RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2022.063
ScienceAsia 48 (2022): **479–487**

Some new super convergence of a quartic integro-spline at the mid-knots of a uniform partition

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Received 29 Jun 2021, Accepted 7 Feb 2022
Available online 15 Apr 2022

ABSTRACT: In this paper, we study some new super convergence of a quartic integro-spline at the mid-knots of a uniform partition. We prove that the quartic integro-spline has super convergence in function values approximation (sixth order convergence), in second-order derivatives approximation (fourth order convergence) and in fourth-order derivatives approximation (second order convergence) at the mid-knots, no matter that the quartic integro-spline is determined by using four exact boundary conditions or is determined by using four approximate boundary conditions. These new super convergence properties also have been numerically examined.

KEYWORDS: super convergence, quartic integro-spline, mid-knot, integral interpolation, error analysis

MSC2020: 65D07 65D05 41A15

INTRODUCTION

Let $\Delta := \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a uniform partition of $[a, b]$ with step length $h = (b-a)/n$,

$$I_j := \int_{x_j}^{x_{j+1}} y(x)\,\mathrm{d}x \qquad (j = 0, 1, \ldots, n-1) \tag{1}$$

be the given integral values of an unknown function $y = y(x)$. Approximating $y = y(x)$ and its derivatives by using the integral values (1) is called integro-approximation. Splines have been widely used for this problem, see the works of Behforooz [1, 2], Zhanlav [3–5], Mijiddorj [6, 7], Lang [8–10], Xu [11, 12], Haghighi [13, 14], and Wu [15–17]. Generally, the obtained integro-splines have good approximation abilities.

For example, in [8], we studied a quartic integro-spline $s = s(x)$ satisfying

$$\int_{x_j}^{x_{j+1}} s(x)\,\mathrm{d}x = I_j \qquad (j = 0, 1, \ldots, n-1) \tag{2}$$

and four boundary conditions

$$s(x_0) = y(x_0), \tag{3}$$
$$s(x_1) = y(x_1), \tag{4}$$
$$s(x_{n-1}) = y(x_{n-1}), \tag{5}$$
$$s(x_n) = y(x_n). \tag{6}$$

We reported that the quartic integro-spline $s = s(x)$ possesses super convergence in function values approximation (sixth order convergence) and in second-order derivatives approximation (fourth order convergence) at the knots x_j $(j = 0, 1, \ldots, n)$, i.e.,

$$s^{(k)}(x_j) - y^{(k)}(x_j) = O(h^{6-k}), \quad k = 0, 2. \tag{7}$$

Obviously, the convergence orders of these two approximations at the knots are all one order higher than the ordinary cases of a quartic spline. Furthermore, it was also proved in [8] that the super convergence (7) still hold even if the exact boundary function values $y(x_0)$, $y(x_1)$, $y(x_{n-1})$, $y(x_n)$ in (3), (4), (5) and (6) are replaced respectively by the following approximate boundary function values

$$\begin{aligned}\widetilde{y}(x_0) = \tfrac{1}{60h}(&147I_0 - 213I_1 + 237I_2 - 163I_3\\ &+ 62I_4 - 10I_5),\end{aligned} \tag{8}$$

$$\begin{aligned}\widetilde{y}(x_1) = \tfrac{1}{60h}(&10I_0 + 87I_1 - 63I_2 + 37I_3\\ &- 13I_4 + 2I_5),\end{aligned} \tag{9}$$

$$\begin{aligned}\widetilde{y}(x_{n-1}) = \tfrac{1}{60h}(&10I_{n-1} + 87I_{n-2} - 63I_{n-3}\\ &+ 37I_{n-4} - 13I_{n-5} + 2I_{n-6}),\end{aligned} \tag{10}$$

$$\begin{aligned}\widetilde{y}(x_n) = \tfrac{1}{60h}(&147I_{n-1} - 213I_{n-2} + 237I_{n-3}\\ &- 163I_{n-4} + 62I_{n-5} - 10I_{n-6}).\end{aligned} \tag{11}$$

Later, the super convergence of some other integro-splines at the knots of a uniform partition has also been studied. The super convergence of sextic integro-spline in approximating $y^{(k)}(x_j)$ $(k = 0, 2, 4)$ was presented in [15] and the super convergence of quintic integro-spline in approximating $y^{(k)}(x_j)$ $(k = 1, 3)$ was given in [3, 9, 10].

Do some integro-splines have super convergence properties at some other points? The answer is YES. In [12], we have proved that some quadratic integro-splines have super convergence in function values approximation and in second-order derivatives approximation at the mid-knots $\tau_j = (x_j + x_{j+1})/2$, $j = 0, 1, \ldots, n-1$. Considering quadratic integro-splines have super convergence at mid-points, it is natural to ask that whether or not the above-mentioned quartic integro-spline also has some new super convergence at the mid-knots except for the existing ones (7) at the knots. In this paper, we will answer this question.

We assume that $y = y(x)$ belongs to the class $C^6[a,b]$. We will prove that the above-mentioned quartic integro-spline, no matter it is determined by using I_j $(j = 0,1,\dots,n-1)$ along with the exact boundary function values $y(x_0)$, $y(x_1)$, $y(x_{n-1})$, $y(x_n)$ or it is determined by using I_j $(j = 0,1,\dots,n-1)$ along with the approximate boundary function values $\widetilde{y}(x_0)$, $\widetilde{y}(x_1)$, $\widetilde{y}(x_{n-1})$, $\widetilde{y}(x_n)$, also has some new super convergence at the mid-knots.

BRIEF PRELIMINARIES OF THE QUARTIC INTEGRO-SPLINE

The quartic integro-spline $s = s(x)$ determined by (2) and (3), (4), (5), (6) is a piecewise quartic polynomial, which is three times continuously differentiable over $[a,b]$ (see [8]). It is an element of the $(n+4)$-dimensional quartic spline space associated with the interval $[a,b]$ and the partition Δ (see [18–20]). It can be represented as

$$s(x) = \sum_{i=-2}^{n+1} c_i B_i(x), \tag{12}$$

where $B_i(x) =$

$$\frac{1}{24h^4}\begin{cases}(x-x_{i-2})^4, & x\in[x_{i-2},x_{i-1}],\\ (x-x_{i-2})^4-5(x-x_{i-1})^4, & x\in[x_{i-1},x_i],\\ (x-x_{i-2})^4-5(x-x_{i-1})^4+10(x-x_i)^4, & x\in[x_i,x_{i+1}],\\ (x-x_{i+3})^4-5(x-x_{i+2})^4, & x\in[x_{i+1},x_{i+2}],\\ (x-x_{i+3})^4, & x\in[x_{i+2},x_{i+3}],\\ 0, & \text{else},\end{cases} \tag{13}$$

$(i=-2,-1,\dots,n+1)$ are the quartic B-splines [21, 22].

