Finite-time Synchronization of Large-scale Hyper Chaotic Systems with Parametric Uncertainties and Time-varying Delay

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ABSTRACT: In this paper, the finite-time synchronization of large-scale hyper chaotic systems with parametric uncertainties and time-varying delay is investigated. Based on Lyapunov stability theory and finite-time control technique, a suitable time-delay state feedback controller is designed, and a sufficient condition of finite-time synchronization is put forward. Finally, the numerical simulation is presented to illustrate the applicability and effectiveness of the proposed method.

KEYWORDS: finite-time control technique; combined chaotic system; uncertain parameter; nonlinear control

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INTRODUCTION

Due to its potential applications for secure communication, engineering science, and other fields, the synchronization of chaotic systems has been an active research object[1]. However, most of the previous researches were primarily concerned with the asymptotic stability of the synchronization error system[2]. In reality, optimizing the synchronization time is more valuable than achieving asymptotic synchronization. To realize fast synchronization, the exponential synchronization which can ensure the response system converge to the drive system at an exponential rate has been proposed[3]. Later, the finite-time control technique arises naturally in time-optimal control[4]. Its key feature is to drive the controlled system to its target within a finite time and to keep them there then after[5].

In many practical applications, the influence of parametric uncertainties can not be ignored, otherwise, the stability of the synchronization error system would be destroyed[6]. The problem of finite-time synchronization between two chaotic systems with unknown parameters has been addressed[7], however, the time-delay has not been considered. In fact, time-delays exist widely in reality because of finite signal transformation times and memory effects. For delayed chaotic system, the relationship between the drive and the response systems are

more complex. The finite-time synchronization of chaotic systems with time-delay has been studied[8], but the parametric uncertainty has not been taken into account, and what is more, it only considers the simple case in which the time-delay is constant. It is concluded from the recent research that time-delayed system is still vulnerable for communication because the time delay τ can be exposed by several measures, such as filling factor[9], autocorrelation[10], one step prediction error[11] and so on. If the delay time τ is known, the time delayed system becomes quite a simple one, and the message encoded by the chaotic signal can be extracted by the common attack method[12]. Therefore, to realize the finite-time synchronization of chaotic systems with parametric uncertainties and time-varying delay, the aforementioned methods are no longer suitable, and a new analytical control scheme should be presented.

Furthermore, it is worth pointing out that all the synchronization methods mentioned above are involved in one drive and one response system. However, in practice, since the configuration of synchronization is limited to a single pair of one way coupled oscillations, this method can not be applied for multiuser communication systems[13]. Since it can further improve the security when the synchronization technique is applied in the secure communication, the technique of multiplexing is

therefore a very important issue for high-capacity communications. In addition, the level of the communication security depends on the complexity level of the drive dynamical system and signal loading scheme. If the driving system involves more chaotic subsystems and the way of combination among subsystems is more complex, then the geometry topological manifold of the compound driving system will become more complex, and the transmitted signal will be separated into several parts carried by different subsystems to improve the anti-attack ability and anti-translated capability. R. Luo proposed the combination synchronization which contains two drive systems and one response system [14]. Recently, the dual synchronization were investigated which involves two drive systems and two response systems [15]. Combining the advantages of the two methods mentioned above, dual combination synchronization which involves four drive systems and two response systems has been put forward [16]. However, the disadvantage of these combination methods is that the subsystems involved must have the same dimension and the dimension of the combined system does not change. In fact, the higher the dimension of chaotic system, the more complex of the chaotic topology. This inspires our work.

By combining k ordinary chaotic systems into one chaotic system (the dimensions of the subsystems can be the same or different), we can obtain a new drive system. Because the compound system involves multiple subsystems and its dimension increases significantly, we call it large-scale hyper chaotic system. Using the same method, we construct another large-scale hyper chaotic system by a group of ordinary chaotic systems as the response system. It is obvious that the problem of the synchronization between these two large-scale hyper chaotic systems are more general and of more practical significance.

