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Synchronization stability for recurrent neural networks with time-varying delays

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ABSTRACT: This paper studies the general decay synchronization (GDS) of a class of recurrent neural networks (RNNs) with general activation functions and time-varying delays. By constructing suitable Lyapunov-Krasovskii functionals and employing useful inequality techniques, some sufficient conditions on the GDS of considered RNNs are established via a type of nonlinear control. In addition, an example with numerical simulations is presented to illustrate the obtained theoretical results.

KEYWORDS: general decay synchronization, general activation functions

MSC2010: 34D06 93B52

INTRODUCTION

Neural network dynamical systems have become one of the hot topics in modern applied mathematics. Its dynamic types of behaviour often include asymptotic stability, robust stability, local or global stability, synchronization stability, exponential stability, the existence of periodic solution, and polynomial stability[1–10]. As a mature and widely-accepted network system, recurrent neural network system has become one of the most important topic both in theory and applications, such as classification of image processing, pattern recognition, signal processing, associative memories, optimization problem[1–16].

However, the time delays inevitably exist in natural and man-made systems and cannot be neglected. We can see from the results that the time delays have a great destabilizing influence on the implementation of neural networks[1–20]. There has been a lot of literature related to the study of recurrent neural networks with time delays[6–16].

It is worth noting that the synchronization problem in neural network systems is one of the most basic and important concerns when we investigate the dynamical types of behaviour of recurrent neural networks (RNNs). Furthermore, the synchronization play an extremely important role in many fields of science including biology, climatology, sociology, ecology[13–21]. In view of the significance of the synchronization for delayed recurrent cellular neural networks (RCNNs), there are many important works have been developed to stabilize or synchronize neural networks and nonlinear systems[13–24].

It is well known that the estimate of the convergent rate of synchronization is very interesting and useful for studying the synchronization of chaotic systems. In some cases, the convergence rates of the synchronization are not shown or very difficult to estimate. For example, consider the equation[21] $\dot{y}(x) = -y^3/2,\ x \geqslant 0$. Although the equation is asymptotically stable, it is very difficult to estimate the convergent rate of the solution.

However, researchers have recently investigated the synchronization problem for classes of chaotic neural networks (NNs) with continuous activations by introducing a new concept of synchronization, called general decay synchronization (GDS)[21–24]. This leads us to consider a new type of convergence rate, such as convergence with general decay. Furthermore, studies on the general decay synchronization for RNNs with time-varying delays are fairly rare. Hence based on the above analysis and reasons, we consider in this study the following n-dimensional RNNs with time-varying delays

$$\begin{aligned}\dot{x}_i(t) = {} & -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) \\ & + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau_{ij}(t))) \\ & + \sum_{j=1}^{n} d_{ij} h_j(x_j(t-\sigma_{ij}(t))) + I_i,\end{aligned} \tag{1}$$

where $i \in \Upsilon \triangleq \{1, 2, \dots, n\}$, $n \geqslant 2$ denotes the number of neurons in the neural networks; $x_i(t)$ corresponds to the state variable of the ith unit at time t; $c_i > 0$ denotes the rate with which the ith neuron resets its potential to the resting state when isolated from the other neurons and inputs; a_{ij}, b_{ij}, and d_{ij} are the connection weights between the ith and jth neurons at time t; $f_i(\cdot)$, $g_i(\cdot)$, and $h_i(\cdot)$ are the nonlinear activation functions; I_i is the external input vector; and $\tau_{ij}(t)$ and $\sigma_{ij}(t)$ are the transmission time-varying delays satisfying $0 \leqslant \tau_{ij}(t) \leqslant \tau_{ij}$ and $0 \leqslant \sigma_{ij}(t) \leqslant \sigma_{ij}$, respectively.

The main purpose of the study is to construct suitable Lyapunov-Krasovskii functionals and apply a method to establish some new sufficient conditions on the general decay synchronization[22,23] for the system (1).

PRELIMINARIES

In this study, we use $\Upsilon = \{1, 2, \dots, n\}$ and $\mathbb{R}_0^+ = [0, \infty)$, unless otherwise stated. The initial conditions associated with the system (1) are given by

$$x_i(s) = \varphi_i(s), \quad s \in [-\tau, 0], \quad i = 1, 2, \dots, n$$

where $\tau = \max_{i,j\in\Upsilon}\{\tau_{ij}, \sigma_{ij}\}$ and $\varphi(s) = (\varphi_1(s), \varphi_2(s), \dots, \varphi_n(s)) \in C([-\tau, 0], \mathbb{R})$, the Banach space of all continuous functions with norm

$$\|\varphi\| = \sum_{i=1}^{n} \sup_{s\in[-\tau,0]} |\varphi_i(s)|.$$

Let $v = (v_1, v_1, \dots, v_n) \in \mathbb{R}^n$ with norm

$$\|v\| = \left(\sum_{i=1}^{n} |v_i|^2\right)^{1/2} \quad \text{or} \quad \|\mathrm{v}\| = \max_{\mathrm{i}\in\Upsilon} |\mathrm{v_i}|.$$

Throughout this paper, we assume that the following assumptions are satisfied.