The coefficients c_i $(i = -2,-1,\dots,n+1)$ of the quartic integro-spline in (12) can be obtained by solving the linear system (see [8])

$$\begin{pmatrix}1 & 11 & 11 & 1 & 0 & & \\ 0 & 1 & 11 & 11 & 1 & & \\ 1 & 26 & 66 & 26 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & 26 & 66 & 26 & 1\\ & & 1 & 11 & 11 & 1 & 0\\ & & 0 & 1 & 11 & 11 & 1\end{pmatrix}\begin{pmatrix}c_{-2}\\ c_{-1}\\ c_0\\ \vdots\\ c_{n-1}\\ c_n\\ c_{n+1}\end{pmatrix} = \begin{pmatrix}24s(x_0)\\ 24s(x_1)\\ \frac{120}{h}I_0\\ \vdots\\ \frac{120}{h}I_{n-1}\\ 24s(x_{n-1})\\ 24s(x_n)\end{pmatrix}. \tag{14}$$

To study the super convergence at the mid-knots, the values $B_i^{(k)}(\tau_j)$ $(k = 0,1,2,3,4)$ are needed. These values can be obtained by using (13). We list them in Table 1.

Table 1 The values of the quartic B-spline and its first four derivatives at the mid-knots.

	τ_{i-2}	τ_{i-1}	τ_i	τ_{i+1}	τ_{i+2}	else
$B_i(x)$	$\frac{1}{384}$	$\frac{76}{384}$	$\frac{230}{384}$	$\frac{76}{384}$	$\frac{1}{384}$	0
$B_i'(x)$	$\frac{1}{48h}$	$\frac{22}{48h}$	$\frac{0}{48h}$	$-\frac{22}{48h}$	$-\frac{1}{48h}$	0
$B_i''(x)$	$\frac{1}{8h^2}$	$\frac{4}{8h^2}$	$-\frac{10}{8h^2}$	$\frac{4}{8h^2}$	$\frac{1}{8h^2}$	0
$B_i'''(x)$	$\frac{1}{2h^3}$	$-\frac{2}{2h^3}$	$\frac{0}{2h^3}$	$\frac{2}{2h^3}$	$-\frac{1}{2h^3}$	0
$B_i''''(x)$	$\frac{1}{h^4}$	$-\frac{4}{h^4}$	$\frac{6}{h^4}$	$-\frac{4}{h^4}$	$\frac{1}{h^4}$	0

By using (12) and the data in Table 1, for $j = 0,1,\dots,n-1$, we have the following formulae

$$s(\tau_j) = \sum_{i=j-2}^{j+2} c_i B_i(\tau_j) = \frac{1}{384}(c_{j-2}+76c_{j-1}+230c_j+76c_{j+1}+c_{j+2}), \tag{15}$$

$$s'(\tau_j) = \sum_{i=j-2}^{j+2} c_i B_i'(\tau_j) = \frac{1}{48h}(-c_{j-2}-22c_{j-1}+22c_{j+1}+c_{j+2}), \tag{16}$$

$$s''(\tau_j) = \sum_{i=j-2}^{j+2} c_i B_i''(\tau_j) = \frac{1}{8h^2}(c_{j-2}+4c_{j-1}-10c_j+4c_{j+1}+c_{j+2}), \tag{17}$$

$$s'''(\tau_j) = \sum_{i=j-2}^{j+2} c_i B_i'''(\tau_j) = \frac{1}{2h^3}(-c_{j-2}+2c_{j-1}-2c_{j+1}+c_{j+2}), \tag{18}$$

$$s''''(\tau_j) = \sum_{i=j-2}^{j+2} c_i B_i''''(\tau_j) = \frac{1}{h^4}(c_{j-2}-4c_{j-1}+6c_j-4c_{j+1}+c_{j+2}). \tag{19}$$

NEW INHERENT RELATIONS OF THE QUARTIC INTEGRO-SPLINE

First, we present some new inherent relations between I_j and $s(\tau_j)$, $s'(\tau_j)$, $s''(\tau_j)$, $s'''(\tau_j)$, $s''''(\tau_j)$ of the quartic integro-spline.

Lemma 1 *For $j = 2,3,\dots,n-3$, we have*

$$s(\tau_{j-2})+26s(\tau_{j-1})+66s(\tau_j)+26s(\tau_{j+1})+s(\tau_{j+2}) = \frac{5}{16h}(I_{j-2}+76I_{j-1}+230I_j+76I_{j+1}+I_{j+2}), \tag{20}$$

$$s'(\tau_{j-2})+26s'(\tau_{j-1})+66s'(\tau_j)+26s'(\tau_{j+1})+s'(\tau_{j+2}) = \frac{5}{2h^2}(-I_{j-2}-22I_{j-1}+22I_{j+1}+I_{j+2}), \tag{21}$$

$$s''(\tau_{j-2})+26s''(\tau_{j-1})+66s''(\tau_j)+26s''(\tau_{j+1})+s''(\tau_{j+2}) = \frac{15}{h^3}(I_{j-2}+4I_{j-1}-10I_j+4I_{j+1}+I_{j+2}), \tag{22}$$

$$s'''(\tau_{j-2})+26s'''(\tau_{j-1})+66s'''(\tau_j)+26s'''(\tau_{j+1}) + s'''(\tau_{j+2}) = \tfrac{60}{h^4}(-I_{j-2}+2I_{j-1}-2I_{j+1}+I_{j+2}), \quad (23)$$

$$s''''(\tau_{j-2})+26s''''(\tau_{j-1})+66s''''(\tau_j)+26s''''(\tau_{j+1}) + s''''(\tau_{j+2}) = \tfrac{120}{h^5}(I_{j-2}-4I_{j-1}+6I_j-4I_{j+1}+I_{j+2}). \quad (24)$$

Proof: By referring to (14), for $j=0,1,\dots,n-1$, we have

$$I_j = \tfrac{h}{120}(c_{j-2}+26c_{j-1}+66c_j+26c_{j+1}+c_{j+2}). \quad (25)$$

These relations can be proved by comparing the coefficients of c_j ($j=-2,-1,\dots,n+1$) by using (15), (16), (17), (18), (19) and (25). □

Next, we give another proof of Lemma 1.

