On the meromorphic solutions of generalized q-difference equations

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ABSTRACT: A generalized q-difference analogue of the lemma on the logarithmic derivative is introduced, which is an improvement of previous result due to Korhonen, and was applied to the growth of meromorphic solutions of some linear difference equations. Examples are given to illustrate the sharpness of the results.

KEYWORDS: meromorphic solutions, q-difference equations, growth

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INTRODUCTION

When applying Nevanlinna theory to the growth and value distribution of meromorphic solutions of differential equations in the complex plane, estimates involving logarithmic derivatives have often proved useful, see [1,2]. A difference analogue of the lemma on the logarithmic derivative for finite-order meromorphic functions in $\mathbb{C}$ was obtained independently by Halburd and Korhonen [3,4] and Chiang and Feng [5]. Halburd et al [6] showed that if the hyper-order $\sigma_2(f)$ of f satisfies $\sigma_2(f) = \sigma_2 < 1$ and $\varepsilon > 0$, then

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\sigma_2-\varepsilon}}\right) \qquad (1)$$

for all r outside a set of finite logarithmic measure.

Here it is assumed that the reader is familiar with Nevanlinna theory, see [7,8]. In what follows, the growth order, the type and the exponent of convergence of the poles of a meromorphic function $f(z)$ are represented by $\sigma(f), \tau(f)$ and $\lambda(\frac{1}{f})$, respectively. For $p \in \mathbb{N}$, we define the iterated order, and the iterated type of $f(z)$, respectively, as follows:

$$\sigma_p(f) = \limsup_{r\to\infty} \frac{\log_p T(r,f)}{\log r},$$

$$\tau_p(f) = \limsup_{r\to\infty} \frac{\log_{p-1} T(r,f)}{r^{\sigma_p(f)}},$$

see [9]. In particular, $\sigma_1(f) = \sigma(f)$ is the order,

$\tau_1(f) = \tau(f)$ is the type, and $\sigma_2(f)$ is the hyper-order of f. By a meromorphic function we always mean a function meromorphic in $\mathbb{C}$.

Korhonen [10] obtained a generalization of (1) by proving that if $\omega(z) = cz^n + p_{n-1}z^{n-1} + \cdots + p_0$ and $\varphi(z) = cz^n + q_{n-1}z^{n-1} + \cdots + q_0$ are non-constant polynomials, then for any non-rational meromorphic function f of hyper-order less than $\frac{1}{n^2}$,

$$m\left(r, \frac{f \circ \omega}{f \circ \varphi}\right) = o\big(T(|c|r^n, f)\big) \qquad (2)$$

as $r \to \infty$ outside an exceptional set of finite logarithmic measure.

The purpose of this paper is to improve the estimate (2) by giving an explicit error term.

Theorem 1 *Let f be a non-rational meromorphic function, let $\omega(z) = cz^n + p_{n-1}z^{n-1} + \cdots + p_0$ and $\varphi(z) = cz^n + q_{n-1}z^{n-1} + \cdots + q_0$ be non-constant polynomials. If $\sigma_2(f) = \sigma_2 < \frac{1}{n^2}$ and $\varepsilon > 0$, then*

$$m\left(r, \frac{f \circ \omega}{f \circ \varphi}\right) = o\left(\frac{T(|c|r^n, f)}{r^{\frac{1}{n} - n\sigma_2 - \varepsilon}}\right)$$

for all r outside an exceptional set of finite logarithmic measure.

Remark 1 Theorem 1 is a generalization of Theorem 5.1 in [6] for meromorphic functions composed with polynomials.

Theorem 1 is a useful tool in the growth of finite-order transcendental meromorphic solutions

of functional equations with polynomial compositions. In the remainder of this paper, we will apply it to the growth of meromorphic solutions of linear difference equations in the homogeneous and the non-homogeneous cases, when there is a dominating coefficient.

APPLICATIONS TO FUNCTIONAL EQUATIONS

By the use of recent new difference versions of Nevanlinna theory, many studies [1, 5, 11–20] of the properties of meromorphic solutions of the difference equation

$$a_k(z)f(z+c_k) + \cdots + a_0(z)f(z) = 0, \quad k \in \mathbb{N}, \quad (3)$$

and its special case $c_j = j$, $j = 1, \ldots, k$, where c_j are distinct non-zero complex constants and $a_j(z)$ are meromorphic functions, have appeared during recent years. Chiang and Feng [5] showed that if $a_j(z)(j = 0, 1, \ldots, k)$ are entire functions and if there exists an integer $l \in \{0, \ldots, k\}$ such that $\sigma(a_l) > \max_{0 \leqslant j \leqslant k, j \neq l}\{\sigma(a_j)\}$, then every meromorphic solution $f(\not\equiv 0)$ of (3) with $c_j = j$ satisfies $\sigma(f) \geqslant \sigma(a_l) + 1$. In the case when there is more than one coefficients having the maximal order, Laine and Yang [11] proved that if $a_j(z)(j = 0, 1, \ldots, k)$ are entire functions of finite order such that among those having the maximal order $\sigma = \max_{0 \leqslant j \leqslant k}\{\sigma(a_j)\}$, exactly one has its type strictly greater than the others, then for every meromorphic solution $f(\not\equiv 0)$ of (3), we have $\sigma(f) \geqslant \sigma + 1$.

Liu and Qi [16] investigated the growth of meromorphic solutions of linear q-difference equation

$$a_k(z)f(q_k z + \eta_k) + \cdots + a_0(z)f(q_0 z + \eta_0) = 0, \quad (4)$$

where $\eta_0, \ldots, \eta_k$ are non-zero complex constants.