(H$_1$) For each $j \in \Upsilon$, the activation functions $f_j(u), g_j(u), h_j(u)$ are continuous and there exist constants $L_j, H_j, K_j, N_j, M_j, O_j > 0$ such that for all $v_1, v_2 \in \mathbb{R}$,

$$|f_j(v_1) - f_j(v_2)| \leqslant L_j |v_1 - v_2| + N_j,$$
$$|g_j(v_1) - g_j(v_2)| \leqslant H_j |v_1 - v_2| + M_j,$$
$$|h_j(v_1) - h_j(v_2)| \leqslant K_j |v_1 - v_2| + O_j.$$

(H$_2$) Time-varying delays $\tau_{ij}(t)$ and $\sigma_{ij}(t)$ are differentiable and there exist real numbers $0 \leqslant \zeta_{ij}, \gamma_{ij} < 1$ such that for any $t \geqslant 0$,

$$0 \leqslant \dot{\tau}_{ij}(t) \leqslant \zeta_{ij} \quad \text{and} \quad 0 \leqslant \dot{\sigma}_{\mathrm{ij}}(\mathrm{t}) \leqslant \gamma_{\mathrm{ij}}.$$

In this paper, we consider the system (1) as the driven system, and the response system is given as

$$\begin{aligned} \dot{y}_i(t) = &-c_i y_i(t) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t)) \\ &+ \sum_{j=1}^{n} b_{ij} g_j(y_j(t - \tau_{ij}(t))) \\ &+ \sum_{j=1}^{n} d_{ij} h_j(y_j(t - \sigma_{ij}(t))) + I_i + u_i(t), \end{aligned} \tag{2}$$

where $u_i(t)$ is the controller to be designed.

Let $e_i(t) = y_i(t) - x_i(t)$. Then from (1) and (2), the error of the dynamical system is

$$\begin{aligned} \dot{e}_i(t) = &-c_i e_i(t) + \sum_{j=1}^{n} a_{ij} \tilde{f}_j + \sum_{j=1}^{n} b_{ij} \tilde{g}_{ij} \\ &+ \sum_{j=1}^{n} d_{ij} \tilde{h}_{ij} + u_i(t), \end{aligned} \tag{3}$$

where

$$\tilde{f}_j = f_j(y_j(t)) - f_j(x_j(t)),$$
$$\tilde{g}_{ij} = g_j(y_j(t - \tau_{ij}(t))) - g_j(x_j(t - \tau_{ij}(t))),$$
$$\tilde{h}_{ij} = h_j(y_j(t - \sigma_{ij}(t))) - h_j(x_j(t - \sigma_{ij}(t))).$$

We now give the definitions of ψ-type functions and GDS.

Definition 1 [Refs. 22, 23]. A function $\psi : \mathbb{R}_0^+ \to [1, \infty)$ is said to be ψ-type function if it satisfies the following conditions:

(i) It is differentiable and nondecreasing;
(ii) $\psi(0) = 1$ and $\psi(\infty) = \infty$;
(iii) $\tilde{\psi}(t) = \dot{\psi}(t)/\psi(t)$ is nondecreasing and $\psi^* = \sup_{t\geqslant 0} \tilde{\psi}(t) < \infty$, where $\dot{\psi}(t)$ is the time derivative of $\psi(t)$;
(iv) for any $t, s \geqslant 0$, $\psi(t + s) \leqslant \psi(t)\psi(s)$.

It is not difficult to check that functions $\psi(t) = \mathrm{e}^{\alpha t}$ and $\psi(t) = (1 + t)^{\alpha}$ for any $\alpha > 0$ satisfy the above four conditions, thus are ψ-type functions.

Definition 2 [Refs. 22, 23]. The drive-response systems (1) and (2) are said to be general decay synchronized if there exists a constant $\varepsilon > 0$ and a ψ-type function ψ such that for any solutions $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ and $y(t) = (y_1(t), y_2(t), \dots, y_n(t))$ of the systems (1) and (2), respectively,

$$\limsup_{t\to\infty} \frac{\log\|y(t) - x(t)\|}{\log \psi(t)} \leqslant \varepsilon,$$

where ε is the convergence rate when synchronization error approaches zero.

($\mathbf{H}_3$) For the functions $\psi(t), \tilde{\psi}(t)$ given in Definition 1, there exist a function $\varrho(t) \in C(\mathbb{R}, \mathbb{R}_0^+)$ and a constant δ such that for any $t \geqslant 0$

$$\tilde{\psi}(t) \leqslant 1, \qquad \sup_{t\in[0,\infty)} \int_0^t \psi^{\delta}(s)\varrho(s)\,\mathrm{d}s < \infty. \quad (4)$$

We now present a useful lemma which is essential to this study.