Proof: For a quartic integro-spline, we study it on $[x_{j-2},x_{j+3}]$. It is quartic over every subintervals and is three times continuously differentiable across the inner knots x_{j-1}, x_j, x_{j+1} and x_{j+2}, therefore, it has and only has nine independent quantities. All the other quantities relative with the interval $[x_{j-2},x_{j+3}]$ can be expressed by using the nine independent quantities. For example, we may take $s(\tau_i)$ ($i=j-2,\dots,j+2$) and I_i ($i=j-2,\dots,j+1$) as the nine independent quantities. By using the coefficients of the B-splines, we can express I_{j+2} by using $s(\tau_i)$ ($i=j-2,\dots,j+2$) and I_i ($i=j-2,\dots,j+1$) as follows.

From (15) and (25), we have

$$c_{i-2}+76c_{i-1}+230c_i+76c_{i+1}+c_{i+2} = 384s(\tau_i), \quad i=j-2,\dots,j+2;$$
$$c_{i-2}+26c_{i-1}+66c_i+26c_{i+1}+c_{i+2} = \tfrac{120}{h}I_i, \quad i=j-2,\dots,j+1.$$

We write the system as $AC=R$, where

$$A=\begin{pmatrix} 1 & 76 & 230 & 76 & 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 76 & 230 & 76 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 76 & 230 & 76 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 76 & 230 & 76 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 & 76 & 230 & 76 & 1\\ 1 & 26 & 66 & 26 & 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 26 & 66 & 26 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 26 & 66 & 26 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 26 & 66 & 26 & 1 & 0 \end{pmatrix},$$

$$C=\begin{pmatrix} c_{j-4} & c_{j-3} & \cdots & c_{j+3} & c_{j+4}\end{pmatrix}^{\mathrm{T}},$$
$$R=\begin{pmatrix} 384s(\tau_{j-2}) & \cdots & 384s(\tau_{j+2}) & \tfrac{120}{h}I_{j-2} & \cdots & \tfrac{120}{h}I_{j+1}\end{pmatrix}^{\mathrm{T}}.$$

Hence, we have

$$\begin{aligned} I_{j+2} &= \tfrac{h}{120}(c_j+26c_{j+1}+66c_{j+2}+26c_{j+3}+c_{j+4})\\ &= \tfrac{h}{120}(0,0,0,0,1,26,66,26,1)C\\ &= \tfrac{h}{120}(0,0,0,0,1,26,66,26,1)A^{-1}R\\ &= \tfrac{h}{120}(1,26,66,26,1,-1,-76,-230,-76)R\\ &= \tfrac{16h}{5}(s(\tau_{j-2})+26s(\tau_{j-1})+66s(\tau_j)+26s(\tau_{j+1})\\ &\quad +s(\tau_{j+2}))-(I_{j-2}+76I_{j-1}+230I_j+76I_{j+1}). \quad (26)\end{aligned}$$

Rearranging the terms and the coefficients, we can get (20) immediately. The others can be obtained similarly. □

Next, we present some new inherent relations between four boundary I_j, four boundary $s(\tau_j)$ and two boundary $s(x_j)$ of the quartic integro-spline.

Lemma 2

$$111s(\tau_0)+34s(\tau_1)+\tfrac{7}{5}s(\tau_2) = \tfrac{2473}{16h}I_0+\tfrac{65}{2h}I_1+\tfrac{7}{16h}I_2-\tfrac{959}{40}s(x_0)-\tfrac{137}{8}s(x_1), \quad (27)$$

$$31s(\tau_0)-31s(\tau_1)-\tfrac{123}{5}s(\tau_2)-s(\tau_3) = \tfrac{703}{16h}I_0-\tfrac{625}{16h}I_1-\tfrac{373}{16h}I_2-\tfrac{5}{16h}I_3-\tfrac{137}{20}s(x_0), \quad (28)$$

$$-s(\tau_{n-4})-\tfrac{123}{5}s(\tau_{n-3})-31s(\tau_{n-2})+31s(\tau_{n-1}) = \tfrac{703}{16h}I_{n-1}-\tfrac{625}{16h}I_{n-2}-\tfrac{373}{16h}I_{n-3}-\tfrac{5}{16h}I_{n-4}-\tfrac{137}{20}s(x_n), \quad (29)$$

$$\tfrac{7}{5}s(\tau_{n-3})+34s(\tau_{n-2})+111s(\tau_{n-1}) = \tfrac{2473}{16h}I_{n-1} +\tfrac{65}{2h}I_{n-2}+\tfrac{7}{16h}I_{n-3}-\tfrac{959}{40}s(x_n)-\tfrac{137}{8}s(x_{n-1}). \quad (30)$$

Proof: By using (12) and (13), we have for $j=0,1,\dots,n$,

$$s(x_j) = \tfrac{1}{24}(c_{j-2}+11c_{j-1}+11c_j+c_{j+1}). \quad (31)$$

These relations can be proved by using (15), (25) and (31).

Moreover, (27) also can be obtained as follows. For a quartic integro-spline, we study it on $[x_0,x_3]$. It has and only has seven independent quantities. Here, we choose $s(\tau_0)$, $s(\tau_1)$, $s(\tau_2)$, I_0, I_1, $s(x_0)$ and $s(x_1)$ as the seven independent quantities. All the other quantities relative with $[x_0,x_3]$ can be expressed by using the seven independent quantities, I_2 is not an exception. By using the coefficients of the B-splines, from (15), (25) and (31), we have

$$c_{i-2}+76c_{i-1}+230c_i+76c_{i+1}+c_{i+2}=384s(\tau_i), \quad i=0,1,2;$$
$$c_{i-2}+26c_{i-1}+66c_i+26c_{i+1}+c_{i+2}=\tfrac{120}{h}I_i, \quad i=0,1;$$
$$c_{i-2}+11c_{i-1}+11c_i+c_{i+1}=24s(x_i), \quad i=0,1.$$