Inspired by the discussion above, in this paper, we deal with the finite-time synchronization of large-scale hyper chaotic systems with parametric uncertainties and time-varying delay. On the basic of the finite-time control technique, a nonlinear controller is designed. Furthermore, a sufficient condition of finite-time synchronization in the form of linear matrix inequalities (LMIs) is established. Finally, the numerical simulation is put forward to demonstrate the feasibility and correctness of the advanced scheme.

Compared with the existing literatures, there are two advantages which make the proposed com-

munication scheme attractive. Firstly, both the drive system and the response system are the compound of multiple chaotic subsystems. In the process of secret communication, the transmitted signals can be split into several different parts and loaded in different subsystems to improve the security of signal transmission. In addition, the dimension of the compound system increases significantly, which lead to the chaotic topological manifold becoming more complex. Secondly, by adding a nonlinear term to the traditional linear controller, fast synchronization is effectively achieved.

PROBLEM DESCRIPTION

Taking into account the parametric uncertainties and time-varying delay, we consider a group of chaotic systems as the master dynamics, which is described by

$$
\begin{aligned}
\dot{x}^i(t) = {} & (A^i + \Delta A^i)x^i(t) + (B^i + \Delta B^i)x^i(t - \tau(t)) \\
& + (C^i + \Delta C^i)f^i(x^i(t)) \\
& + (D^i + \Delta D^i)f^i(x^i(t - \tau(t))) \quad (1 \leqslant i \leqslant k),
\end{aligned}
\tag{1}
$$

where $x^i(t) = \left[x_1^i(t) \cdots x_n^i(t) \right]^T \in \mathbb{R}^n$ is the state vector of the ith master system, and $f^i(x^i(\cdot)) = \left[f_1^i(x^i(\cdot)) \cdots f_n^i(x^i(\cdot)) \right]^T \in \mathbb{R}^n$ is continuous nonlinear function, where A^i, B^i, C^i and D^i are known $n \times n$ constant matrices, ΔA^i, ΔB^i, ΔC^i and ΔD^i are parametric uncertainties, and $\tau(t)$ is the transmission time-varying delay.

Correspondingly, the response system is given by

$$
\begin{aligned}
\dot{y}^i(t) = {} & (A^i + \Delta \hat{A}^i)y^i(t) + (B^i + \Delta \hat{B}^i)y^i(t - \tau(t)) \\
& + (C^i + \Delta \hat{C}^i)f^i(y^i(t)) + (D^i + \Delta \hat{D}^i) \\
& \times f^i(y^i(t - \tau(t))) + u^i(t) \quad (1 \leqslant i \leqslant k),
\end{aligned}
\tag{2}
$$

where $y^i(t) = \left[y_1^i(t) \cdots y_n^i(t) \right]^T \in \mathbb{R}^n$ is the state vector of the ith response system, $\Delta \hat{A}^i, \Delta \hat{B}^i, \Delta \hat{C}^i$ and $\Delta \hat{D}^i$ are parametric uncertainties, and $u(t) = \left[u_1(t) \cdots u_n(t) \right]^T \in \mathbb{R}^n$ is the vector of control input.

For convenience, define

$$
\begin{aligned}
x(\cdot) &= [(x^1(\cdot))^T, \dots, (x^k(\cdot))^T]^T, \\
y(\cdot) &= [(y^1(\cdot))^T, \dots, (y^k(\cdot))^T]^T, \\
f(\cdot) &= [(f^1(\cdot))^T, \dots, (f^k(\cdot))^T]^T, \\
u(\cdot) &= [(u^1(\cdot))^T, \dots, (u^k(\cdot))^T]^T, \\
A &= \mathrm{diag}\{A^1, A^2, \dots, A^k\},
\end{aligned}
$$