Lemma 1 (Refs. 22, 23) . *Under the assumption* ($\mathbf{H}_3$)*, assume that the synchronization error $e(t) = y(t) - x(t)$ of the driver-response systems* (1) *and* (2) *satisfy the differential equation $\dot{e}(t) = g(t, e_t)$, where $e_t = e(t+s)$ for $s \in [-\tau, 0]$ and the function $g(t, e_t)$ is locally bounded. If there exists a differentiable functional $V(t, e_t) : \mathbb{R}_0^+ \times C \to \mathbb{R}_0^+$ and positive constants λ_1, λ_2 such that for any $(t, e_t) \in \mathbb{R}_0^+ \times C$*

$$\begin{aligned} &(\lambda_1 \|e(t)\|)^2 \leqslant V(t, e_t) \\ &\frac{\mathrm{d}V}{\mathrm{d}t}(t, e_t)\Big|_{(3)} \leqslant -\delta V(t, e_t) + \lambda_2 \varrho(t), \end{aligned} \quad (5)$$

where $x(t)$ and $y(t)$ are solutions of the systems (1) *and* (2)*, respectively, and $\delta > 0$ and $\varrho(t)$ are defined in* ($\mathbf{H}_3$)*. Then the driver-response systems* (1) *and* (2) *are general decay synchronized in the sense of Definition 2, and the convergence rate is $\delta/2$.*

MAIN RESULTS

In this section, we will obtain some sufficient conditions to insure the GDS of the systems (1) and (2). First letting C_{ij} and D_{ij} be numbers greater than zero, and under assumption ($\mathbf{H}_3$) designing the controller $u_i(t)$ of the response system (2) as follows.

$$u_i(t) = -\alpha_i \operatorname{sgn}(e_i(t)) - \frac{\beta_i \|e(t)\|^2 e_i(t)}{e_i^2(t) + \varrho(t)}, \quad (6)$$

where β_i and α_i are control gains satisfying

$$\begin{aligned} &-c_i - \beta_i + \sum_{j=1}^n \Big(\frac{A_{ji}}{1-\zeta_{ji}} + \frac{B_{ji}}{1-\gamma_{ji}} + \tau_{ji} C_{ji} + \sigma_{ji} D_{ji} \\ &\quad + \frac{|a_{ij}|L_j + |a_{ji}|L_i + |b_{ij}|H_j + |d_{ij}|K_j}{2} \Big) < 0, \\ &-\alpha_i + \sum_{j=1}^n \big(|a_{ij}|N_j + |b_{ij}|M_j + |d_{ij}|O_j \big) < 0, \end{aligned} \quad (7)$$

where $A_{ij} = |b_{ij}|H_j/2$ and $B_{ij} = |d_{ij}|K_j/2$, $i, j \in \Upsilon$.

Based on the nonlinear controller (6), the following theorem is obtained.

Theorem 1 *Suppose* ($\mathbf{H}_1$)–($\mathbf{H}_3$) *hold. Then the response network* (2) *is general decay synchronized with the drive network* (1) *under the nonlinear controller* (6) *if the control gains β_i satisfy* (7).

Proof: Firstly, we construct the following Lyapunov-Krasovskii functional.

$$\begin{aligned} V_1(t) = &\sum_{i=1}^n \frac{1}{2} e_i^2(t) + \sum_{i=1}^n \sum_{j=1}^n \int_{t-\tau_{ij}(t)}^t \frac{A_{ij} e_j^2(s)}{1-\zeta_{ij}} \,\mathrm{d}s \\ &+ \sum_{i=1}^n \sum_{j=1}^n \int_{t-\sigma_{ij}(t)}^t \frac{B_{ij} e_j^2(s)}{1-\gamma_{ij}} \,\mathrm{d}s. \end{aligned} \quad (8)$$

Calculating the derivative of $V_1(t)$ along the trajectory of the system (3), we obtain

$$\begin{aligned} \dot{V}_1(t) = &\sum_{i=1}^n \Big\{ e_i(t) \Big[-c_i e_i(t) + \sum_{j=1}^n a_{ij} \tilde{f}_j + \sum_{j=1}^n b_{ij} \tilde{g}_{ij} \\ &+ \sum_{j=1}^n d_{ij} \tilde{h}_{ij} - \alpha_i \operatorname{sgn}(e_i(t)) - \frac{\beta_i \|e(t)\|^2 e_i(t)}{e_i^2(t) + \varrho(t)} \Big] \Big\} \\ &+ \sum_{i=1}^n \sum_{j=1}^n A_{ij} \Big[\frac{e_j^2(t)}{1-\zeta_{ij}} - \frac{1-\dot{\tau}_{ij}(t)}{1-\zeta_{ij}} e_j^2(t-\tau_{ij}(t)) \Big] \\ &+ \sum_{i=1}^n \sum_{j=1}^n B_{ij} \Big[\frac{e_j^2(t)}{1-\gamma_{ij}} - \frac{1-\dot{\sigma}_{ij}(t)}{1-\gamma_{ij}} e_j^2(t-\sigma_{ij}(t)) \Big] \\ \leqslant &\sum_{i=1}^n \Big[-c_i e_i^2(t) + \sum_{j=1}^n \big| a_{ij} e_i(t) \tilde{f}_j \big| + \sum_{j=1}^n \big| b_{ij} e_i(t) \tilde{g}_{ij} \big| \\ &+ \sum_{j=1}^n \big| d_{ij} e_i(t) \tilde{h}_{ij} \big| - \alpha_i \|e_i(t)\| - \frac{\beta_i \|e(t)\|^2 e_i^2(t)}{e_i^2(t) + \varrho(t)} \Big] \\ &+ \sum_{i=1}^n \sum_{j=1}^n \Big(\frac{A_{ij}}{1-\zeta_{ij}} + \frac{B_{ij}}{1-\gamma_{ij}} \Big) e_j^2(t) \\ &- \sum_{i=1}^n \sum_{j=1}^n A_{ij} e_j^2(t-\tau_{ij}(t)) - \sum_{i=1}^n \sum_{j=1}^n B_{ij} e_j^2(t-\sigma_{ij}(t)). \end{aligned}$$