By using the same method of (26), we have

$$I_2 = \tfrac{h}{120}(c_0+26c_1+66c_2+26c_3+c_4)$$
$$=\tfrac{h}{120}\begin{pmatrix}0\\0\\1\\26\\66\\26\\1\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}1&76&230&76&1&0&0\\0&1&76&230&76&1&0\\0&0&1&76&230&76&1\\1&26&66&26&1&0&0\\0&1&26&66&26&1&0\\1&11&11&1&0&0&0\\0&1&11&11&1&0&0\end{pmatrix}^{-1}\begin{pmatrix}384s(\tau_0)\\384s(\tau_1)\\384s(\tau_2)\\\frac{120}{h}I_0\\\frac{120}{h}I_1\\24s(x_0)\\24s(x_1)\end{pmatrix}$$
$$=\frac{h}{120}\begin{pmatrix}\frac{555}{7}\\\frac{170}{7}\\1\\-\frac{2473}{7}\\-\frac{520}{7}\\274\\\frac{1370}{7}\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}384s(\tau_0)\\384s(\tau_1)\\384s(\tau_2)\\\frac{120}{h}I_0\\\frac{120}{h}I_1\\24s(x_0)\\24s(x_1)\end{pmatrix}.$$

Rearranging the terms and their coefficients, we can get (27) without any difficulty. The others can be obtained similarly. □

In the following, the inherent relations between four boundary I_j, four boundary $s''(\tau_j)$, two boundary $s(x_j)$, and the inherent relations between four boundary I_j, four boundary $s''''(\tau_j)$, two boundary $s(x_j)$ of the quartic integro-spline are given below. These relations can be proved by using the same methods of Lemma 1 and Lemma 2.

Lemma 3

$$143s''(\tau_0)+74s''(\tau_1)+3s''(\tau_2) = -\tfrac{1005}{h^3}I_0+\tfrac{120}{h^3}I_1+\tfrac{45}{h^3}I_2+\tfrac{630}{h^2}s(x_0)+\tfrac{210}{h^2}s(x_1), \quad (32)$$

$$95s''(\tau_0)+9s''(\tau_1)-23s''(\tau_2)-s''(\tau_3) = -\tfrac{675}{h^3}I_0+\tfrac{285}{h^3}I_1-\tfrac{15}{h^3}I_2-\tfrac{15}{h^3}I_3+\tfrac{420}{h^2}s(x_0), \quad (33)$$

$$-s''(\tau_{n-4})-23s''(\tau_{n-3})+9s''(\tau_{n-2})+95s''(\tau_{n-1}) =-\tfrac{675}{h^3}I_{n-1}+\tfrac{285}{h^3}I_{n-2}-\tfrac{15}{h^3}I_{n-3}-\tfrac{15}{h^3}I_{n-4}+\tfrac{420}{h^2}s(x_n), \quad (34)$$

$$3s''(\tau_{n-3})+74s''(\tau_{n-2})+143s''(\tau_{n-1}) = -\tfrac{1005}{h^3}I_{n-1} + \tfrac{120}{h^3}I_{n-2}+\tfrac{45}{h^3}I_{n-3}+\tfrac{630}{h^2}s(x_n)+\tfrac{210}{h^2}s(x_{n-1}). \quad (35)$$

Lemma 4

$$35s''''(\tau_0)+173s''''(\tau_1)+77s''''(\tau_2)+3s''''(\tau_3) =-\tfrac{3000}{h^5}I_0+\tfrac{2760}{h^5}I_1-\tfrac{1560}{h^5}I_2+\tfrac{360}{h^5}I_3+\tfrac{1440}{h^4}s(x_0), \quad (36)$$

$$13s''''(\tau_0)+22s''''(\tau_1)+s''''(\tau_2) = -\tfrac{2040}{h^5}I_0 - \tfrac{960}{h^5}I_1+\tfrac{120}{h^5}I_2+\tfrac{720}{h^4}s(x_0)+\tfrac{2160}{h^4}s(x_1), \quad (37)$$

$$s''''(\tau_{n-3})+22s''''(\tau_{n-2})+13s''''(\tau_{n-1}) = -\tfrac{2040}{h^5}I_{n-1} - \tfrac{960}{h^5}I_{n-2}+\tfrac{120}{h^5}I_{n-3}+\tfrac{720}{h^4}s(x_n)+\tfrac{2160}{h^4}s(x_{n-1}), \quad (38)$$

$$3s''''(\tau_{n-4})+77s''''(\tau_{n-3})+173s''''(\tau_{n-2})+35s''''(\tau_{n-1}) = -\tfrac{3000}{h^5}I_{n-1}+\tfrac{2760}{h^5}I_{n-2}-\tfrac{1560}{h^5}I_{n-3} + \tfrac{360}{h^5}I_{n-4}+\tfrac{1440}{h^4}s(x_n). \quad (39)$$

SUPER CONVERGENCE AT THE MID-KNOTS

Let $s = s(x)$ be the quartic integro-spline satisfying (2) and (3), (4), (5), (6). For $j = 0,1,\dots,n-1$, let $e^{(k)}(\tau_j) = s^{(k)}(\tau_j) - y^{(k)}(\tau_j)$ $(k = 0,1,2,3,4)$ be the errors. From Lemma 1, we have the following results.

Lemma 5 *For $j = 2,3,\dots,n-3$, we have*

$$e^{(k)}(\tau_{j-2})+26e^{(k)}(\tau_{j-1})+66e^{(k)}(\tau_j)+26e^{(k)}(\tau_{j+1}) + e^{(k)}(\tau_{j+2}) = O(h^{6-k}), \qquad k = 0,2,4; \quad (40)$$

$$e^{(k)}(\tau_{j-2})+26e^{(k)}(\tau_{j-1})+66e^{(k)}(\tau_j)+26e^{(k)}(\tau_{j+1}) + e^{(k)}(\tau_{j+2}) = O(h^{5-k}), \qquad k = 1,3. \quad (41)$$