Assuming that $P_i(z)(i = 0, 1, \ldots, k)$ are polynomials with degree $n(\geqslant 1)$, we consider the equation

$$a_k(z)f(P_k(z)) + \cdots + a_0(z)f(P_0(z)) = 0, \quad (5)$$

which contains (3) and (4) as special cases. When (5) has only one dominating coefficient and all coefficients are entire functions, we obtain the following result.

Theorem 2 Let $P_j(z) = cz^n + c_{j_{n-1}}z^{n-1} + \ldots + c_{j_0}(c \neq 0)$, $j = 0, 1, \ldots, k$, be polynomials with degree $n \geqslant 1$ and let H be a complex set satisfying $\overline{\mathrm{dens}}\{r = |z| : z \in H\} > 0$, and let $a_j(z)$, $j = 0, 1, \ldots, k$ be entire functions satisfying $\max_{0 \leqslant j \leqslant k}\{\sigma(a_j)\} \leqslant \sigma$. If there

exists an integer $l \in \{0, \ldots, k\}$ such that for some constants $0 \leqslant \alpha < \beta$ and $\delta > 0$ sufficiently small,

$$m(r, a_l(z)) \geqslant \beta r^{\sigma-\delta}, \quad m(r, a_j(z)) \leqslant \alpha r^{\sigma-\delta}$$

for all $j = 0, 1, \ldots, k$, $j \neq l$, as $z \to \infty$ for $z \in H$, then every transcendental meromorphic solution $f(z)(\not\equiv 0)$ of (5) satisfies $\sigma(f) \geqslant \sigma(a_l)/n + 1/n^2$.

Note that in Theorem 2, the condition that there exists more than one coefficient having the maximal order and (5) has only one dominating coefficient. When there is only one dominating coefficient and all coefficients are meromorphic functions in (5), we obtain the following result.

Theorem 3 Let $P_j(z) = cz^n + c_{j_{n-1}}z^{n-1} + \ldots + c_{j_0}$ ($c \neq 0$), $j = 0, 1, \ldots, k$, be polynomials with degree $n \geqslant 1$ and $a_j(z)$, $j = 0, 1, \ldots, k$ be meromorphic functions. If there exists an integer $l \in \{0, \ldots, k\}$ such that $a_l(z)$ satisfies

$$\lambda(\tfrac{1}{a_l}) < \sigma(a_l) < \infty,$$
$$\max\{\sigma(a_j) : j = 0, 1, \ldots, k, j \neq l\} \leqslant \sigma(a_l),$$
$$\sum_{\sigma(a_j)=\sigma(a_l), j \neq l} \tau(a_j) < \tau(a_l) < \infty,$$

then every transcendental meromorphic solution $f(z)(\not\equiv 0)$ of (5) satisfies $\sigma(f) \geqslant \sigma(a_l)/n + 1/n^2$.

Note that in Theorem 3, the condition that there exists more than one coefficient having the maximal order and (5) has only one coefficient having the maximal type.

Finally, we consider the case of the non-homogeneous equation

$$a_k(z)f(P_k(z)) + \cdots + a_0(z)f(P_0(z)) = F(z), \quad (6)$$

where $P_i(z)(i = 0, 1, \ldots, k)$ are non-constant polynomials with different leading coefficients.

Theorem 4 Let $P_j(z) = c_{jp_j}z^{p_j} + c_{jp_j-1}z^{p_j-1} + \ldots + c_{j0}(c_{jp_j} \neq 0)$, $j = 0, 1, \ldots, k$, be polynomials with degree $p_j \geqslant 1$, let $a_j(z)$, $j = 0, 1, \ldots, k$, $F(z)$ be meromorphic functions, and let l satisfy the conditions of Theorem 3. We have

(i) if $\sigma(F) < \sigma(a_l)$, or $\sigma(F) = \sigma(a_l)$ and $\sum_{\sigma(a_j)=\sigma(a_l), j \neq l} \tau(a_j) + \tau(F) < \tau(a_l)$, or $\sigma(F) = \sigma(a_l)$ and $\sum_{\sigma(a_j)=\sigma(a_l)} \tau(a_j) < \tau(F)$, then every meromorphic solution $f(z)(\not\equiv 0)$ of (6) satisfies $\sigma(f) \geqslant \dfrac{\sigma(a_l)}{\max\{p_j\}}$;

(ii) *if* $F(z) \equiv 0$, *then every meromorphic solution*
$f(z)(\not\equiv 0)$ *of* (6) *satisfies* $\sigma(f) \geqslant \dfrac{\sigma(a_l)}{\max\{p_j\}}$;

(iii) *if* $\sigma(F) > \sigma(a_l)$, *then every meromorphic solution* $f(z)(\not\equiv 0)$ *of* (6) *satisfies* $\sigma(f) \geqslant \dfrac{\sigma(F)}{\max\{p_j\}}$.

Comparing Theorem 4 to Theorem 2 and Theorem 3, $P_j(z)(j = 0, 1, \ldots, k)$ are non-constant polynomials with different leading coefficients in (6).

EXAMPLES

The following Example 1 is related to the sharpness of Theorem 1.

Example 1 The functions $f(z) = e^z$, $\omega(z) = z^2 + z + 1$ and $\varphi(z) = z^2$ satisfy the hypothesis of Theorem 1. Now $\dfrac{T(r^2, f)}{r^{\frac{1}{2} - \varepsilon}} = \dfrac{r^{\frac{3}{2} + \varepsilon}}{\pi}$ and

$$m\left(r, \frac{f \circ \omega}{f \circ \varphi}\right) = m(r, e^{z+1}) = \frac{r}{\pi} + O(1) = o\left(\frac{T(r^2, f)}{r^{\frac{1}{2} - \varepsilon}}\right).$$

The following Example 2 shows that the strict inequality $\sigma(f) > \sigma(a_l)/n + 1/n^2$ can occur in Theorem 2.