$$B = \text{diag}\{B^1, B^2, \dots, B^k\},$$
$$C = \text{diag}\{C^1, C^2, \dots, C^k\},$$
$$D = \text{diag}\{D^1, D^2, \dots, D^k\},$$
$$\Delta A = \text{diag}\{\Delta A^1, \cdots, \Delta A^k\},$$
$$\Delta B = \text{diag}\{\Delta B^1, \cdots, \Delta B^k\},$$
$$\Delta C = \text{diag}\{\Delta C^1, \dots, \Delta C^k\},$$
$$\Delta D = \text{diag}\{\Delta D^1, \dots, \Delta D^k\},$$
$$\Delta \tilde{A} = \text{diag}\{\Delta \tilde{A}^1, \dots, \Delta \tilde{A}^k\},$$
$$\Delta \tilde{B} = \text{diag}\{\Delta \tilde{B}^1, \dots, \Delta \tilde{B}^k\},$$
$$\Delta \tilde{C} = \text{diag}\{\Delta \tilde{C}^1, \dots, \Delta \tilde{C}^k\},$$
$$\Delta \tilde{D} = \text{diag}\{\Delta \tilde{D}^1, \dots, \Delta \tilde{D}^k\},$$

and

$$\bar{A} = A + \Delta A, \qquad \bar{B} = B + \Delta B,$$
$$\bar{C} = C + \Delta C, \qquad \bar{D} = D + \Delta D,$$
$$\Delta \tilde{A}^i = \Delta A^i - \Delta \hat{A}^i, \qquad \Delta \tilde{B}^i = \Delta B^i - \Delta \hat{B}^i,$$
$$\Delta \tilde{C}^i = \Delta C^i - \Delta \hat{C}^i, \qquad \Delta \tilde{D}^i = \Delta D^i - \Delta \hat{D}^i.$$

Then, the systems (1) and (2) are reduced respectively in the following forms

$$\dot{x}(t) = \bar{A}x(t) + \bar{B}x(t - \tau(t)) + \bar{C}f(x(t))$$
$$\qquad\qquad + \bar{D}f(x(t - \tau(t))), \qquad (3)$$
$$\dot{y}(t) = (\bar{A} - \Delta \tilde{A})y(t) + (\bar{B} - \Delta \tilde{B})y(t - \tau(t))$$
$$\qquad + (\bar{C} - \Delta \tilde{C})f(y(t))$$
$$\qquad + (\bar{D} - \Delta \tilde{D})f(y(t - \tau(t))) + u(t), \quad (4)$$

where $x(\cdot), y(\cdot), f(\cdot), u(\cdot) \in \mathbb{R}^{\bar{n}}, \bar{A}, \bar{B}, \bar{C}, \bar{D}, \Delta \tilde{A}, \Delta \tilde{B},$ $\Delta \tilde{C}, \Delta \tilde{D} \in \mathbb{R}^{\bar{n} \times \bar{n}}$, and $\bar{n} = kn$.

In order to obtain the main result, some reasonable assumptions and necessary lemmas are introduced below.

Lemma 1 (Ref. 3) *Let $\Lambda \in \mathbb{R}^{n \times n}$ is a positive symmetric matrix, then, for any real matrices $X_1, X_2 \in \mathbb{R}^{n \times m}$, the following inequality holds*

$$X_1^T X_2 + X_2^T X_1 \leqslant X_1^T \Lambda^{-1} X_1 + X_2^T \Lambda X_2.$$

Specially, choosing $\Lambda = \varepsilon I$, we obtain

$$X_1^T X_2 + X_2^T X_1 \leqslant \varepsilon^{-1} X_1^T X_1 + \varepsilon X_2^T X_2,$$

where ε is a positive scalar.