Proof: By using Taylor formula, for $j = 0,1,\dots,n-1$, we have

$$I_j = \int_{x_j}^{x_{j+1}} y(x)\,\mathrm{d}x$$
$$=\int_{x_j}^{x_{j+1}}\Big[y(\tau_j)+y'(\tau_j)(x-\tau_j)+\tfrac{y''(\tau_j)}{2!}(x-\tau_j)^2+\tfrac{y'''(\tau_j)}{3!}(x-\tau_j)^3 + \tfrac{y''''(\tau_j)}{4!}(x-\tau_j)^4+\tfrac{y^{(5)}(\tau_j)}{5!}(x-\tau_j)^5+\tfrac{y^{(6)}(\xi_j)}{6!}(x-\tau_j)^6\Big]\mathrm{d}x$$
$$=y(\tau_j)h+\tfrac{1}{24}y''(\tau_j)h^3+\tfrac{1}{1920}y''''(\tau_j)h^5+O(h^7). \quad (42)$$

Similarly, for $j = 1,2,\dots,n-2$, we have

$$\sum_{i=j-1}^{j+1} I_i = 3y(\tau_j)h+\tfrac{9}{8}y''(\tau_j)h^3 + \tfrac{81}{640}y''''(\tau_j)h^5+O(h^7), \quad (43)$$

and for $j = 2,3,\dots,n-3$, we have

$$\sum_{i=j-2}^{j+2} I_i = 5y(\tau_j)h+\tfrac{125}{24}y''(\tau_j)h^3 + \tfrac{625}{384}y''''(\tau_j)h^5+O(h^7). \quad (44)$$

From (20), for $j = 2,3,\dots,n-3$, by using (42), (43) and (44), we have

$$s(\tau_{j-2})+26s(\tau_{j-1})+66s(\tau_j)+26s(\tau_{j+1})+s(\tau_{j+2}) = \tfrac{5}{16h}(I_{j-2}+76I_{j-1}+230I_j+76I_{j+1}+I_{j+2})$$
$$= \frac{5}{16h}\Big(\sum_{i=j-2}^{j+2} I_i+75\sum_{i=j-1}^{j+1} I_i+154I_j\Big)$$
$$= 120y(\tau_j)+30y''(\tau_j)h^2+\tfrac{7}{2}y''''(\tau_j)h^4+O(h^6). \quad (45)$$

At the same time, for $j = 2, 3, \ldots, n-3$, we have

$$y(\tau_{j-2}) + 26y(\tau_{j-1}) + 66y(\tau_j) + 26y(\tau_{j+1}) + y(\tau_{j+2}) \\ = 120y(\tau_j) + 30y''(\tau_j)h^2 + \tfrac{7}{2}y''''(\tau_j)h^4 + O(h^6). \quad (46)$$

Hence, by using (45) and (46), we have

$$e(\tau_{j-2}) + 26e(\tau_{j-1}) + 66e(\tau_j) + 26e(\tau_{j+1}) + e(\tau_{j+2}) \\ = (y(\tau_{j-2}) + 26y(\tau_{j-1}) + 66y(\tau_j) + 26y(\tau_{j+1}) + y(\tau_{j+2})) \\ - (s(\tau_{j-2}) + 26s(\tau_{j-1}) + 66s(\tau_j) + 26s(\tau_{j+1}) + s(\tau_{j+2})) \\ = O(h^6).$$

Thus, the case of $k = 0$ is proved. The other cases can be proved by using the same manner. □

Eq. (41) implies that the quartic integro-spline has no super convergence in the first-order derivatives approximation and the third-order derivatives approximation at the mid-knots. In fact, by using (41) and the results in [8], we can only have

$$e^{(k)}(\tau_j) = s^{(k)}(\tau_j) - y^{(k)}(\tau_j) = O(h^{5-k}), \ k = 1, 3. \quad (47)$$

Obviously, these two convergence orders are ordinary for a quartic spline.

Based on Lemma 2, we get the next lemma.

Lemma 6

$$111e(\tau_0) + 34e(\tau_1) + \tfrac{7}{5}e(\tau_2) = O(h^6), \quad (48)$$
$$31e(\tau_0) - 31e(\tau_1) - \tfrac{123}{5}e(\tau_2) - e(\tau_3) = O(h^6), \quad (49)$$
$$-e(\tau_{n-4}) - \tfrac{123}{5}e(\tau_{n-3}) - 31e(\tau_{n-2}) + 31e(\tau_{n-1}) = O(h^6), \quad (50)$$
$$\tfrac{7}{5}e(\tau_{n-3}) + 34e(\tau_{n-2}) + 111e(\tau_{n-1}) = O(h^6). \quad (51)$$

Proof: Identity (49) can be proved as follows. From (28), since $s(x_0) = y(x_0)$, we get

$$31e(\tau_0) - 31e(\tau_1) - \tfrac{123}{5}e(\tau_2) - e(\tau_3) \\ = \tfrac{83}{h}I_0 - \tfrac{63}{4h}\sum_{i=0}^{1} I_i - \tfrac{189}{8h}\sum_{i=0}^{2} I_i - \tfrac{5}{16h}\sum_{i=0}^{3} I_i - \tfrac{137}{20}y(x_0) \quad (52) \\ - (31y(\tau_0) - 31y(\tau_1) - \tfrac{123}{5}y(\tau_2) - y(\tau_3)) \quad (53) \\ = O(h^6), \quad (54)$$

where the $O(h^6)$ term in (54) can be obtained by substituting the following results

$$\sum_{l=0}^{m-1} I_l = \sum_{i=0}^{5} \frac{(mh)^{i+1}}{(i+1)!} y^{(i)}(x_0) + O(h^7) \qquad (m = 1, 2, 3, 4)$$

into (52) and expanding the term of (53) at x_0 by using the Taylor formula. (48), (50) and (51) can be proved similarly. □

Besides, based on Lemma 3 and Lemma 4, we also have the following two lemmas. The proofs are omitted.

Lemma 7

$$143e''(\tau_0) + 74e''(\tau_1) + 3e''(\tau_2) = O(h^4), \quad (55)$$
$$95e''(\tau_0) + 9e''(\tau_1) - 23e''(\tau_2) - e''(\tau_3) = O(h^4), \quad (56)$$
$$-s''(\tau_{n-4}) - 23s''(\tau_{n-3}) + 9s''(\tau_{n-2}) + 95s''(\tau_{n-1}) = O(h^4), \quad (57)$$
$$3e''(\tau_{n-3}) + 74e''(\tau_{n-2}) + 143e''(\tau_{n-1}) = O(h^4). \quad (58)$$

Lemma 8

$$35e''''(\tau_0) + 173e''''(\tau_1) + 77e''''(\tau_2) + 3e''''(\tau_3) = O(h^2), \quad (59)$$
$$13e''''(\tau_0) + 22e''''(\tau_1) + e''''(\tau_2) = O(h^2), \quad (60)$$
$$e''''(\tau_{n-3}) + 22e''''(\tau_{n-2}) + 13e''''(\tau_{n-1}) = O(h^2), \quad (61)$$
$$3e''''(\tau_{n-4}) + 77e''''(\tau_{n-3}) + 173e''''(\tau_{n-2}) + 35e''''(\tau_{n-1}) \\ = O(h^2). \quad (62)$$

In the following, the super convergence of quartic integro-spline at the mid-knots will be presented.

