we can build solutions for Eq. (4) by Lemma 4 and Remark 1.

Case 1. *Rational solutions*. According to (17), we assume that the form of rational solutions of Eq. (4) with a pole at $z_0 \in \mathbb{C}$ are given by

$$w(z)=\frac{c_{-6}}{(z-z_0)^6}+\frac{c_{-5}}{(z-z_0)^5}+\cdots+c_0. \quad (26)$$

Then substituting Eq. (26) into Eq. (4), and equating the similar power terms of z to zero, we have the following approximately expression recursively:

$$w(z)=\frac{-110880\alpha}{(z-z_0)^6}+\frac{2772}{5(z-z_0)^4}+\left(\frac{11319}{16375\alpha}-\frac{924}{131}\right)\frac{1}{(z-z_0)^2}-\frac{1}{6}\lambda-\frac{227}{13100\alpha}+\frac{8183}{2456250\alpha^2}. \quad (27)$$

But if we put Eq. (27) into Eq. (4), we have

$$\left.\begin{aligned} &-\frac{177502248}{2145125\alpha}+\frac{1835064}{17161}+\frac{3355868439}{268140625\alpha^2}=0\\ &\frac{314622}{429025}+\frac{92623377}{6703515625\alpha^2}-\frac{22830423}{107256250\alpha}=0\\ &\mu+\frac{154587}{171610000\alpha^2}-\frac{1857541}{5362812500\alpha^3}+\\ &\qquad\frac{66961489}{2011054687500\alpha^4}-\frac{\lambda^2}{12}=0.\end{aligned}\right\} \quad (28)$$

Eq. (28) has no algebraic solution about α, λ, μ. Therefore Eq. (4) has no rational solution.

Case 2. *Elliptic function solutions.* By Eq. (16) and $c_{-1}=0$, we can assume that the form of elliptic solutions of Eq. (4) are given by

$$w(z)=\frac{(-1)^6c_{-6}}{5!}\wp''''(z,g_2,g_3)+\frac{(-1)^4c_{-4}}{3!}\wp''(z,g_2,g_3)+\frac{(-1)^2c_{-2}}{1!}\wp(z,g_2,g_3)+c_0. \quad (29)$$

From former discussion, we have $c_{-6}=-110880\alpha$, $c_{-4}=\frac{2772}{5}$, $c_{-2}=-\frac{924}{131}+\frac{11319}{16375\alpha}$, $c_{-1}=c_{-3}=c_{-5}=0$, where c_0 is a complex constant. Then putting Eq. (29) into Eq. (4), we have

$$w(z)=-924\alpha\wp''''(z-z_0,g_2,g_3)+\frac{462}{5}\wp''(z-z_0,g_2,g_3)-\frac{115500\alpha-11319}{16375\alpha}\wp(z-z_0,g_2,g_3), \quad (30)$$

with a movable pole with multiplicity 6 with an arbitrary complex constant z_0, where

$$\left.\begin{aligned} g_2&=\frac{3890700000\alpha^3g_3-818750\alpha^2\lambda-85125\alpha+16366}{45391500\alpha^2}\\ g_3&=\frac{1}{\alpha^3}\Big(\frac{\alpha^2\lambda}{4752}+\frac{\alpha}{69168}+\frac{2429}{389070000}\pm\frac{\sqrt{m}}{15562800000}\Big)\end{aligned}\right\} \quad (31)$$

provided that

$$m=991047750000\alpha^2\lambda-5775000000\alpha^2+80264415000\alpha+4399258710,$$

$$18019050000\sqrt{m}\alpha^3\lambda+991047750000000\alpha^4\lambda-6848139952500000\alpha^3\lambda+1268211000\sqrt{m}\alpha^2+71567265000000\alpha^3+928027842\sqrt{m}\alpha-525296825592000\alpha^2-65882439471156\alpha=0,$$

$$-413086721250000000\alpha^4\lambda^2-3091125125000000\mu\alpha^4+20593200000\sqrt{m}\alpha^2\lambda-3741098000000000\alpha^3\lambda+266250000\sqrt{m}\alpha^2+12546875000000\alpha^3-5034411023500000\alpha^2\lambda+708925000\sqrt{m}\alpha-75573885000000\alpha^2+662733661\sqrt{m}-313056476146500\alpha-50385107208926=0.$$