Example 2 For Theorem 2, $f(z) = e^z$ satisfies the difference equation

$$e^{2z} f(3z^2 - 2z + 1) + e^z f(3z^2 - z + 1) - 2f(3z^2 + 1) = 0,$$

where $a_2(z) = e^{2z}$, $a_1(z) = e^z$, $a_0(z) = -2$ and $H = \{z : \arg z = 0\}$. Clearly, $\sigma(a_2) = \sigma(a_1) = 1$, $\sigma(a_0) = 0$ and $l = n = 2$. Here $\sigma(f) = 1 > \frac{3}{4} = \frac{1}{2} + \frac{1}{2^2}$.

The following Examples 3 and 4 show that the equality $\sigma(f) = \sigma(a_l)/n + 1/n^2$ can occur in Theorems 2 and 3.

Example 3 For Theorem 2, $f(z) = e^{-z^2}$ satisfies the difference equation

$$e^{z+1} f(z+1) - e^{-z} f(z) = 0,$$

where $a_1(z) = e^{z+1}$, $a_0(z) = -e^{-z}$ and $H = \{z : \arg z = 0\}$. Then $l = n = 1$ and $\sigma(a_1) = \sigma(a_0) = 1$, we have $\sigma(f) = 2 = \sigma(a_1) + 1$.

Example 4 For Theorem 3, $f(z) = e^{z^3} \tan z$ satisfies the difference equation

$$a_2(z) f(z + \tfrac{3}{2}\pi) + a_1(z) f(z + \pi) + a_0(z) f(z) = 0,$$

where $a_2(z) = e^{-(\frac{9}{2}\pi z^2 + \frac{27}{4}\pi^2 z + \frac{27}{8}\pi^3)} \tan^2 z$, $a_1(z) = 2 e^{-(3\pi z^2 + 3\pi^2 z + \pi^3)}$ and $a_0(z) = -1$. Then, $a_j(z)$, $j = 0, 1, 2$ satisfy

$$\lambda(\tfrac{1}{a_2}) = 1 < 2 = \sigma(a_2),$$
$$\max\{\sigma(a_0), \sigma(a_1)\} = 2 = \sigma(a_2),$$
$$\tau(a_1) = 3 < \tfrac{9}{2} = \tau(a_2),$$

where $l = n = 1$. Here $\sigma(f) = 3 = \sigma(a_2) + 1$.

The following Example 5 shows that the result in Theorem 4 is sharp.

Example 5 For Theorem 4, we consider the meromorphic functions

$$f(z) = e^{3z^2} \tan z \quad \text{and} \quad g(z) = e^{z^3} \tan z.$$

Case 1. If $\sigma(F) < \sigma(a_l)$ and $F(z)(\not\equiv 0)$, then $f(z)$ satisfies the difference equation

$$a_2(z) f(z + 3\pi) + a_1(z) f(z + 2\pi) + a_0(z) f(z) = F(z), \ (7)$$

where $a_2(z) = e^{-3z^2} \cot z$, $a_1(z) = e^{z^2 - 12\pi z - 12\pi^2}$, $a_0(z) = -e^{z^2}$ and $F(z) = e^{18\pi z + 27\pi^2}$. Then, $a_j(z)$, $j = 0, 1, 2$ and $F(z)$ satisfy

$$\lambda(\tfrac{1}{a_2}) = 1 < 2 = \sigma(a_2),$$
$$\sigma(F) = 1 < 2 = \max\{\sigma(a_0), \sigma(a_1)\} = \sigma(a_2),$$
$$\tau(a_0) + \tau(a_1) = \tfrac{1}{\pi} + \tfrac{1}{\pi} = \tfrac{2}{\pi} < \tfrac{3}{\pi} = \tau(a_2),$$

where $l = 2$ and $\max\{p_2, p_1, p_0\} = 1$. Here $\sigma(f) = \sigma(a_2) = 2$.

Case 2. If $\sigma(F) = \sigma(a_l)$ and $\sum_{\sigma(a_j) = \sigma(a_l), j \neq l} \tau(a_j) + \tau(F) < \tau(a_l)$, then $f(z)$ satisfies (7), where $a_2(z) = e^{-(7z^2 + 18\pi z + 27\pi^2)} \cot z$, $a_1(z) = e^{z^2 - 12\pi z - 12\pi^2}$, $a_0(z) = -e^{z^2}$ and $F(z) = e^{-4z^2}$. Then, $a_j(z)$, $j = 0, 1, 2$ and $F(z)$ satisfy

$$\lambda(\tfrac{1}{a_2}) = 1 < 2 = \sigma(a_2),$$
$$\sigma(F) = 2 = \max\{\sigma(a_0), \sigma(a_1)\} = \sigma(a_2),$$
$$\tau(a_0) + \tau(a_1) + \tau(F) = \tfrac{1}{\pi} + \tfrac{1}{\pi} + \tfrac{4}{\pi} = \tfrac{6}{\pi} < \tfrac{7}{\pi} = \tau(a_2),$$

where $l = 2$ and $\max\{p_2, p_1, p_0\} = 1$. Here $\sigma(f) = \sigma(a_2) = 2$.