Theorem 1 *Assume that $y(x) \in C^6[a, b]$ and $s = s(x)$ be the quartic integro-spline determined by (2) and (3), (4), (5), (6). For $j = 0, 1, \ldots, n-1$, we have*

$$e^{(k)}(\tau_j) = s^{(k)}(\tau_j) - y^{(k)}(\tau_j) = O(h^{6-k}), \ k = 0, 2, 4. \quad (63)$$

Proof: We first prove the case of $k = 0$. We add (48) multiplied by $-31/111$ to (49) and add (51) multiplied by $-31/111$ to (50). We obtain

$$-\tfrac{4495}{111}e(\tau_1) - \tfrac{2774}{111}e(\tau_2) - e(\tau_3) = O(h^6), \quad (64)$$
$$-e(\tau_{n-4}) - \tfrac{2774}{111}e(\tau_{n-3}) - \tfrac{4495}{111}e(\tau_{n-2}) = O(h^6). \quad (65)$$

By using (40), (48), (51), (64) and (65), we have

$$\begin{pmatrix} 111 & 34 & \frac{7}{5} & & & & \\ 0 & -\frac{4495}{111} & -\frac{2774}{111} & -1 & & & \\ 1 & 26 & 66 & 26 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & 26 & 66 & 26 & 1 \\ & & & -1 & -\frac{2774}{111} & -\frac{4495}{111} & 0 \\ & & & & \frac{7}{5} & 34 & 111 \end{pmatrix} \times \begin{pmatrix} e(\tau_0) \\ e(\tau_1) \\ e(\tau_2) \\ \vdots \\ e(\tau_{n-3}) \\ e(\tau_{n-2}) \\ e(\tau_{n-1}) \end{pmatrix} = \begin{pmatrix} O(h^6) \\ O(h^6) \\ O(h^6) \\ \vdots \\ O(h^6) \\ O(h^6) \\ O(h^6) \end{pmatrix}.$$

The coefficient matrix is strictly diagonally dominant. It implies that the infinite norm of its inverse matrix is bounded. In fact, the infinite norm of its inverse matrix is independent on n and is less than $1/12$. So we get (63) for $k = 0$.

Next, we are aimed to prove (63) for $k = 2$. Add (55) multiplied by $-95/143$ to (56) and also add (58) multiplied by $-95/143$ to (57), we obtain

$$-\tfrac{5743}{143}e''(\tau_1)-\tfrac{3574}{143}e''(\tau_2)-e''(\tau_3)=O(h^4), \quad (66)$$
$$-s''(\tau_{n-4})-\tfrac{3574}{143}s''(\tau_{n-3})-\tfrac{5743}{143}s''(\tau_{n-2})=O(h^4). \quad (67)$$

By using (40), (55), (58), (66) and (67), we have

$$\begin{pmatrix} 143 & 74 & 3 & & & & \\ 0 & -\frac{5743}{143} & -\frac{3574}{143} & -1 & & & \\ 1 & 26 & 66 & 26 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & 26 & 66 & 26 & 1 \\ & & & -1 & -\frac{3574}{143} & -\frac{5743}{143} & 0 \\ & & & & 3 & 74 & 143 \end{pmatrix} \times \begin{pmatrix} e''(\tau_0) \\ e''(\tau_1) \\ e''(\tau_2) \\ \vdots \\ e''(\tau_{n-3}) \\ e''(\tau_{n-2}) \\ e''(\tau_{n-1}) \end{pmatrix} = \begin{pmatrix} O(h^4) \\ O(h^4) \\ O(h^4) \\ \vdots \\ O(h^4) \\ O(h^4) \\ O(h^4) \end{pmatrix}.$$

The coefficient matrix is also strictly diagonally dominant. The infinite norm of its inverse matrix also is independent on n and is less than $1/12$. So we get (63) for $k=2$.

Similarly, by using (40) and Lemma 8, we have

$$\begin{pmatrix} 35 & 173 & 77 & 3 & & & \\ 13 & 22 & 1 & 0 & & & \\ 1 & 26 & 66 & 26 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & 26 & 66 & 26 & 1 \\ & & & 0 & 1 & 22 & 13 \\ & & & 3 & 77 & 173 & 35 \end{pmatrix} \times \begin{pmatrix} e''''(\tau_0) \\ e''''(\tau_1) \\ e''''(\tau_2) \\ \vdots \\ e''''(\tau_{n-3}) \\ e''''(\tau_{n-2}) \\ e''''(\tau_{n-1}) \end{pmatrix} = \begin{pmatrix} O(h^2) \\ O(h^2) \\ O(h^2) \\ \vdots \\ O(h^2) \\ O(h^2) \\ O(h^2) \end{pmatrix}.$$

We add the first equation multiplied by $-13/35$ to the second equation and add the n-th equation multiplied by $-13/35$ to the $(n-1)$-th equation. Moreover, we add the first equation multiplied by $-1/35$ to the third equation and add the n-th equation multiplied by $-1/35$ to the $(n-2)$-th equation. Not considering the first equation (59) and the last equation (62) temporarily, we get

$$\begin{pmatrix} -\frac{1479}{35} & -\frac{138}{5} & -\frac{39}{35} & & & & \\ \frac{737}{35} & \frac{319}{5} & \frac{907}{35} & 1 & & & \\ 1 & 26 & 66 & 26 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & 26 & 66 & 26 & 1 \\ & & & 1 & \frac{907}{35} & \frac{319}{5} & \frac{737}{35} \\ & & & & -\frac{39}{35} & -\frac{138}{5} & -\frac{1479}{35} \end{pmatrix} \times \begin{pmatrix} e''''(\tau_1) \\ e''''(\tau_2) \\ e''''(\tau_3) \\ \vdots \\ e''''(\tau_{n-4}) \\ e''''(\tau_{n-3}) \\ e''''(\tau_{n-2}) \end{pmatrix} = \begin{pmatrix} O(h^2) \\ O(h^2) \\ O(h^2) \\ \vdots \\ O(h^2) \\ O(h^2) \\ O(h^2) \end{pmatrix}.$$