Case 3. *Exponential function solutions.* By Eq. (18), we assume that the form of the simply periodic solutions of Eq. (4) are given by

$$w(z)=\frac{c_{-6}}{(\exp(\theta z)+\exp(\theta z_0))^6}+\frac{c_{-5}}{(\exp(\theta z)+\exp(\theta z_0))^5}+\cdots+c_0, \quad (32)$$

where c_{-i}, θ are unknown, $z_0 \in \mathbb{C}$ is arbitrary. Substituting Eq. (32) into Eq. (4), then we get the following simply periodic solutions:

$$w(z)=\frac{5544000}{591361}\frac{(\mathrm{e}^{\frac{5}{\sqrt{1538}}(z-z_0)})^6}{(\mathrm{e}^{\frac{10}{\sqrt{1538}}(z-z_0)}+1)^6}-\frac{\lambda}{6}-\frac{30000}{591361}, \quad (33)$$

where $\alpha=\frac{769}{2500}$ and $\mu=\frac{1}{12}\lambda^2-\frac{2700000000}{349707832321}$.

Moreover, we can rewrite Eq. (33) into the following form of solitons:

$$w(z)=\frac{86625}{591361}\operatorname{sech}^6\Big(\pm\frac{5}{\sqrt{1538}}(z-z_0)\Big)-\frac{\lambda}{6}-\frac{30000}{591361}, \quad (34)$$

where $\alpha=\frac{769}{2500}$ and $\mu=\frac{1}{12}\lambda^2-\frac{2700000000}{349707832321}$. Hence, the proof of Theorem 2 is complete.

The computer simulation of solutions (30) with $z_0=0$, $\alpha=1$, $\lambda=1$ in the complex domain are described in Fig. 1

The computer simulation of solutions (33) with $z_0=0$, $\lambda=1$ and Eq. (34) with $z_0=0$, $\lambda=-1$ in the complex domain are described in Fig. 2. These figures depict periodic properties of the new exact solutions of Eq. (4).

Remark 3 In the sense of Nevanlinna's value distribution theory, the growth order of meromorphic solutions of Eq. (4) which belong to the class W is no greater than two.

Remark 4 In this section, we built some new explicit solutions for Eq. (4). Particularly, solutions (34) are similar to the solutions proposed in Ma [6] and Wazwaz [7]. To the best of our knowledge, we build new Weierstrass elliptic solutions (30) and new exponential function solutions (33) [5].

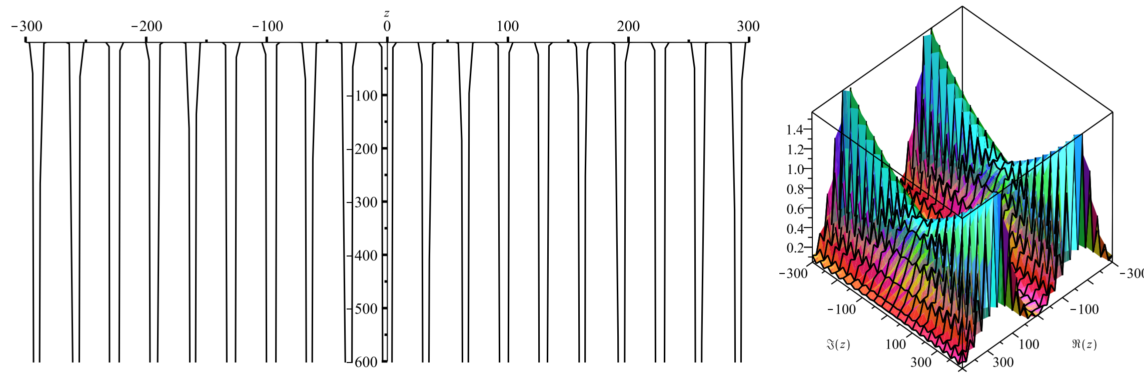

Fig. 1 The 2D and 3D plots of solutions (30) (with $z_0 = 0$, $\alpha = 1$, $\lambda = 1$).