Using ($\mathbf{H}_1$) and $ab \leqslant (a^2+b^2)/2$, we have

$$\begin{aligned} &\sum_{i=1}^n \sum_{j=1}^n \big| a_{ij} e_i(t) \tilde{f}_j \big| \\ &\quad \leqslant \sum_{i=1}^n \sum_{j=1}^n |a_{ij} e_i(t)| (L_j |e_j(t)| + N_j) \\ &\quad \leqslant \sum_{i=1}^n \sum_{j=1}^n |a_{ij}| \Big[\frac{L_j}{2} \big(e_i^2(t) + e_j^2(t) \big) + N_j |e_i(t)| \Big]. \end{aligned}$$

Similarly, we have

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|b_{ij}e_i(t)\tilde{g}_{ij}\right| &\leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}e_i(t)|(H_j|e_j(t-\tau_{ij}(t))|+M_j)\\ &\leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|\left[\frac{H_j}{2}\big(e_i^2(t)+e_j^2(t-\tau_{ij}(t))\big)+M_j|e_i(t)|\right]\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|d_{ij}e_i(t)\tilde{h}_{ij}\right| &\leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}|d_{ij}e_i(t)|(K_j|e_j(t-\sigma_{ij}(t))|+O_j)\\ &\leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}|d_{ij}|\left[\frac{K_j}{2}\big(e_i^2(t)+e_j^2(t-\sigma_{ij}(t))\big)+O_j|e_i(t)|\right].\end{aligned}$$

We construct the Lyapunov-Krasovskii functional,

$$\begin{aligned}V_2(t)=\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\tau_{ij}}^{0}\int_{t+s}^{t}C_{ij}e_j^2(\epsilon)\,\mathrm{d}\epsilon\,\mathrm{d}s\\ +\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\sigma_{ij}}^{0}\int_{t+s}^{t}D_{ij}e_j^2(\epsilon)\,\mathrm{d}\epsilon\,\mathrm{d}s.\end{aligned}$$

Calculating the derivative of $V_2(t)$, we obtain

$$\begin{aligned}\dot{V}_2(t)\leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\big(\tau_{ij}C_{ij}+\sigma_{ij}D_{ij}\big)e_j^2(t)\\ -\sum_{i=1}^{n}\sum_{j=1}^{n}\left[C_{ij}\int_{t-\tau_{ij}}^{t}e_j^2(\epsilon)\,\mathrm{d}\epsilon+D_{ij}\int_{t-\sigma_{ij}}^{t}e_j^2(\epsilon)\,\mathrm{d}\epsilon\right].\end{aligned}$$

Finally, we construct the following Lyapunov-Krasovskii functional, $V(t)=V_1(t)+V_2(t)$. Then, there exists a scalar $\chi>1$ such that

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^{n}e_i^2(t)\leqslant V(t),\\ &V(t)\leqslant \chi\sum_{i=1}^{n}e_i^2(t)+\frac{\chi}{E}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[C_{ij}\int_{t-\tau_{ij}}^{t}e_j^2(\epsilon)\,\mathrm{d}\epsilon\right.\\ &\qquad\left.+D_{ij}\int_{t-\sigma_{ij}}^{t}e_j^2(\epsilon)\,\mathrm{d}\epsilon\right], \qquad (9)\end{aligned}$$

where $E=\min_{i\in\Upsilon}\{E_i\}$ with

$$\begin{aligned}E_i=c_i+\beta_i-\sum_{j=1}^{n}\left[\frac{A_{ji}}{1-\zeta_{ji}}+\frac{B_{ji}}{1-\gamma_{ji}}+\tau_{ji}C_{ji}+\sigma_{ji}D_{ji}\right.\\ \left.+\frac{|a_{ij}|L_j+|a_{ji}|L_i+|b_{ij}|H_j+|d_{ij}|K_j}{2}\right]>0.\end{aligned}$$