Lemma 2 (Ref. 17) *Assume $V(t)$ is a continuous and positive definite function which satisfies the following differential inequality*

$$\dot{V}(t) \leqslant -b_1 V^\rho(t) - b_2 V(t), \ t \geqslant t_0, \ V(t_0) \geqslant 0$$

where $b_1 > 0, b_2 > 0$ and $0 < c < 1$ are constants. Then, when $V^{1-\rho}(t_0) \leqslant b_1/b_2$, the following results are true

$$V(t) \begin{cases} \leqslant e^{b_2(t-t_0)}[V^{1-\rho}(t_0) + \dfrac{b_1}{b_2} \\ \quad -\dfrac{b_1}{b_2}e^{-b_2(1-\rho)(t-t_0)}]^{1/(1-\rho)} & \text{if } t_0 \leqslant t < T, \\ = 0 & \text{if } t \geqslant T, \end{cases}$$

with T given by

$$T = t_0 + \frac{1}{b_2(1-\rho)} \ln\left(1 + \frac{b_2 V^{1-\rho}(t_0)}{b_1}\right).$$

Assumption 1. For the time-varying delay $\tau(t)$, there exist non-negative constants h and q, i.e.

$$0 \leqslant \tau(t) \leqslant h, \quad 0 \leqslant \dot{\tau}(t) \leqslant q < 1.$$

Assumption 2. Parametric uncertainties ΔA, ΔB, ΔC, ΔD, $\Delta \hat{A}$, $\Delta \hat{B}$, $\Delta \hat{C}$ and $\Delta \hat{D}$ are bounded, and they can be decomposed as

$$\Delta A = H_1 F_1(t) E_1, \qquad \Delta B = H_2 F_2(t) E_2,$$
$$\Delta C = H_3 F_3(t) E_3, \qquad \Delta D = H_4 F_4(t) E_4,$$
$$\Delta \hat{A} = H_1 \hat{F}_1(t) E_1, \qquad \Delta \hat{B} = H_2 \hat{F}_2(t) E_2,$$
$$\Delta \hat{C} = H_3 \hat{F}_3(t) E_3, \qquad \Delta \hat{D} = H_4 \hat{F}_4(t) E_4,$$

where H_j, E_j are known constant matrices with appropriate dimensions, $F_j(t)$ and $\hat{F}_j(t)$ are uncertain matrices satisfying

$$F_j(t)^T F_j(t) \leqslant I, \quad \hat{F}_j(t)^T \hat{F}_j(t) \leqslant I, \quad j = 1, 2, 3, 4.$$

Assumption 3. The difference between the corresponding parametric uncertainties of systems (1) and (2) are norm-bounded, i.e.

$$\left\|\Delta \tilde{A}\right\| \leqslant \alpha_1, \qquad \left\|\Delta \tilde{B}\right\| \leqslant \alpha_2,$$
$$\left\|\Delta \tilde{C}\right\| \leqslant \alpha_3, \qquad \left\|\Delta \tilde{D}\right\| \leqslant \alpha_4,$$

where $\alpha_j > 0$ ($j = 1, 2, 3, 4$) are known real constants, and $\|\cdot\|$ represents the 2-norm.

Assumption 4. The parametric uncertainties and the time-varying delay do not destroy the chaotic behavior of the drive and response systems.

Definition 1 For the large-scale drive system (3) and the large-scale response system (4), it is said that these two systems are complete synchronization, if

$$\lim_{t \to \infty} \|x(t) - y(t)\| = 0. \qquad (5)$$

Definition 2 For the large-scale drive system (3) and the large-scale response system (4), if there exists a constant $T > 0$, such that

$$
\begin{cases}
\lim_{t \to T} \|x(t) - y(t)\| = 0, \\
\|x(t) - y(t)\| = 0 \quad \text{if } t \geqslant T,
\end{cases}
\tag{6}
$$

then it is said that these two large-scale hyper chaotic systems are finite-time synchronization.