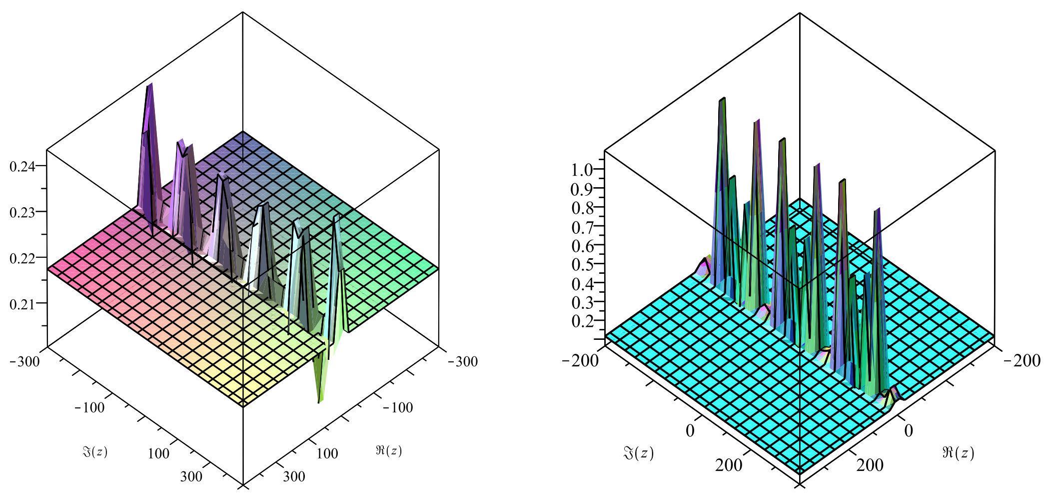

Fig. 2 The 3D plots of solutions (33) (the first with $\lambda = 1$) and (34) (the second with $\lambda = -1$).

APPLICATIONS

In this section, we intend to apply Nevanlinna's value distribution theory and the extended complex method to the well known nonlinear dispersive Kawahara equation and find some meromorphic solutions and the exponential function solution (42) seems to be new. Kawahara equation [14, 28] reads

$$u_t + \kappa u u_x + \alpha u_{xxx} - \beta u_{xxxxx} = 0, \qquad (35)$$

which describes magneto-acoustic waves in a cold collision-free plasma, αu_{xxx} and βu_{xxxxx} are dispersive terms, κ is the strength of the nonlinearity [29]. The higher-order dispersion β and the nonlinear effect κ cannot degenerate to zero, otherwise the type of KE equation will be changed. By a travelling wave transformation $u(z) = u(x - ct)$, c is the velocity of the travelling wave in the x-direction at time t, we obtain the following algebraic differential equation

$$cu' - \kappa u u' - \alpha u''' + \beta u''''' = 0. \qquad (36)$$

Integrating Eq. (36) with respect to z, we obtain the following fourth-order equation

$$cu' - \frac{\kappa}{2}u^2 - \alpha u'' + \beta u'''' + C = 0. \qquad (37)$$

Rewrite Eq. (37) into the following form

$$cu' - \alpha u'' + \beta u'''' + C = \frac{\kappa}{2}u^2, \qquad (38)$$

where β is the balance coefficient to the nonlinear effect κ. By Clunie's lemma, we can prove that all meromorphic solutions of Eq. (37) belong to the class W. Furthermore, If $\alpha = 0$, Eq. (37) is a Briot-Bouquet equation, by Lemma 3, all meromorphic solutions belong to the class W. Here we assume $\kappa \neq 0$. Substituting the Laurent series (13) into Eq. (37), we have $p = 1$, $q = 4$, and