Case 3. If $\sigma(F) = \sigma(a_l)$ and $\sum_{\sigma(a_j) = \sigma(a_l), j \neq l} \tau(a_j) + \tau(a_l) < \tau(F)$, then $f(z)$ satisfies (7), where $a_2(z) = e^{z^2 - 18\pi z - 27\pi^2} \cot z$, $a_1(z) = e^{-(z^2 + 12\pi z + 12\pi^2)}$, $a_0(z) = -e^{-z^2}$, $F(z) = e^{4z^2}$. Then, $a_j(z)$, $j = 0, 1, 2$ and $F(z)$ satisfy

$$\lambda(\tfrac{1}{a_2}) = 1 < 2 = \sigma(a_2),$$
$$\sigma(F) = 2 = \max\{\sigma(a_0), \sigma(a_1)\} = \sigma(a_2),$$
$$\tau(a_0) + \tau(a_1) + \tau(a_2) = \tfrac{1}{\pi} + \tfrac{1}{\pi} + \tfrac{1}{\pi} = \tfrac{3}{\pi} < \tfrac{4}{\pi} = \tau(F),$$

where $l = 2$ and $\max\{p_2, p_1, p_0\} = 1$. Here $\sigma(f) = \sigma(a_2) = 2$.

Case 4. If $F(z) \equiv 0$, then $g(z)$ satisfies the difference equation

$$a_2(z)g(z + \tfrac{5\pi}{2}) + a_1(z)g(z + 2\pi) + a_0(z)g(z) = 0,$$

where $a_2(z) = \mathrm{e}^{-(\frac{15\pi}{2}z^2 + \frac{75\pi^2}{4}z + \frac{125\pi^3}{8})}\tan^2 z$, $a_1(z) = 2\,\mathrm{e}^{-(6\pi z^2 + 12\pi^2 z + 8\pi^3)}$ and $a_0(z) = -1$. Then, $a_j(z)$, $j = 0, 1, 2$ satisfy

$$\lambda(\tfrac{1}{a_2}) = 1 < 2 = \sigma(a_2),$$
$$\max\{\sigma(a_0), \sigma(a_1)\} = 2 = \sigma(a_2),$$
$$\tau(a_1) = 6 < \tfrac{15}{2} = \tau(a_2),$$

where $l = 2$ and $\max\{p_2, p_1, p_0\} = 1$. Here $\sigma(g) = 3 > 2 = \sigma(a_2)$.

Case 5. If $\sigma(F) > \sigma(a_l)$, then $g(z)$ satisfies (7), where $F(z) = \mathrm{e}^{z^3}$, $a_2(z) = \mathrm{e}^{-(9\pi z^2 + 27\pi^2 z + 27\pi^3)}\cot z$, $a_1(z) = \mathrm{e}^{-(6\pi z^2 + 12\pi^2 z + 8\pi^3)}$ and $a_0(z) = -1$. Then, $a_j(z)$, $j = 0, 1, 2$ and $F(z)$ satisfy

$$\sigma(F) = 3 > 2 = \max\{\sigma(a_0), \sigma(a_1)\} = \sigma(a_2),$$

where $l = 2$ and $\max\{p_2, p_1, p_0\} = 1$. Here $\sigma(g) = \sigma(F) = 3$. Moreover, $a_j(z)$, $j = 0, 1, 2$ satisfy

$$\lambda(\tfrac{1}{a_2}) = 1 < 2 = \sigma(a_2) \text{ and } \tau(a_1) = 6 < 9 = \tau(a_2),$$

these two conditions are not necessary for Case 5.

AUXILIARY LEMMAS

Cherry and Ye [21] showed the following Borel-Nevanlinna Growth Lemma.

Lemma 1 ([21]) *Let $F(r)$ and $\phi(r)$ be positive, non-decreasing, continuous functions defined for $r_0 \leqslant r < \infty$, and assume that $F(r) \geqslant \mathrm{e}$ for $r \geqslant r_0$. Let $\xi(x)$ be a positive, non-decreasing, continuous function defined for $\mathrm{e} \leqslant x < \infty$. Let $C > 1$ be a constant, and let E be the closed subset of $[r_0, \infty)$ defined by*

$$E = \left\{ r \in [r_0, \infty) : F\left(r + \frac{\phi(r)}{\xi(F(r))}\right) \leqslant CF(r) \right\}.$$

Then, for all $R < \infty$,

$$\int_{E \cap [r_0, R]} \frac{\mathrm{d}r}{\phi(r)} \leqslant \frac{1}{\xi(\mathrm{e})} + \frac{1}{\log C} \int_{\mathrm{e}}^{F(R)} \frac{\mathrm{d}x}{x\xi(x)}.$$

The following lemma is an analogue of the the lemma on the logarithmic derivative for meromorphic functions composed with polynomials.

Lemma 2 ([10]) *Let f be a meromorphic function such that $f(0) \neq 0, \infty$, let $n \in \mathbb{N}$, and let $\alpha > 1$ and $0 < \delta < 1$. If the polynomials $\omega(z) = cz^n + p_{n-1}z^{n-1} + \cdots + p_0$ and $\varphi(z) = cz^n + q_{n-1}z^{n-1} + \cdots + q_0$ are distinct and non-constant, then there exists an $r_0 > 0$ such that, for all $r = |z| \geqslant r_0$,*

$$m\left(r, \frac{f \circ \omega}{f \circ \varphi}\right) \leqslant \frac{K(\alpha, \delta, \omega, \varphi)}{r^{\delta/n}} \times \left(T(\alpha|c|r^n, f) + \log^+ \frac{1}{|f(0)|} \right), \quad (8)$$

where

$$K(\alpha, \delta, \omega, \varphi) = \frac{8\alpha C(\delta(\alpha + 1) + n(6\alpha + 2))}{\delta(1 - \delta)|c|^{\delta/n}(\alpha - 1)}$$

with $C = 1 + |p_{n-1}| + |q_{n-1}|$.