Assumption 5. The nonlinear functions $f_i(\cdot)$ are bounded, and satisfy the Lipschitz condition, namely there exist positive constants $l_1, \cdots, l_{\bar{n}}$ such that

$$
|f_i(a) - f_i(b)| \leqslant l_i |a - b|, \quad i = 1, 2, \cdots, \bar{n}.
$$

Denote $l = \max\{l_1, \cdots, l_{\bar{n}}\}$, it follows that

$$
\|f(x(\cdot)) - f(y(\cdot))\| \leqslant l \|x(\cdot) - y(\cdot)\| = l \|e(\cdot)\|.
$$

The main object in this paper is to design an appropriate controller $u(t)$ to guarantee the large-scale hyper chaotic systems (3) and (4) are finite-time synchronization.

We define the synchronization error vector

$$
e(t) = x(t) - y(t).
\tag{7}
$$

Combining systems (3),(4) with the error vector (7), the following error dynamical system is obtained

$$
\begin{aligned}
\dot{e}(t) &= \dot{x}(t) - \dot{y}(t) \\
&= \bar{A}x(t) + \bar{B}x(t - \tau(t)) + \bar{C}f(x(t)) \\
&\quad + \bar{D}f(x(t - \tau(t))) - [(\bar{A} - \Delta\tilde{A})y(t) \\
&\quad + (\bar{B} - \Delta\tilde{B})y(t - \tau(t)) + (\bar{C} - \Delta\tilde{C})f(y(t)) \\
&\quad + (\bar{D} - \Delta\tilde{D})f(y(t - \tau(t))) + u(t)],
\end{aligned}
$$

that is

$$
\begin{aligned}
\dot{e}(t) &= \bar{A}e(t) + \bar{B}e(t - \tau(t)) + \bar{C}[f(x(t)) \\
&\quad - f(y(t))] + \bar{D}[f(x(t - \tau(t))) \\
&\quad - f(y(t - \tau(t)))] + \Delta\tilde{A}y(t) \\
&\quad + \Delta\tilde{B}y(t - \tau(t)) + \Delta\tilde{C}f(y(t)) \\
&\quad + \Delta\tilde{D}f(y(t - \tau(t))) - u(t),
\end{aligned}
\tag{8}
$$

based on what will be proposed in the main results.

MAIN RESULTS

It is clear that the finite-time synchronization of these two large-scale hyper chaotic systems is directly equivalent to the finite-time stabilization of the error system (8). In this section, we will pay our attention to design a suitable feedback controller to ensure the error trajectory converges to zero within a finite time. Without loss of generality, we employ the time-delay state feedback controller written by

$$
\begin{aligned}
u(t) &= K_1 e(t) + K_2 e(t - \tau(t)) \\
&\quad + \operatorname{sgn} e(t) [\alpha_1 Y(t) + \alpha_2 Y(t - \tau(t)) \\
&\qquad + \alpha_3 F(y(t)) + \alpha_4 F(y(t - \tau(t)))] \\
&\quad + \frac{\gamma}{2} \Big[e^T(t) P e(t) \\
&\qquad + \int_{t-\tau(t)}^{t} e^T(s) Q e^{-\beta(t-s)} e(s) ds \Big]^{\eta} \frac{Pe(t)}{\|Pe(t)\|^2},
\end{aligned}
\tag{9}
$$

with

$$
\begin{cases}
Y(t) = [|y_1(t)|, \cdots, |y_{\bar{n}}(t)|]^T, \\
Y(t - \tau(t)) = [|y_1(t - \tau(t))|, \\
\qquad\qquad \dots, |y_{\bar{n}}(t - \tau(t))|]^T, \\
F(y(t)) = [|f_1(y(t))|, \cdots, |f_{\bar{n}}(y(t))|]^T, \\
F(y(t - \tau(t))) = [|f_1(y(t - \tau(t)))|, \\
\qquad\qquad \dots, |f_{\bar{n}}(y(t - \tau(t)))|]^T, \\
\operatorname{sgn} e(t) = \operatorname{diag}\{\operatorname{sgn} e_1(t), \cdots, \operatorname{sgn} e_{\bar{n}}(t)\},
\end{cases}
\tag{10}
$$

where K_1, K_2 P and Q are control gain matrices with appropriate dimensions, $\gamma > 0$ and $0 < \eta < 1$ are known constants.