$$u(z) = \frac{1680\beta}{\kappa}\frac{1}{z^4} - \frac{280\alpha}{13\kappa}\frac{1}{z^2} - \frac{280c}{69\kappa}\frac{1}{z} - \frac{31\alpha^2}{507\beta\kappa} + \cdots. \qquad (39)$$

Substituting $u(z)$ into Eq. (37) we have

$$-\frac{961\alpha^4}{514098\beta^2\kappa}+C-\frac{8680c\alpha^2}{34983\kappa z\beta}+\frac{1}{z^2}\left(-\frac{19880c^2}{4761\kappa}-\frac{8680\alpha^3}{6591\kappa\beta}\right)-\frac{32480c\alpha}{897\kappa z^3}=0, \quad (40)$$

then $c=\alpha=0$, and we obtain the rational solution $u(z)=1680\frac{\beta}{\kappa(z-z_0)^4}$, where $\alpha=0$, $c=0$ and $C=0$.

In order to find periodic solutions, we substitute the following form into Eq. (37)

$$u(z)=\frac{c_{-4}}{(e^{\theta z}-1)^4}+\frac{c_{-2}}{(e^{\theta z}-1)^2}+\frac{c_{-1}}{e^{\theta z}-1}+c_0, \quad (41)$$

where $\theta\neq 0$, then we obtain $-4435200\frac{\beta^2\theta^8(e^{\theta z})^7}{\kappa(e^{\theta z}-1)^7}=0$, hence $\beta=0$. Furthermore, in the case of $\beta=0$, we obtain the following form of exponential function solutions:

$$u(z)=-\frac{12\alpha\theta^2(e^{\theta z})^2}{\kappa(e^{\theta z}-e^{\theta z_0})^2}+\frac{12e^{\theta z}\theta(5\alpha\theta-c)}{5\kappa(e^{\theta z}-e^{\theta z_0})}-\frac{25\alpha^2\theta^2-30\alpha c\theta-c^2}{25\alpha\kappa}, \quad (42)$$

provided that $\theta c(25\alpha^2\theta^2-c^2)=0$, and $-625\alpha^4\theta^4+1500\alpha^3c\theta^3-850\alpha^2c^2\theta^2-60\alpha c^3\theta+1250C\alpha^2\kappa-c^4=0$, where z_0 is an arbitrary complex number.

In order to clarify the elliptic function solutions, noting that if $c_{-1}=-\frac{280c}{69\kappa}\neq 0$, it follows Lemma 4 that Eq. (37) has no elliptic function solution. If $c_{-1}=-\frac{280c}{69\kappa}=0$, then $c=0$, by using a similar operation, we obtain the following form of elliptic solutions

$$u(z)=1680\frac{\beta\wp^2}{\kappa}-\frac{280\alpha\wp}{13\kappa}-\frac{31\alpha^2}{507\beta\kappa}, \quad (43)$$

where $c=0$, $57122C\beta^2\kappa-1457\alpha^4=0$, $g_2=0$, and $g_3=-\frac{31\alpha^3}{4745520\beta^3}$.

CONCLUSION

By travelling wave transformation, a PDE can be reduced into a CDE. We can check the solutions belonging to the class W by using Nevanlinna's value distribution theory. Then, construct meromorphic solutions of the CDE by using the extended complex method with the aid of Painlevé analysis. Furthermore, these meromorphic solutions contain exponential function solutions and elliptic function solutions. This systematic method should be applied to build doubly and simply periodic function solutions of nonlinear higher-order PDEs in applied science or mathematical physics.

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