Obviously, its coefficient matrix is also strictly diagonally dominant. The infinite norm of its inverse matrix also is independent on n and is less than $1/12$. Hence, we get

$$e''''(\tau_j)=O(h^2), \qquad j=1,2,\dots,n-2.$$

At this stage, we reconsider the first equation (59) and the last equation (62). It is easy to check that

$$e''''(\tau_0)=\tfrac{1}{35}\big[O(h^2)-173e''''(\tau_1)-77e''''(\tau_2)-3e''''(\tau_3)\big]$$
$$=O(h^2).$$

Similarly, we can get $e''''(\tau_{n-1})=O(h^2)$. So we finish the proof of (63) for $k=4$. □

In [8], it was proved that the super convergence (7) at the knots still hold even if the exact boundary function values $y(x_0)$, $y(x_1)$, $y(x_{n-1})$, $y(x_n)$ in (3), (4), (5) and (6) are replaced respectively by the approximate boundary function values $\widetilde{y}(x_0)$, $\widetilde{y}(x_1)$, $\widetilde{y}(x_{n-1})$, $\widetilde{y}(x_n)$ given in (8), (9), (10) and (11).

The next theorem guarantees that the super convergence (63) at the mid-knots also still holds when four approximate boundary function values are used.

Theorem 2 *Assume that $y(x)\in C^6[a,b]$ and $s=s(x)$ be the quartic integro-spline determined by* (2) *and* (3)*,* (4)*,* (5)*,* (6) *with the approximate values $\widetilde{y}(x_0)$, $\widetilde{y}(x_1)$, $\widetilde{y}(x_{n-1})$, $\widetilde{y}(x_n)$ given in* (8)*,* (9)*,* (10) *and* (11)*. For $j=0,1,\dots,n-1$, we still have* (63)*.*

Proof: First, we point out that the presented relations in Lemma 1, Lemma 2, Lemma 3, Lemma 4 are valid for the quartic integro-spline that is determined by using I_j $(j=0,1,\dots,n-1)$ along with $\widetilde{y}(x_0)$, $\widetilde{y}(x_1)$, $\widetilde{y}(x_{n-1})$, $\widetilde{y}(x_n)$.

Second, we remark that Lemma 5, Lemma 6, Lemma 7 and Lemma 8 still hold. To show it, we take (49) of Lemma 6 as an example. From (8), we have

$$\widetilde{y}(x_0)=y(x_0)+O(h^6). \qquad (68)$$

Table 2 The MAEs of the quartic integro-splines of y_1 and y_2 (exact boundary conditions).

	$E_0(y_1,n)$	$E_2(y_1,n)$	$E_4(y_1,n)$	$E_0(y_2,n)$	$E_2(y_2,n)$	$E_4(y_2,n)$
$n=10$	4.826×10^{-3}	3.715×10^{-0}	$2.305\times10^{+3}$	1.100×10^{-2}	7.654×10^{-0}	$6.562\times10^{+3}$
$n=20$	2.424×10^{-4}	6.836×10^{-1}	$7.676\times10^{+2}$	1.319×10^{-4}	3.749×10^{-1}	$7.919\times10^{+2}$
$n=40$	4.952×10^{-6}	5.591×10^{-2}	$2.356\times10^{+2}$	2.051×10^{-6}	2.345×10^{-2}	$2.013\times10^{+2}$
$n=80$	9.827×10^{-8}	4.075×10^{-3}	$7.262\times10^{+1}$	3.120×10^{-8}	1.431×10^{-3}	$5.713\times10^{+1}$
$n=160$	1.422×10^{-9}	2.607×10^{-4}	$1.894\times10^{+1}$	4.855×10^{-10}	8.911×10^{-5}	$1.501\times10^{+1}$
$n=320$	2.233×10^{-11}	1.638×10^{-5}	4.785×10^{-0}	7.604×10^{-12}	5.566×10^{-6}	3.835×10^{-0}
$n=640$	3.691×10^{-13}	1.041×10^{-6}	1.214×10^{-0}	1.840×10^{-13}	3.761×10^{-7}	9.718×10^{-1}

Table 3 The NCOs of the quartic integro-splines of y_1 and y_2 (exact boundary conditions).

	$O_0(y_1,n_1\to n_2)$	$O_2(y_1,n_1\to n_2)$	$O_4(y_1,n_1\to n_2)$	$O_0(y_2,n_1\to n_2)$	$O_2(y_2,n_1\to n_2)$	$O_4(y_2,n_1\to n_2)$
$10\to20$	4.3	2.4	1.6	6.3	4.3	3.0
$20\to40$	5.6	3.6	1.7	6.0	4.0	2.0
$40\to80$	5.8	3.8	1.7	6.0	4.0	1.8
$80\to160$	6.0	4.0	1.9	6.0	4.0	1.9
$160\to320$	6.0	4.0	2.0	6.0	4.0	2.0
$320\to640$	5.9	4.0	2.0	5.4	3.9	2.0

By using (28), (52), (53), (54), (68) and noticing $s(x_0)=\widetilde{y}(x_0)$, we get

$$\begin{aligned}
&31e(\tau_0)-31e(\tau_1)-\tfrac{123}{5}e(\tau_2)-e(\tau_3)\\
&=(\tfrac{83}{h}I_0-\tfrac{63}{4h}\sum_{i=0}^{1}I_i-\tfrac{189}{8h}\sum_{i=0}^{2}I_i-\tfrac{5}{16h}\sum_{i=0}^{3}I_i-\tfrac{137}{20}\widetilde{y}(x_0))\\
&\quad-(31y(\tau_0)-31y(\tau_1)-\tfrac{123}{5}y(\tau_2)-y(\tau_3))\\
&=O(h^6)-\tfrac{137}{20}O(h^6)\\
&=O(h^6).
\end{aligned}$$

So, we get (49) of Lemma 6 for the quartic integro-spline that is determined by using I_j $(j=0,1,\dots,n-1)$ along with $\widetilde{y}(x_0)$, $\widetilde{y}(x_1)$, $\widetilde{y}(x_{n-1})$, $\widetilde{y}(x_n)$. The others of Lemma 6, Lemma 7 and Lemma 8 can be proved similarly by using $s(x_1)=\widetilde{y}(x_1)$, $s(x_{n-1})=\widetilde{y}(x_{n-1})$, $s(x_n)=\widetilde{y}(x_n)$ and

$$\begin{aligned}
\widetilde{y}(x_1)&=y(x_1)+O(h^6),\\
\widetilde{y}(x_{n-1})&=y(x_{n-1})+O(h^6),\\
\widetilde{y}(x_n)&=y(x_n)+O(h^6),
\end{aligned}$$

which can be derived from (9), (10) and (11).