Lemma 3 ([22]) *Let f be a meromorphic function with iterated order $0 < \sigma_p(f) < \infty$ and iterated type $0 < \tau_p(f) < \infty$. Then for any given $\beta < \tau_p(f)$, there exists a subset E of $[1, \infty)$ that has infinite logarithmic measure, such that $\log_{p-1} T(r, f) > \beta r^{\sigma_p(f)}$ holds for all $r \in E$.*

Chiang and Feng [5] proved that, for any arbitrary $c \in \mathbb{C}$, the Nevanlinna characteristic of any meromorphic function f of finite order satisfies the asymptotic relation $T(r, f(z + c)) \sim T(r, f)(r \to \infty)$. The following lemma is a generalization of their result to a certain type of composite meromorphic functions, including a class of infinite order functions.

Lemma 4 ([10]) *Let $w(z) = cz^n + p_{n-1}z^{n-1} + \cdots + p_0$ be a non-constant polynomial. If f is a meromorphic function such that*

$$\limsup_{r \to \infty} \frac{\log\log T(r, f)}{\log r} < \frac{1}{n^2},$$

then

$$T(r, f \circ w) = (1 + o(1))T(|c|r^n, f),$$

where r approaches infinity outside a possible exceptional set of finite logarithmic measure.

Proof of Theorem 1

Proof: It may be helpful to mention that similar techniques (but with more complexity) can be found in [10], Lemma 2.2, and references therein.

Denote $g(r) := T(|c|r^n, f)$ and $\alpha = \beta^n$. For positive, non-decreasing, continuous function $\xi(x)$

and $\phi(r)$ defined for $e \leqslant x < \infty$ and $r_0 \leqslant r < \infty$, respectively, where r_0 is such that $g(r) \geqslant e$ for all $r \geqslant r_0$, Lemma 1 implies that

$$g\left(r + \frac{\phi(r)}{\xi(g(r))}\right) \leqslant 2g(r)$$

for all r outside a set E satisfying

$$\int_{E \cap [r_0, R]} \frac{dr}{\phi(r)} \leqslant \frac{1}{\xi(e)} + \frac{1}{\log 2} \int_e^{g(R)} \frac{dx}{x\xi(x)},$$

where $R < \infty$. By choosing $\phi(r) = r$ and $\xi(x) = (\log(x))^{1+\varepsilon_1}$ with $\varepsilon_1 > 0$, and defining

$$\beta = 1 + \frac{1}{(\log g(r))^{1+\varepsilon_1}},$$

it follows that

$$T(\alpha|c|r^n, f) = g(\beta r) \leqslant 2g(r) = 2T(|c|r^n, f) \quad (9)$$

for all r outside a set E of finite logarithmic measure. Moreover, by substituting $\alpha = \beta^n$ into (8), it follows that there exists a positive absolute constant C such that

$$K(\alpha, \delta, \omega, \varphi) \leqslant C\big(\log T(|c|r^n, f)\big)^{1+\varepsilon_1} \quad (10)$$

for all r sufficiently large.

Suppose that $\sigma_2(f) = \sigma_2 < 1/n^2$, then by the definition of $\sigma_2(f)$, for any given $\varepsilon_2 > 0$ and all r large enough, we have $\log T(r, f) \leqslant r^{\sigma_2 + \varepsilon_2}$. Hence, by choosing ε_1 sufficiently small in (10), it follows that

$$K(\alpha, \delta, \omega, \varphi) \leqslant C r^{n(\sigma_2 + \varepsilon_2)} \quad (11)$$

for all r sufficiently large. The assertion follows in the case $f(0) \neq 0, \infty$ by choosing $\delta = 1 - \varepsilon_2$ in (8) and combining inequalities (9) and (11). If f has either a zero or pole at the origin, then, for a suitable $k \in \mathbb{Z}$, we may write $f(z) = z^k g(z)$ where $g(z)$ is finite and non-zero at the origin. Hence, by choosing $\varepsilon = \varepsilon(n, \varepsilon_2)$ sufficiently small, it follows that

$$\begin{aligned}
m\left(r, \frac{f \circ \omega}{f \circ \varphi}\right) &\leqslant m\left(r, \frac{g \circ \omega}{g \circ \varphi}\right) + O(\log r) \\
&= o\left(\frac{T(|c|r^n, g)}{r^{\frac{1}{n} - n\sigma_2 - \varepsilon}}\right) + O(\log r) \\
&= o\left(\frac{T(|c|r^n, f)}{r^{\frac{1}{n} - n\sigma_2 - \varepsilon}}\right) + O(\log r)
\end{aligned}$$

outside an exceptional set E' of finite logarithmic measure. Therefore, since f is a non-rational meromorphic function, we have

$$m\left(r, \frac{f \circ \omega}{f \circ \varphi}\right) = o\left(\frac{T(|c|r^n, f)}{r^{\frac{1}{n} - n\sigma_2 - \varepsilon}}\right)$$

as r approaches infinity outside E'. This completes the proof of Theorem 1. $\qquad\square$

Proof of Theorem 2

Proof: Let $f(z)(\not\equiv 0)$ be a transcendental meromorphic solution of (5). Suppose that $\sigma(f) < \sigma(a_l)/n + 1/n^2$, then by Theorem 1 and choosing ε sufficiently small, there exists a set E_1 with finite logarithmic measure such that for all $r \notin E_1$ and $r \to \infty$, we have for $j, l \in \{0, 1, \ldots, k\}$

$$m\left(r, \frac{f \circ P_j}{f \circ P_l}\right) = o\left(\frac{T(|c|r^n, f)}{r^{\frac{1}{n} - \varepsilon}}\right). \quad (12)$$

By the assumptions of Theorem 2, we know that $\sigma(a_l) = \sigma$. For any given sufficiently small $\delta > 0$, there exists a set H_1 satisfying $\overline{\mathrm{dens}}\{r = |z| : z \in H_1\} > 0$ such that for all $z \in H_1$ and $|z| = r \to \infty$, we obtain for $j = 0, 1, \ldots, k$, $j \neq l$

$$m(r, a_j(z)) \leqslant \alpha r^{\sigma - \delta}. \quad (13)$$

We divide (5) by $f(P_l(z))$ to get

$$-a_l(z) = \sum_{j=0, j \neq l}^{k} a_j(z) \frac{f(P_j(z))}{f(P_l(z))}. \quad (14)$$