Calculating the derivative of $V(t)$ and from above results, we obtain

$$\begin{aligned}\dot{V}(t)&\leqslant\sum_{i=1}^{n}\left[-c_i+\sum_{j=1}^{n}\left(\frac{A_{ji}}{1-\zeta_{ji}}+\frac{B_{ji}}{1-\gamma_{ji}}+\tau_{ji}C_{ji}\right.\right.\\ &\quad\left.\left.+\sigma_{ji}D_{ji}+\frac{|a_{ij}|L_j+|a_{ji}|L_i+|b_{ij}|H_j+|d_{ij}|K_j}{2}\right)\right]e_i^2(t)\\ &\quad-\sum_{i=1}^{n}\left[\alpha_i-\sum_{j=1}^{n}\big(|a_{ij}|N_j+|b_{ij}|M_j+|d_{ij}|K_j\big)\right]|e_i(t)|\\ &\quad-\sum_{i=1}^{n}\frac{\beta_i\|e(t)\|^2e_i^2(t)}{e(t)^2+\varrho(t)}-\sum_{i=1}^{n}\sum_{j=1}^{n}\left(C_{ij}\int_{t-\tau_{ij}}^{t}e_j^2(\epsilon)\,\mathrm{d}\epsilon\right.\\ &\quad\left.+D_{ij}\int_{t-\sigma_{ij}}^{t}e_j^2(\epsilon)\,\mathrm{d}\epsilon\right)-\sum_{i=1}^{n}\beta_ie_i^2(t)+\sum_{i=1}^{n}\beta_ie_i^2(t)\\ &\leqslant\sum_{i=1}^{n}\left[-c_i-\beta_i+\sum_{j=1}^{n}\left(\frac{A_{ji}}{1-\zeta_{ji}}+\frac{B_{ji}}{1-\gamma_{ji}}+\tau_{ji}C_{ji}\right.\right.\\ &\quad\left.\left.+\sigma_{ji}D_{ji}+\frac{|a_{ij}|L_j+|a_{ji}|L_i+|b_{ij}|H_j+|d_{ij}|K_j}{2}\right)\right]e_i^2(t)\\ &\quad+\sum_{i=1}^{n}\beta_ie_i^2(t)-\sum_{i=1}^{n}\frac{\beta_i\|e(t)\|^2e_i^2(t)}{\|e(t)\|^2+\varrho(t)}\\ &\quad-\sum_{i=1}^{n}\sum_{j=1}^{n}\left(C_{ij}\int_{t-\tau_{ij}}^{t}e_j^2(\epsilon)\,\mathrm{d}\epsilon+D_{ij}\int_{t-\sigma_{ij}}^{t}e_j^2(\epsilon)\,\mathrm{d}\epsilon\right)\\ &\leqslant\sum_{i=1}^{n}-E_ie_i^2(t)+\max_{i\in\Upsilon}\{\beta_i\}\frac{\|e(t)\|^2\varrho(t)}{\|e(t)\|^2+\varrho(t)}\\ &\quad-\sum_{i=1}^{n}\sum_{j=1}^{n}\left(C_{ij}\int_{t-\tau_{ij}}^{t}e_j^2(\epsilon)\,\mathrm{d}\epsilon+D_{ij}\int_{t-\sigma_{ij}}^{t}e_j^2(\epsilon)\,\mathrm{d}\epsilon\right).\end{aligned}$$

Since

$$\frac{\|e(t)\|^2\varrho(t)}{\|e(t)\|^2+\varrho(t)}\leqslant\frac{\|e(t)\|^2\varrho(t)}{\|e(t)\|^2}=\varrho(t),$$

we have

$$\begin{aligned}\dot{V}(t)\leqslant\sum_{i=1}^{n}-E_ie_i^2(t)+\beta\varrho(t)\\ -\sum_{i=1}^{n}\sum_{j=1}^{n}\left[C_{ij}\int_{t-\tau_{ij}}^{t}e_j^2(\epsilon)\,\mathrm{d}\epsilon+D_{ij}\int_{t-\sigma_{ij}}^{t}e_j^2(\epsilon)\,\mathrm{d}\epsilon\right],\end{aligned} \qquad (10)$$

where $\beta = \max_{i\in\Upsilon}\{\beta_i\} > 0$. Taking δ such that $\delta\chi < E$, (9) and (10) give

$$\begin{aligned}\dot V(t)+\delta V(t) &\leqslant \sum_{i=1}^{n} -E_i e_i^2(t)+\beta\varrho(t)\\ &-\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(C_{ij}\int_{t-\tau_{ij}}^{t} e_j^2(\epsilon)\,\mathrm{d}\epsilon + D_{ij}\int_{t-\sigma_{ij}}^{t} e_j^2(\epsilon)\,\mathrm{d}\epsilon\Big)\\ &+\delta\Big[\chi\sum_{i=1}^{n} e_i^2(t)+\frac{\chi}{E}\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(C_{ij}\int_{t-\tau_{ij}}^{t} e_j^2(\epsilon)\,\mathrm{d}\epsilon\\ &\qquad + D_{ij}\int_{t-\sigma_{ij}}^{t} e_j^2(\epsilon)\,\mathrm{d}\epsilon\Big)\Big]\\ &\leqslant (\delta\chi - E)\sum_{i=1}^{n} e_i^2(t)+\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\frac{\delta\chi}{E}-1\Big)\\ &\times\Big(C_{ij}\int_{t-\tau_{ij}}^{t} e_j^2(\epsilon)\,\mathrm{d}\epsilon + D_{ij}\int_{t-\sigma_{ij}}^{t} e_j^2(\epsilon)\,\mathrm{d}\epsilon\Big)+\beta\varrho(t),\end{aligned}$$

which implies that

$$\dot V(t)+\delta V(t)\leqslant \beta\varrho(t). \tag{11}$$

Then by Lemma 1, the drive-response systems (1) and (2) achieve GDS under the adaptive nonlinear controller (6). The convergence rate of $e(t)$ approaching zero is $\delta/2$. □

Remark 1 The function ψ is used as the decay function, so ψ-type stability is also said to be stability with general decay rate. When $\psi(t) = e^{\alpha t}$ and $\psi(t) = (1+t)^{\alpha}$ for any $\alpha > 0$, ψ-type stability may be specialized as exponential synchronization and polynomial synchronization.