Because all the needed lemmas remain be valid, we conclude that the super convergence properties (63) still hold when the approximate values $\widetilde{y}(x_0)$, $\widetilde{y}(x_1)$, $\widetilde{y}(x_{n-1})$ and $\widetilde{y}(x_n)$ are used. □

In a word, Theorem 1 and Theorem 2 show that the quartic integro-spline possesses the super convergence properties (63) at the mid-knots, no matter exact boundary conditions are used or approximate boundary conditions are used.

NUMERICAL TESTS

In this section, we are aimed to perform some numerical tests by Matlab to verify the super convergence properties (63).

For a test function $y=y(x)$, let $s=s(x)$ be the quartic integro-spline. At the mid-knots, we define three maximum absolute errors (MAEs) as

$$E_k(y,n)=\max_{0\leqslant j\leqslant n-1}|e^{(k)}(\tau_j)|,\quad k=0,2,4.$$

At the same time, we define three numerical convergence orders (NCOs) of the maximum absolute errors as

$$O_k(y,n_1\to n_2)=\frac{\log(E_k(y,n_1)/E_k(y,n_2))}{\log(n_2/n_1)},\ k=0,2,4.$$

The tested functions are $y_1=1/(1+16x^2)$ and $y_2=\cos(10x+1)$, the interval is $[a,b]=[-1,1]$.

We first test the convergence with four exact function values as exact boundary conditions. See Table 2 and Table 3 for the MAEs and the NCOs of the quartic integro-splines of y_1 and y_2. From Table 2, as the step length h becoming its one half, it can be found that the decrease rates of $E_0(y,n)$, $E_2(y,n)$ and $E_4(y,n)$ are about 1/64, 1/16 and 1/4, respectively. It shows $E_0(y,n)=O(h^6)$, $E_2(y,n)=O(h^4)$ and $E_4(y,n)=O(h^2)$. The numerical convergence orders listed in Table 3 are approximately equal to the theoretical ones.

Next, we continue to do some tests with four approximate function values as approximate boundary conditions. See Table 4 and Table 5 for the results. These results are also in accord with the super convergence properties (63). The numerical convergence orders are also approximately equal to the theoretical ones even if approximate boundary conditions are used.

In a word, the super convergence properties (63) have been numerically confirmed.

Table 4 The MAEs of the quartic integro-splines of y_1 and y_2 (approximate boundary conditions).

	$E_0(y_1,n)$	$E_2(y_1,n)$	$E_4(y_1,n)$	$E_0(y_2,n)$	$E_2(y_2,n)$	$E_4(y_2,n)$
$n=50$	1.408×10^{-6}	2.495×10^{-2}	$1.681\times10^{+2}$	9.657×10^{-5}	1.387×10^{-0}	$3.060\times10^{+3}$
$n=100$	2.364×10^{-8}	1.689×10^{-3}	$4.745\times10^{+1}$	1.049×10^{-6}	5.964×10^{-2}	$6.619\times10^{+2}$
$n=200$	3.736×10^{-10}	1.071×10^{-4}	$1.218\times10^{+1}$	2.191×10^{-8}	5.001×10^{-3}	$2.063\times10^{+2}$
$n=300$	3.289×10^{-11}	2.120×10^{-5}	5.441×10^{-0}	2.057×10^{-9}	1.057×10^{-3}	$9.672\times10^{+1}$
$n=400$	5.883×10^{-12}	6.721×10^{-6}	3.068×10^{-0}	3.768×10^{-10}	3.443×10^{-4}	$5.570\times10^{+1}$
$n=500$	1.535×10^{-12}	2.751×10^{-6}	1.963×10^{-0}	1.004×10^{-10}	1.433×10^{-4}	$3.612\times10^{+1}$
$n=600$	5.386×10^{-13}	1.329×10^{-6}	1.362×10^{-0}	3.397×10^{-11}	6.984×10^{-5}	$2.530\times10^{+1}$

Table 5 The NCOs of the quartic integro-splines of y_1 and y_2 (approximate boundary conditions).

	$O_0(y_1,n_1\to n_2)$	$O_2(y_1,n_1\to n_2)$	$O_4(y_1,n_1\to n_2)$	$O_0(y_2,n_1\to n_2)$	$O_2(y_2,n_1\to n_2)$	$O_4(y_2,n_1\to n_2)$
$50\to100$	5.9	3.9	1.8	6.5	4.5	2.2
$100\to200$	6.0	4.0	2.0	5.6	3.6	1.9
$200\to300$	6.0	4.0	2.0	5.8	3.8	1.9
$300\to400$	6.0	4.0	2.0	5.9	3.9	1.9
$400\to500$	6.0	4.0	2.0	5.9	3.9	1.9
$500\to600$	5.7	4.0	2.0	5.9	3.9	2.0

CONCLUSION

In this paper, we have mainly studied some new super convergence of a quartic integro-spline. At the mid-knots, the function values approximation, the second-order derivatives approximation and the fourth-order derivatives approximation of the quartic integro-spline are sixth order convergent, fourth order convergent and second order convergent, respectively. These convergence orders are all one order higher than the ordinary cases of a quartic spline. These new super convergence are also valuable for the quartic integro-spline. We conclude that the quartic integro-spline has two super convergence properties (7) at the knots, and three super convergence properties (63) at the mid-knots. In the future, some other related problems will be considered. Especially, motivated by a reviewer of this paper, we will first investigate some super convergence of the quartic integro-spline at some other special points, such as the Gauss points (see [23, 24] for examples).

Acknowledgements: This work was supported by the National Natural Science Foundation of China (Grant No. 11501533).

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