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# A note on the recognition of $PSL_2(p)$

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**ABSTRACT**: It is proved in this note that a finite group $G$ is isomorphic to the projective special linear group $PSL_2(p)$ or a affine linear group $A\Gamma L_1(8)$ if and only if $|G| = p(p^2-1)/(2,p-1)$ and $(p^2-1)/(2,p-1) \in N(G)$, where $p$ is a prime and $N(G) = \{|x^G| : x \in G\}$.

**KEYWORDS**: group order, conjugacy class length, projective special linear group

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## INTRODUCTION

In recent years, there is an extensive research interest in characterizing non-abelian simple groups by their certain arithmetical properties, such as group order and element order, conjugacy class lengths, group order and degrees of vertices of prime graph. In this note, we concentrate on the order and a speical conjugacy class length of group, which is associated with Thompson's conjecture [1]: *Suppose that $G$ is a finite centerless group and $L$ is a finite non-abelian simple group. Then $G$ is isomorphic to $L$ if $N(G) = N(L)$.*

There has been significant progress in the study of this conjecture [2–4]. Especially, Gorshkov [4] claimed that he had proved Thompson's conjecture. Therefore, it is natural to investigate some problems beyond this conjecture. For example, Li [5] try to characterize non-abelian simple groups by some special conjugacy class lengthes together with the orders of groups. Some families of simple groups had been determined in this way, including sporadic simple groups, simple $K_3$-groups, simple $K_4$-groups, and projective special linear groups $PSL_4(4)$ and $PSL_2(p)$, see [6–8] and references therein. Meanwhile, those simple groups were proved for Thompson's conjecture.

It should be noted that the proof processes of those results above are all dependent on the Classification Theorem of Finite Simple Groups (CFSG) so that too much tedious verification case by case is involved. In this note, by virtue of [9], we avoid using the CFSG to obtain a recognition of $PSL_2(p)$, and greatly simplified the proof process of Theorem 3.2 in [6].

For a finite group $G$, $\pi(G)$ is the set of prime divisors of $|G|$, and $\Gamma(G)$ denotes the *prime graph* of $G$, whose vertices' set is $\pi(G)$ and any two vertices $p$ and $q$ have an edge if and only if $G$ has an element of order $pq$ [10]. Further, $T(G)$ is the set of connected components of $\Gamma(G)$, and $\pi_i(G)$ is the vertices' set of its $i$-th connected component. It follows that $|G|$ is able to be decomposed as a product of $m_1, m_2, \ldots, m_{|T(G)|}$, where the set of prime divisors of $m_i$ is equal to $\pi_i(G)$. Especially, we call these $m_i$'s the *order components* of $G$, and we usually let $\mathrm{OC}(G) = \{m_1, m_2, \ldots, m_{|T(G)|}\}$, and $2 \in \pi_1(G)$ when $2\,\big|\,|G|$. Other notation and terminologies are standard and can be found in [11, 12].

## LEMMAS

In this section, Lemmas 1–3 are taken from [13, 14] and [10], respectively.

**Lemma 1** *Suppose that $G$ is a Frobenius group of even order such that $M$ is its kernel and $N$ is a complement of $M$ in $G$. Then the set $T(G)$ is equal to $\{\pi(M), \pi(N)\}$.*

**Lemma 2** *Assume that $G$ is a 2-Frobenius group of even order such that there is a normal series $1 \leqslant M \leqslant N \leqslant G$, where $N$ and $G/M$ are two Frobenius groups, and $M$ and $N/M$ are their kernels respectively. Then the set $T(G)$ is equal to $\{\pi(M), \pi(G/N) \cup \pi(N/M)\}$, and $|G/N|$ is a factor of $|Out(M/N)|$. Moreover, if $M = Z_p \times Z_p \times \cdots \times Z_p$ and $|M| = p^n$, and $N/M$ is cyclic and $|N/M| = p^n - 1$, then $|G/N|\,\big|\,n$.*

**Lemma 3** *Let $G$ be a finite group with $|T(G)| > 1$. Then*
(i) *$G$ is a Frobenius group, or a 2-Frobenius group;*
(ii) *$G$ has a normal series $1 \leqslant M \leqslant N \leqslant G$ satisfying that $M$ is nilpotent, and $N/M$ is a non-communicative simple group, and $\pi(M) \bigcup \pi(G/N)$ is a subset of $\pi_1(G)$.*

The following result of Brauer and Reynolds is also essential in the proof of our main result, by which one can avoid invoking the CFSG (see [9]).