In addition, the controller (6) in the system (2) becomes

$$u_i(t) = -\frac{\beta_i\|e(t)\|^2 e_i(t)}{e_i^2(t)+\varrho(t)}, \quad i\in\Upsilon. \tag{12}$$

From Theorem 1, we have the following corollary.

Corollary 1 *Suppose* $(\mathbf{H}_1)$*,* $(\mathbf{H}_2)$*,* $(\mathbf{H}_3)$ *hold. Then the response network* (2) *can be general decay synchronized with the drive network* (1) *under the nonlinear controller* (12) *if the control gains* β_i *satisfy the inequality*

$$\begin{aligned}-c_i-\beta_i+\sum_{j=1}^{n}\Big[&\frac{A_{ji}}{1-\zeta_{ji}}+\frac{B_{ji}}{1-\gamma_{ji}}+\tau_{ji}C_{ji}+\sigma_{ji}D_{ji}\\ &+\frac{|a_{ij}|L_j+|a_{ji}|L_i+|b_{ij}|H_j+|d_{ij}|K_j}{2}\Big]<0. \end{aligned}\tag{13}$$

Remark 2 To achieve GDS of the considered master-slave systems, Wang et al[22,23] used the controllers $u_i(t)$ as

$$-G_i e(t) - G_2\,\mathrm{sgn}(e(t)) - \frac{\|A\|_\infty^2\|e(t)\|^2 e(t)}{2(\|A\|_\infty\|e(t)\|^2+\varrho(t))},$$

where $G_1 = \mathrm{diag}(\epsilon_1,\epsilon_2,\dots,\epsilon_n)$ and $G_2 = \mathrm{diag}(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n)$. However, in this study we used the simpler and more efficient controller given in (6) and (12). Hence the results of this study is an improvement and extension of the results obtained in Refs. 22, 23.

If in $(\mathbf{H}_1)$ we assume that the activation functions $f_j(u), g_j(u), h_j(u)$ are globally Lipschitz, i.e., the constants $N_j = M_j = O_j = 0$, the $(\mathbf{H}_1)$ turns to $(\bar{\mathbf{H}}_1)$.

$(\bar{\mathbf{H}}_1)$ $f_j(u), g_j(u), h_j(u)$ are globally Lipschitz continuous, i.e., there exist constants $L_j, H_j, K_j > 0$ such that for all $v_1, v_2\in\mathbb{R}$,

$$\begin{aligned}|f_j(v_1)-f_j(v_2)| &\leqslant L_j|v_1-v_2|,\\ |g_j(v_1)-g_j(v_2)| &\leqslant H_j|v_1-v_2|,\\ |h_j(v_1)-h_j(v_2)| &\leqslant K_j|v_1-v_2|.\end{aligned}$$

In the system (1), if $d_{ij}=0$, then it is reduced to the following n-dimensional RNNs with time-varying delays

$$\begin{aligned}\dot x_i(t) = -c_i x_i(t) &+ \sum_{j=1}^{n} a_{ij} f_j(x_j(t))\\ &+\sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau_{ij}(t))) + I_i. \end{aligned}\tag{14}$$

Accordingly, the response system (2) becomes

$$\begin{aligned}\dot y_i(t) = -c_i y_i(t) &+ \sum_{j=1}^{n} a_{ij} f_j(y_j(t))\\ &+\sum_{j=1}^{n} b_{ij} g_j(y_j(t-\tau_{ij}(t))) + I_i + u_i(t). \end{aligned}\tag{15}$$

Accordingly, the assumptions $(\mathbf{H}_1)$ and $(\bar{\mathbf{H}}_1)$ turn to:

$(\mathbf{H}_1^*)$ For each $j\in\Upsilon$, the activation functions $f_j(u)$ and $g_j(u)$ are continuous and there exist constants $L_j, H_j, N_j, M_j > 0$ such that for $v_1, v_2\in\mathbb{R}$,

$$\begin{aligned}|f_j(v_1)-f_j(v_2)| &\leqslant L_j|v_1-v_2|+N_j,\\ |g_j(v_1)-g_j(v_2)| &\leqslant M_j|v_1-v_2|+M_j.\end{aligned}$$

($\bar{\mathbf{H}}_1^*$) For each $j \in \Upsilon$, the activation functions $f_j(u), g_j(u)$ are globally Lipschitz continuous, i.e., there exist constants $L_j, H_j > 0$ such that for $v_1, v_2 \in \mathbb{R}$,

$$|f_j(v_1) - f_j(v_2)| \leqslant L_j |v_1 - v_2|,$$
$$|g_j(v_1) - g_j(v_2)| \leqslant H_j |v_1 - v_2|.$$

From Theorem 1 we have the following corollaries.