Now, we may choose sufficiently small ε' satisfying $0 < 2\varepsilon' < \min\{\varepsilon, \delta, n\sigma(a_l) - n^2\sigma(f) + 1\}$, and deduce from (12)–(14) that for sufficiently large r satisfying $|z| = r \in H_1 \backslash E_1$, it follows that

$$\begin{aligned}
T(r, a_l) = m(r, a_l) &\leqslant \sum_{j=0, j \neq l}^{k} m(r, a_j) \\
&\quad + \sum_{j=0, j \neq l}^{k} m\left(r, \frac{f(P_j)}{f(P_l)}\right) + O(1) \\
&\leqslant O(\alpha r^{\sigma - \delta}) + o\left(\frac{T(|c|r^n, f)}{r^{\frac{1}{n} - \varepsilon'}}\right) \\
&= o(r^{\sigma(a_l) - \varepsilon'}) + o(r^{n\sigma(f) - \frac{1}{n} + \varepsilon'}) \\
&= o(r^{\sigma(a_l) - \varepsilon'}),
\end{aligned}$$

a contradiction. Hence, we have $\sigma(f) \geqslant \sigma(a_l)/n + 1/n^2$. This completes the proof of Theorem 2. $\quad\square$

Proof of Theorem 3

Proof: Let $f(z)(\not\equiv 0)$ be a transcendental meromorphic solution of (5). Suppose that $\sigma(f) < \sigma(a_l)/n + 1/n^2$. We divide through (5) by $f(P_l(z))$ to get (14).

It follows that

$$T(r, a_l) = m(r, a_l) + N(r, a_l)$$
$$\leqslant \sum_{j=0, j\neq l}^{k} m(r, a_j) + \sum_{j=0, j\neq l}^{k} m\left(r, \frac{f(P_j)}{f(P_l)}\right) + N(r, a_l) + O(1)$$
$$\leqslant \sum_{j=0, j\neq l}^{k} T(r, a_j) + O\left(m\left(r, \frac{f(P_j)}{f(P_l)}\right)\right) + N(r, a_l). \quad (15)$$

By Theorem 1, there exists a set E_2 with finite logarithmic measure such that for all $r \notin E_2$ and $r \to \infty$, we have (12).

Let us choose α_1 such that $\lambda(\frac{1}{a_l}) < \alpha_1 < \sigma(a_l) = \sigma$. Then for any given ε_1 such that $0 < 2\varepsilon_1 < \min\{\alpha_1 - \lambda(\frac{1}{a_l}), \sigma - \alpha_1\}$ and sufficiently large r,

$$N(r, a_l) \leqslant r^{\lambda(\frac{1}{a_l}) + \varepsilon_1} < r^{\alpha_1}. \quad (16)$$

Assume that $\max_{0 \leqslant j \leqslant k, j \neq l}\{\sigma(a_j)\} \leqslant \sigma(a_l)$ and $\sum_{\sigma(a_j) = \sigma} \tau(a_j) < \tau(a_l) < \infty$. Then there exists a set $J_1 \subseteq \{0, 1, \ldots, l-1, l+1, \ldots, k\}$ such that for $j \in J_1$, we have $\sigma(a_j) = \sigma(a_l) = \sigma$ and $\sum_{j \in J_1} \tau(a_j) < \tau(a_l) = \tau$. For $i \in \{0, 1, \ldots, l-1, l+1, \ldots, k\}\backslash J_1$ we have $\sigma(a_i) < \sigma(a_l) = \sigma$. Hence, we can choose $\beta_1, \beta_2, \sigma_0$ satisfying $\sum_{j \in J_1} \tau(a_j) = \tau_0 < \beta_1 < \beta_2 < \tau$, $0 < \sigma_0 < \sigma$ such that for any given ε_2 such that $0 < 2\varepsilon_2 < \min\{\beta_2 - \beta_1, \tau - \beta_2, \beta_1 - \tau_0, \sigma - \sigma_0, \sigma_0\}$ and for sufficiently large r, we have

$$T(r, a_j) \leqslant (\tau(a_j) + \varepsilon_2) r^{\sigma}, \quad j \in J_1, \quad (17)$$

and for $i \in \{0, 1, \ldots, l-1, l+1, \ldots, k\}\backslash J_1$

$$T(r, a_i) \leqslant r^{\sigma_0}. \quad (18)$$

By applying Lemma 3, there exists a subset E_3 of $[1, \infty)$ that has infinite logarithmic measure such that for all $r \in E_3$ and $r \to \infty$, the inequality

$$T(r, a_l) > \beta_2 r^{\sigma} \quad (19)$$

holds. Now, we may choose sufficiently small ε' satisfying $0 < (k+1)\varepsilon' < \min\{\varepsilon_1, \varepsilon_2\}$, and deduce from (12) and (15)–(19) that for $r \in E_3 \backslash E_2$ and $r \to \infty$, we have

$$\beta_2 r^{\sigma} < \sum_{j \in J_1} (\tau(a_j) + \varepsilon') r^{\sigma} + \sum_{i \in \{0, 1, \ldots, l-1, l+1, \ldots, k\}\backslash J_1} r^{\sigma_0}$$
$$+ o\left(\frac{T(|c|r^n, f)}{r^{\frac{1}{n} - \varepsilon'}}\right) + r^{\alpha_1}$$
$$< (\beta_1 + k\varepsilon') r^{\sigma} + O(r^{\sigma_0}) + o(r^{n\sigma(f) - \frac{1}{n} + \varepsilon'}) + r^{\alpha_1}.$$