**Lemma 4** *Assume that $G$ is a non-abelian simple group of finite order. If there exists a prime $p$ such that $p\,\big|\,|G|$ and $p > |G|^{1/3}$, then $G$ is one of the followings:*
(i) *$PSL_2(p-1)$, where $p = 2^n + 1 > 3$;*
(ii) *$PSL_2(p)$, where $p > 3$.*

## A RECOGNITION OF $PSL_2(p)$

**Theorem 1** *Assume that $G$ is a finite group and $p$ is a prime number. Then $|G| = p(p^2-1)/(2,p-1)$ and $(p^2-1)/(2,p-1) \in N(G)$ if and only if $G$ is one of the following groups:*

(i) *Frobenius groups: $S_3$ and $A_4$;*

(ii) *Affine linear group: $A\Gamma L_1(8)$, which is a 2-Frobenius group;*

(iii) *Simple groups: $PSL_2(p)$, $p > 3$.*

*Proof*: Firstly, we prove the necessity of the theorem.

If $G \cong S_3$, then $|G| = 6$, and $G$ has an element $v$ with $\circ(v) = 2$ so that its conjugacy class length is 3 in $G$, which implies that $p = 2$.

If $G \cong A_4$, then $|G| = 12$, and $G$ contains an element $v$ with $\circ(v) = 3$ satisfying that its conjugacy class length is 4 in $G$, which implies that $p = 3$.

If $G \cong A\Gamma L_1(8)$, then $G \cong ((Z_2 \times Z_2 \times Z_2) \rtimes Z_7) \rtimes Z_3$, and thus $G$ has an element $v$ with $\circ(v) = 7$ such that $|v^G| = 24$ according to Small Groups of Magma, which means that $p = 7$.

If $G \cong PSL_2(p)$, $p > 3$, then $G$ has order $p(p^2-1)/2$, and $G$ contains an element $v \in G$ with $\circ(v) = p$ satisfying that $|v^G| = (p^2-1)/2$.

Now, we show the sufficiency of the theorem.

Let $p = 2$. Then $G$ is not a cyclic group of order 6 so that $G$ must be the symmetric group $S_3$, which is a Frobenius group, as wanted.

Let $p = 3$. Then $|G| = 12$, and $G$ has no element of order 6. Checking the classification of the groups of order 12, we know that $G$ must be the alternating group $A_4$, which is a Frobenius group, as desired.

Let $p \geqslant 5$. From the conditions of the theorem, there exists an element $v \in G$ with $\circ(v) = p$ satisfying that $\langle v \rangle$ is self-centralized in $G$. Then each subgroup of order $p$ of $G$ is self-centralized by Sylow theorem. So, $\{p\}$ is a connected component of $\Gamma(G)$, which means $|T(G)| > 1$. Therefore, $G$ has one of the structures in Lemma 3. Note that $p = \max(\pi(G))$ and $p \in \mathrm{OC}(G)$.

Assume that $G$ is a finite Frobenius group satisfying that $M$ is its kernel and $T$ is complement of $M$ in $G$. Then $|T| \big| (|M|-1)$. Let $p$ be the factor of $|M|$. By Lemma 1, one can get that $T$ has order $(p^2-1)/2$ and $M$ has order $p$, which means that $p$ is equal to 1, contradicting $p \geqslant 5$. Thus $p$ must be a divisor of $|T|$, and so $|M| = (p^2-1)/2$ and $|T| = p$, which means $p = 3$, still contradicting $p \geqslant 5$. So, it is impossible that $G$ is a Frobenius group of finite order.

If $G$ is a finite 2-Frobenius group, then $G$ has two normal subgroups $M$ and $N$ so that $1 \leqslant M \leqslant N \leqslant G$, where $N$ and $G/M$ are Frobenius groups, and $M$ and $N/M$ are their kernels respectively. By Lemma 2, one can get that the set $\pi_1(G)$ is equal to $\pi(M) \cup \pi(G/N)$ and $\pi_2(G)$ is equal to $\pi(N/M)$. Thus, $\pi((p+1)/2) \subseteq \pi(M)$ and $|N/M| = p$. Suppose that $p$ is not equal to $2^s - 1$, where $s \geqslant 2$. Then there is an odd prime $f$ with $|M_f| < p$, where $M_f$ is a Sylow $f$-subgroup of $M$. Therefore, $p$ and $|\mathrm{Aut}(M_f)|$ are relatively prime, which means that there is an edge between $f$ and $p$ in $\Gamma(G)$, contradicting $\{p\} \in T(G)$. Thus, $p$ must be equal to $2^s - 1$ for some $s \geqslant 2$, and thus $s$ is a prime. Note that $|G/N|$ is a factor of $p-1$ and $p-1 = 2(2^{s-1}-1)$. Next we discuss $G$ according to whether $|G/N|$ is divisible by 2.