Corollary 2 *Suppose* ($\mathbf{H}_1^*$), ($\mathbf{H}_2$) *and* ($\mathbf{H}_3$) *hold. Then the response network* (15) *can be general decay synchronized with the drive network* (14) *under the nonlinear controller* (6) *if the control gains* β_i *satisfy the inequalities*

$$-c_i - \beta_i + \sum_{j=1}^{n} \left(\frac{A_{ji}}{1-\zeta_{ji}} + \frac{|a_{ij}|L_j + |a_{ji}|L_i + |b_{ij}|H_j}{2} + \tau_{ji} C_{ji} \right) < 0,$$
$$-\alpha_i + \sum_{j=1}^{n} \left(|a_{ij}| N_j + |b_{ij}| M_j \right) < 0.$$

Corollary 3 *Suppose* ($\bar{\mathbf{H}}_1^*$), ($\mathbf{H}_2$) *and* ($\mathbf{H}_3$) *hold. Then the response network* (15) *can be general decay synchronized with the drive network* (14) *under the nonlinear controller* (12) *if the control gains* β_i *satisfy the inequality*

$$-c_i - \beta_i + \sum_{j=1}^{n} \left(\frac{A_{ji}}{1-\zeta_{ji}} + \frac{|a_{ij}|L_j + |a_{ji}|L_i + |b_{ij}|H_j}{2} + \tau_{ji} C_{ji} \right) < 0.$$

NUMERICAL SIMULATIONS

In this section, an example is given to illustrate the effectiveness of the obtained results.

Example 1 For $n = 2$, we consider the following chaotic recurrent neural network system with time-varying delays

$$\begin{aligned} \dot{x}_i(t) = &- c_i x_i(t) + \sum_{j=1}^{2} a_{ij} f_j(x_j(t)) \\ &+ \sum_{j=1}^{2} b_{ij} g_j(x_j(t - \tau_{ij}(t))) \\ &+ \sum_{j=1}^{2} d_{ij} h_j(x_j(t - \sigma_{ij}(t))) + I_i, \end{aligned} \tag{16}$$

where $f_1(u) = f_2(u) = \tanh(u)$, $g_1(u) = g_2(u) = \tanh(u) - 0.1\sinh(u)$, and $h_1(u) = h_2(u) = \tanh(u) + 0.1\cosh(u)$. The parameters of the system (16) are $c_1 = c_2 = 1$, $a_{11} = 1.55$, $a_{12} = -0.1$, $a_{21} = -1$, $a_{22} = 0.4$, $b_{11} = -1.5$, $b_{12} = -0.6$, $b_{21} = 0.5$, $b_{22} = -0.95$, $d_{11} = -0.9$, $d_{12} = -0.4$, $d_{21} = -0.5$, $d_{22} = -0.85$, $\tau_{ij}(t) = \mathrm{e}^t/(1+\mathrm{e}^t)$, $\sigma_{ij}(t) = \mathrm{e}^t/(2+\mathrm{e}^t)$, and $I_i = 0$ for $i = 1, 2$.

The numerical simulation of the system (16) with initial values $x_1(s) = 0.4$ and $x_2(s) = 0.5$ for $s \in [-1, 0]$ is represented in Fig. 1. We can see that the system (16) has a chaotic attractor.

The corresponding response system is

$$\begin{aligned} \dot{y}_i(t) = &-c_i y_i(t) + \sum_{j=1}^{2} a_{ij} f_j(y_j(t)) \\ &+ \sum_{j=1}^{2} b_{ij} g_j(y_j(t - \tau_{ij}(t))) \\ &+ \sum_{j=1}^{2} d_{ij} h_j(y_j(t - \sigma_{ij}(t))) + I_i + u_i(t), \end{aligned} \tag{17}$$

where $c_i, a_{ij}, b_{ij}, d_{ij}, f_j(t), g_j(t), h_j(t), \tau_{ij}(t), \sigma_{ij}(t)$, and I_i are the same as in the system (16), and the nonlinear controllers $u_i(t)$ are designed as

$$u_i(t) = -\alpha_i \operatorname{sgn}(e_i(t)) - \frac{\beta_i \|e(t)\|^2 e_i(t)}{e_i^2(t) + \varrho(t)}, \tag{18}$$

where $e_i(t) = y_i(t) - x_i(t)$ for $i = 1, 2$.