It follows that

$$(\beta_2 - \beta_1 - k\varepsilon') r^{\sigma} < O(r^{\sigma_0}) + o(r^{n\sigma(f) - \frac{1}{n} + \varepsilon'}) + r^{\alpha_1}. \quad (20)$$

Since $0 < (k+1)\varepsilon' < \min\{\varepsilon_1, \varepsilon_2\}$, we obtain from (20) that $\sigma(a_l) = \sigma \leqslant n\sigma(f) - 1/n$. Hence, we have $\sigma(f) \geqslant \sigma(a_l)/n + 1/n^2$. This completes the proof of Theorem 3. $\qquad\square$

Proof of Theorem 4

Proof: Let $f(z)(\not\equiv 0)$ be a meromorphic solution of (6). Note that $m = \max\{p_j\}$, $j = 0, 1, \ldots, k$. We divide through (6) by $f(P_l(z))$ to get

$$-a_l(z) = \sum_{j=0, j\neq l}^{k} a_j(z) \frac{f(P_j(z))}{f(P_l(z))} - \frac{F(z)}{f(P_l(z))}. \quad (21)$$

It follows that

$$T(r, a_l) = m(r, a_l) + N(r, a_l)$$
$$\leqslant \sum_{j=0, j\neq l}^{k} m(r, a_j) + \sum_{j=0, j\neq l}^{k} m\left(r, \frac{f(P_j)}{f(P_l)}\right)$$
$$+ m(r, F) + m\left(r, \frac{1}{f(P_l)}\right) + N(r, a_l) + O(1)$$
$$\leqslant \sum_{j=0, j\neq l}^{k} T(r, a_j) + O\left(\sum_{j=0}^{k} T(r, f(P_j))\right) + T(r, F) + N(r, a_l). \quad (22)$$

Lemma 4 implies that there exists a set E_4 with finite logarithmic measure such that for all $r \notin E_4$ and $r \to \infty$, we have

$$T(r, f(P_j(z))) = (1 + o(1)) T(|c_{jp_j}| r^{p_j}, f)$$
$$\leqslant (1 + o(1)) T(cr^{p_j}, f)$$
$$\leqslant (1 + o(1)) T(cr^m, f), \quad (23)$$

where $c = \max\{|c_{0p_0}|, |c_{1p_1}|, \ldots, |c_{kp_k}|\}$.

Let us choose α_2 such that $\lambda(\frac{1}{a_l}) < \alpha_2 < \sigma(a_l) = \sigma$. Then for any given ε_1 such that $0 < 2\varepsilon_1 < \min\{\alpha_2 - \lambda(\frac{1}{a_l}), \sigma - \alpha_2\}$ and sufficiently large r,

$$N(r, a_l) \leqslant r^{\lambda(\frac{1}{a_l}) + \varepsilon_1} < r^{\alpha_2}. \quad (24)$$

Assume that $\max_{0 \leqslant j \leqslant k, j \neq l}\{\sigma(a_j)\} \leqslant \sigma(a_l)$ and $\sum_{\sigma(a_j) = \sigma} \tau(a_j) < \tau(a_l) < \infty$. Then there exists a set $J_2 \subseteq \{0, 1, \ldots, l-1, l+1, \ldots, k\}$ such that for $j \in J_2$, we have $\sigma(a_j) = \sigma(a_l) = \sigma$ and $\sum_{j \in J_2} \tau(a_j) < \tau(a_l) = \tau$ and for $i \in \{0, 1, \ldots, l-1, l+1, \ldots, k\}\backslash J_2$ we have $\sigma(a_i) < \sigma(a_l) = \sigma$. Hence, we can choose $\beta_3, \beta_4, \sigma'$ satisfying $\sum_{j \in J_2} \tau(a_j) = \tau_1 < \beta_3 < \beta_4 < \tau$, $0 < \sigma' < \sigma$ such that for any given ε_2 such that $0 < 2\varepsilon_2 < \min\{\beta_4 - \beta_3, \tau - \beta_4, \beta_3 - \tau_1, \sigma - \sigma', \sigma'\}$ and sufficiently large r, the inequalities

$$T(r, a_j) \leqslant (\tau(a_j) + \varepsilon_2) r^{\sigma}, \quad j \in J_2 \quad (25)$$

and for $i \in \{0, 1, \ldots, l-1, l+1, \ldots, k\} \backslash J_2$

$$T(r, a_i) \leqslant r^{\sigma'} \tag{26}$$

hold. By applying Lemma 3, there exists a subset E_5 of $[1, \infty)$ that has infinite logarithmic measure such that for all $r \in E_5$ and $r \to \infty$, we have

$$T(r, a_l) > \beta_4 r^{\sigma}. \tag{27}$$

Case 1. If $\sigma(F) < \sigma(a_l)$, we can choose α_3 such that $\sigma(F) < \alpha_3 < \sigma(a_l) = \sigma$. Then for any given ε_3 such that $0 < 2\varepsilon_3 < \min\{\alpha_3 - \sigma(F), \sigma - \alpha_3\}$ and sufficiently large r, we obtain

$$T(r, F) \leqslant r^{\sigma(F) + \varepsilon_3} < r^{\alpha_3}. \tag{28}$$

Now, we choose sufficiently small ε satisfying $0 < (k+1)\varepsilon < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, and deduce from (22)–(28) that for $r \in E_5 \backslash E_4$ and $r \to \infty$, we have

$$\beta_4 r^{\sigma} < \sum_{j \in J_2} (\tau(a_j) + \varepsilon_2) r^{\sigma} + \sum_{i \in \{0,1,\ldots,l-1,l+1,\ldots,k\} \backslash J_2} r^{\sigma_0}$$
$$+ O(T(cr^m, f)) + r^{\alpha_2} + r^{\alpha_3}$$
$$< (\beta_3 + k\varepsilon) r^{\sigma} + O(r^{\sigma'}) + O(r^{m\sigma(f) + \varepsilon}) + r^{\alpha_2} + r^{\alpha_3}.$$