If $2 \big| |G/N|$, then $|M_2| = 2^{s-1}$, and thus there exists an element of order $p$ in $G$ that can act freely on $M_2$, which means that $G$ has one element of order $2p$, contradicting $\{p\} \in T(G)$.

If $2 \not\big| |G/N|$, then $|G/N|$ is a factor of $2^{s-1} - 1$ and $M_2$ has order $2^s$. Further, $M_2$ must be an elementary abelian 2-group. Otherwise, $p$ and $|\mathrm{Aut}(M_2)|$ are coprime, which means that $G$ has one element of order $2p$, a contradiction. Thus $M_2$ is elementary. Assume that $M \neq M_2$. Then there is an odd prime number $r$ such that $r$ is a common factor of $|M|$ and $(p-1)/2$, and thus an element of order $p$ in $G$ can act freely on $M_r$. It follows that $r$ and $p$ have an edge in $\Gamma(G)$, contradicting that $\{p\} \in T(G)$. Hence $M = M_2$, and $|G/N| = 2^{s-1} - 1$. By Lemma 2, $|G/N| \big| s$, and so $s \in \{2, 3\}$. Let $s = 2$. Then $|G| = 12$, and by checking the structure of group of order 12, $G$ cannot be a finite 2-Frobenius group, contradicting the assumption. Thus $s = 3$, and so $p = 7$. Thus, $|G| = 168$, and $G$ has eight subgroups of order 7. By virtue of [15], $G$ is one of the following groups: $A\Gamma L_1(8)$, $AGL_1(8) \times Z_3$, or $PSL_2(7)$. Just that $G$ is a 2-Frobenius group. Thus, $G$ is the group $A\Gamma L_1(8)$ as wanted.

Therefore, $G$ is isomorphic to the 2-Frobenius group: $A\Gamma L_1(8)$.

Now, by the preceding arguments and Lemma 3, one can obtain that $G$ has a series $1 \trianglelefteq M \trianglelefteq N \trianglelefteq G$ satisfying that $M$ is nilpotent, and $N/M$ is a non-communicative simple group, and $\pi(M) \bigcup \pi(G/N) \subseteq \pi_1(G)$. Further, $N/M \leqslant G/M \leqslant \mathrm{Aut}(N/M)$, and $|G/N| \big| |\mathrm{Out}(N/M)|$, and $p \in \mathrm{OC}(N/M)$. Because of $|N/M| \leqslant |G| = p(p^2-1)/2$, one has that $|N/M|^{1/3} < p$, and by Lemma 4, $N/M$ is isomorphic to $PSL_2(p)$, where $p > 3$ or $PSL_2(p-1)$, where $p = 2^n + 1 > 3$.

If $N/M \cong PSL_2(p-1)$, where $p = 2^n + 1 > 3$, then $N/M$ has order of $p(p-1)(p-2)$ and $(p-2) \big| (p+1)/2$ by $|N/M| \big| |G|$. Hence $(p+1)/2 = p-2$. Therefore $p = 5$, which implies that $N/M$ is isomorphic to $PSL_2(4)$, and $N/M$ and $G$ have same order 60, which imply that $G \cong PSL_2(4)$. In view of $PSL_2(4) \cong A_5 \cong PSL_2(5)$, we have $G \cong PSL_2(5)$, as desired.

If $N/M \cong PSL_2(p)$, where $p > 3$, then one has $PSL_2(p) \leqslant G/M \leqslant PGL_2(p)$, becuase $N/M \leqslant G/M \leqslant \mathrm{Aut}(N/M)$. It follows that $G/M$ is isomorphic to $PSL_2(p)$ or $PGL_2(p)$. Let $G/M \cong PGL_2(p)$. Then $|M| = 2$, which means that $M \subseteq Z(G)$, contradicting that $G$ has more than one connected branches. Thus, $G/M$ is isomorphic to $PSL_2(p)$, and then $M = 1$. Therefore, $G \cong PSL_2(p)$, where $p$ is a prime bigger than three, as expected. □

The next results are the corollaries of Theorem 1.