It is not difficult to estimate that $L_j = H_j = K_j = 1$, $N_j = 0.01$, $M_j = 0.02$, $O_j = 0.015$, and $\tau_{ij} = \sigma_{ij} = 1$. Thus the assumptions ($\mathbf{H}_1$) and ($\mathbf{H}_2$) are satisfied. Letting $\varrho(t) = \mathrm{e}^{-0.1t}$, $\psi(t) = \mathrm{e}^t$ and choosing $\alpha_1 = 3.5$, $\alpha_2 = 3.2$, $\beta_1 = 4$, $\beta_2 = 3.7$. Then the assumption ($\mathbf{H}_3$) and (7) of Theorem 1 are satisfied. Hence the drive-response systems (16) and (17) can achieve GDS under the controller (18). The time evolution of synchronization errors between the systems (16) and (17) are demonstrated in Fig. 2, where the initial values of the response system (17) are chosen to be $y_1(s) = 0.5$ and $y_2(s) = 0$ for $s \in [-1, 0]$. The synchronization curves between the systems (16) and (17) are shown in Fig. 3.

CONCLUSIONS

In this study, we investigate the GDS problem for a class of RNNs with general activation functions and time-varying delays. Some sufficient conditions on the general decay synchronization of the drive-response systems (1) and (2) are obtained by constructing suitable Lyapunov-Krasovskii functionals

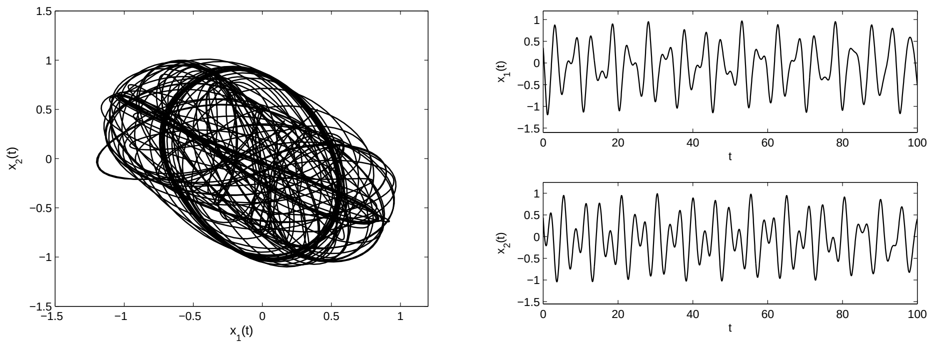

Fig. 1 The chaotic behaviour of delayed recurrent neural network system (16).

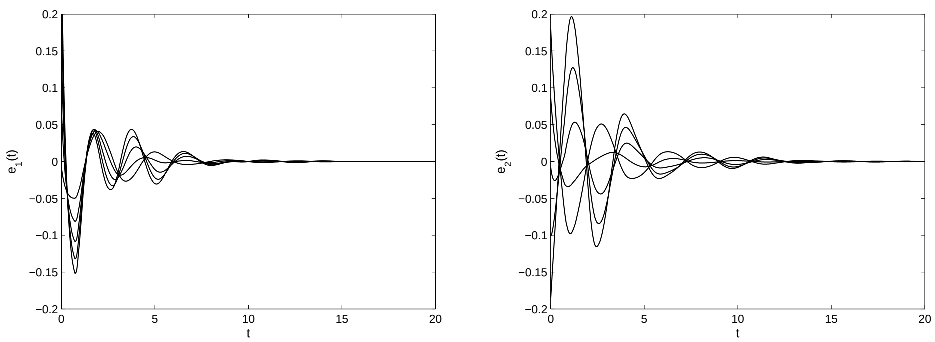

Fig. 2 The evaluation of synchronization error $e_1(t)$ and $e_2(t)$ in Example 1.

and employing useful inequalities. In addition, an example and its numerical simulations are given to validate the theoretical results in this study. Furthermore, it is believed that our approaches and obtained results may bring some new guidance for the synchronization stability study of other type neural networks with delays such as, delayed RNNs with discontinuous activations[25], delayed fuzzy cellular neural networks with discontinuous activations[26,27] and some kinds of delayed complex-valued neural networks[28–30]. The GDS problem for the above mentioned neural networks with delays may be of interest to other researchers.

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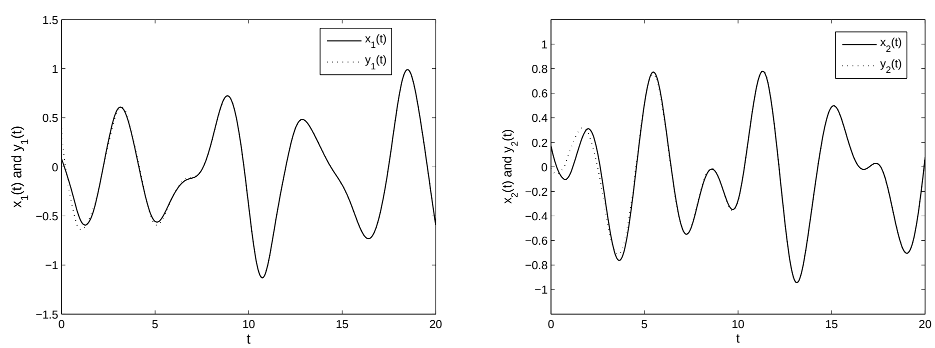

Fig. 3 Synchronization curves of $x_1(t), y_1(t)$ and $x_2(t), y_2(t)$ in Example 1.

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