It follows that

$$(\beta_4 - \beta_3 - k\varepsilon) r^{\sigma} < O(r^{\sigma'}) + O(r^{m\sigma(f) + \varepsilon}) + r^{\alpha_2} + r^{\alpha_3}. \tag{29}$$

Since $0 < (k+1)\varepsilon < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, we obtain from (29) that $\sigma(a_l) = \sigma \leqslant m\sigma(f)$. Hence, we have $\sigma(f) \geqslant \sigma(a_l) / \max\{p_j\}$.

If $\sigma(F) = \sigma(a_l)$ and $\sum_{\sigma(a_j) = \sigma(a_l), j \neq l} \tau(a_j) + \tau(F) < \tau(a_l)$, then for the ε above and sufficiently large r, we have

$$T(r, F) \leqslant (\tau(F) + \varepsilon) r^{\sigma}. \tag{30}$$

Now, we may choose sufficiently small ε satisfying $0 < (k+2)\varepsilon < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, and deduce from (22)–(27) and (30) that for $r \in E_5 \backslash E_4$ and $r \to \infty$,

$$\beta_4 r^{\sigma} < \sum_{j \in J_2} (\tau(a_j) + \varepsilon_2) r^{\sigma} + \sum_{i \in \{0,1,\ldots,l-1,l+1,\ldots,k\} \backslash J_2} r^{\sigma'}$$
$$+ O(T(cr^m, f)) + r^{\alpha_2} + (\tau(F) + \varepsilon) r^{\sigma}$$
$$< (\beta_3 + k\varepsilon) r^{\sigma} + O(r^{\sigma'}) + O(r^{m\sigma(f) + \varepsilon})$$
$$+ r^{\alpha_2} + (\tau(F) + \varepsilon) r^{\sigma}.$$

It follows that

$$(\beta_4 - \beta_3 - \tau(F) - (k+1)\varepsilon) r^{\sigma}$$
$$< O(r^{\sigma'}) + O(r^{m\sigma(f) + \varepsilon}) + r^{\alpha_2}. \tag{31}$$

Since $0 < (k+2)\varepsilon < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, we obtain from (31) that $\sigma(a_l) = \sigma \leqslant m\sigma(f)$. Hence, we have $\sigma(f) \geqslant \sigma(a_l) / \max\{p_j\}$.

If $\sigma(F) = \sigma(a_l)$ and $\sum_{\sigma(a_j) = \sigma(a_l), j \neq l} \tau(a_j) + \tau(a_l) < \tau(F)$. From (6) it follows that for sufficiently large r,

$$T(r, F) \leqslant \sum_{j=0}^{k} T(r, a_j f(P_j)) + O(1)$$

$$\leqslant \sum_{j=0, j \neq l}^{k} T(r, a_j) + O\left(\sum_{j=0}^{n} T(r, f(P_j))\right) + T(r, a_l). \tag{32}$$

Then by Lemma 3, for the ε above, there exists a subset E_6 of $[1, \infty)$ that has infinite logarithmic measure such that for all $r \in E_6$ and $r \to \infty$,

$$T(r, F) > (\tau(F) - \varepsilon) r^{\sigma}. \tag{33}$$

By the definition of $\tau(a_l)$, we have that for the ε above and sufficiently large r, the inequality

$$T(r, a_l) \leqslant (\tau + \varepsilon) r^{\sigma} \tag{34}$$

holds. Now, we may choose sufficiently small ε satisfying $0 < (k+3)\varepsilon < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, and deduce from (23), (25), (26) and (32)–(34) that for $r \in E_6 \backslash E_4$ and $r \to \infty$, we have

$$(\tau(F) - \varepsilon) r^{\sigma} < \sum_{j \in J_2} (\tau(a_j) + \varepsilon_2) r^{\sigma} + \sum_{i \in \{0,1,\ldots,l-1,l+1,\ldots,k\} \backslash J_2} r^{\sigma'}$$
$$+ O(T(cr^m, f)) + (\tau + \varepsilon) r^{\sigma}$$
$$< (\beta_3 + k\varepsilon) r^{\sigma} + O(r^{\sigma'}) + O(r^{m\sigma(f) + \varepsilon}) + (\tau + \varepsilon) r^{\sigma}.$$

It follows that

$$(\tau(F) - \tau - \beta_3 - (k+2)\varepsilon) r^{\sigma} < O(r^{\sigma'}) + O(r^{m\sigma(f) + \varepsilon}). \tag{35}$$

Since $0 < (k+3)\varepsilon < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, we obtain from (35) that $\sigma(a_l) = \sigma \leqslant m\sigma(f)$. Hence, we have $\sigma(f) \geqslant \sigma(a_l) / \max\{p_j\}$.

Case 2. If $F(z) \equiv 0$, then by using a similar reasoning as the one in Theorem 3, we have $\sigma(f) \geqslant \sigma(a_l) / \max\{p_j\}$.

Case 3. If $\sigma(F) > \sigma(a_l)$, then we may suppose that $\sigma(f) < \sigma(F) / \max\{p_j\}$ on the contrary. By (6) and Lemma 4, we obtain

$$\sigma(a_k(z) f(P_k(z)) + \cdots + a_0(z) f(P_0(z))) < \sigma(F),$$

a contradiction. Hence, we have $\sigma(f) \geqslant \sigma(F) / \max\{p_j\}$. This completes the proof of Theorem 4. □

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