**Corollary 1** *Let $G$ be a group of finite order. Then $G$ is one of groups: $PSL_2(7)$ and $A\Gamma L_1(8)$ if and only if $|G| = 2^3 \cdot 3 \cdot 7$ and $24 \in N(G)$.*

**Corollary 2** *Let $G$ be a finite group with $p$ a prime not equal to 7. Then $G \cong PSL_2(p)$ if and only if the order of $G$ is $p(p^2-1)/(2,p-1)$, and $(p^2-1)/(2,p-1)$ is an element of $N(G)$.*

*Proof*: Since $S_3 \cong PSL_2(2)$ and $A_4 \cong PSL_2(3)$, one can easily get the corollary from the proof of Theorem 1. □

**Corollary 3** *The projective special linear groups $PSL_2(p)$ can be characterized by its order components, where $p$ is a prime.*

*Proof*: Let $\mathrm{OC}(G) = \mathrm{OC}(PSL_2(p))$. Then $G$ has order $p(p^2-1)/(2,p-1)$, and $p$ is its one order component. By Theorem 1, $G$ is isomorphic to $PSL_2(p)$ for all primes $p$ except $p = 7$, and $G$ is isomorphic to either $PSL_2(7)$ or $A\Gamma L_1(8)$ when $p = 7$. Since $G$ has three order components when $p = 7$, and by Lemma 2, $A\Gamma L_1(8)$ has two, which implies that $G \ncong A\Gamma L_1(8)$. Therefore, $G \cong PSL_2(7)$ when $p$ is equal to 7. □

**Corollary 4** *Thompson's conjecture is valid for projective special linear groups $PSL_2(p)$ with $p$ a prime.*

*Proof*: Let $G$ be a finite centerless group with $N(G) = N(PSL_2(p))$. By [2], $G$ has the same order components with $PSL_2(p)$. Therefore, the corollary follows directly from Corollary 3. □

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## REFERENCES

1. Khukhro EI, Mazurov VD (2014) *Unsolved Problems in Group Theory: The Kourovka Notebook*, 18th edn, Sobolev Institute of Mathematics, Novosibirsk, Russia. Available at: http://eprints.maths.manchester.ac.uk/2087/1/18tkt.pdf.
2. Chen GY (1996) On Thompson's conjecture. *J Algebra* **185**, 184–193.
3. Ahanjideh N (2011) On Thompson's conjecture for some finite simple groups. *J Algebra* **344**, 205–228.
4. Gorshkov IB (2019) On Thompson's conjecture for finite simple groups. *Comm Algebra* **47**, 5192–5206.
5. Li JB (2012) Finite groups with special conjugacy class sizes or generalized permutable subgroups. PhD thesis, Southwest Univ, Chongqing, China. [in Chinese]
6. Chen YH, Chen GY (2012) Recognizing $L_2(p)$ by its order and one special conjugacy class size. *J Inequal Appl* **2012**, ID 310.
7. Chen YH, Chen GY (2012) Recognization of $A_{10}$ and $L_4(4)$ by two special conjugacy class size. *Ital J Pure Appl Math* **29**, 387–394.
8. Chen YH, Chen GY, Li JB (2015) Recognizing simple $K_4$-groups by few special conjugacy class sizes. *Bull Malays Math Sci Soc* **38**, 51–72.
9. Brauer R, Reynolds WF (1958) On a problem of E. Artin. *Ann Math* **68**, 713–720.
10. Williams JS (1981) Prime graph components of finite groups. *J Algebra* **69**, 487–513.
11. Conway JH, Curtis RT, Norton SP, Parker RA, Wilson RA (1985) *Atlas of Finite Groups*, Clarendon Press, UK.
12. Gorensten D (1980) *Finite Groups*, 2nd edn, Chelesea, New York, NY.
13. Chen GY (1995) On Frobenius and 2-Frobeniusgroup. *J Southwest China Normal Univ* **20**, 485–487. [in Chinese]
14. Zhang LC, Liu XF (2009) Charachterization of the projective general linear group $PGL(2,q)$ by their orders and degree patterns. *Int J Algebra Comput* **19**, 873–889.
15. Miller GA (1902) Determination of all the groups of order 168. *Am Math Mon* **9**, 1–5.