Remark 1 According to Eqs. (9), the control input $u(t)$ contains the factor $Pe(t)/\|Pe(t)\|^2$. In fact, when the error trajectory $e(t) = 0$, it is obvious that $\|Pe(t)\|^2 = 0$, which means $Pe(t)/\|Pe(t)\|^2$ is singular. In order to overcome this disadvantage, the control law (9) is modified as follows

$$
\begin{aligned}
u(t) &= K_1 e(t) + K_2 e(t - \tau(t)) \\
&\quad + \operatorname{sgn} e(t) [\alpha_1 Y(t) + \alpha_2 Y(t - \tau(t)) \\
&\qquad + \alpha_3 F(y(t)) + \alpha_4 F(y(t - \tau(t)))] \\
&\quad + \frac{\gamma}{2} \Big[e^T(t) P e(t) \\
&\qquad + \int_{t-\tau(t)}^{t} e^T(s) Q e^{-\beta(t-s)} e(s) ds \Big]^{\eta} \Delta
\end{aligned}
\tag{11}
$$

with

$$
\Delta = \begin{cases}
\frac{Pe(t)}{\|Pe(t)\|^2} & \text{if } \|Pe(t)\| \geqslant \delta, \\
0 & \text{if } \|Pe(t)\| < \delta,
\end{cases}
\tag{12}
$$

where the switching gain δ is a sufficiently small positive constant which can be chosen according to the designer's requirement.

Remark 2 Another effective approach is using the function $Pe(t)/(\|Pe(t)\|^2 + \xi)$ to approximate $Pe(t)/\|Pe(t)\|^2$, where ξ is a small positive constant. Then the control law (9) is modified as

$$
\begin{aligned}
u(t) = {} & K_1 e(t) + K_2 e(t - \tau(t)) \\
& + \operatorname{sgn} e(t)[\alpha_1 Y(t) + \alpha_2 Y(t - \tau(t)) \\
& \qquad + \alpha_3 F(y(t)) + \alpha_4 F(y(t - \tau(t)))] \\
& + \frac{\gamma}{2}\Big[e^T(t) P e(t) \\
& + \int_{t-\tau(t)}^{t} e^T(s) Q e^{-\beta(t-s)} e(s) ds \Big]^{\eta} \frac{P e(t)}{\|P e(t)\|^2 + \xi}.
\end{aligned}
\tag{13}
$$

Based on the above controller, a sufficient condition of the finite-time synchronization in terms of LMIs is established as follows.

Theorem 1 *If there exist symmetrical positive matrices P, Q, R, T, positive diagonal matrices M, N, control gain matrices K_1, K_2 with appropriate dimensions, and positive real scalars $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$, such that the following LMI holds*

$$
\tilde{\Omega} = \begin{bmatrix}
\tilde{\Omega}_1 & \tilde{\Omega}_2 & \tilde{\Omega}_3 & \tilde{\Omega}_4 & \tilde{\Omega}_5 \\
* & -\varepsilon_1 I & 0 & 0 & 0 \\
* & * & -\varepsilon_2 I & 0 & 0 \\
* & * & * & -\varepsilon_3 I & 0 \\
* & * & * & * & -\varepsilon_4 I
\end{bmatrix} < 0
\tag{14}
$$

where

$$
\tilde{\Omega}_1 = \begin{bmatrix}
\tilde{\Omega}_{11} & \tilde{\Omega}_{12} & PC & PD \\
* & \tilde{\Omega}_{22} & 0 & 0 \\
* & * & \tilde{\Omega}_{33} & 0 \\
* & * & * & \tilde{\Omega}_{44}
\end{bmatrix}
$$

$$
\tilde{\Omega}_2 = \left[H_1^T P, 0, 0, 0 \right]^T,
$$
$$
\tilde{\Omega}_3 = \left[H_2^T P, 0, 0, 0 \right]^T,
$$
$$
\tilde{\Omega}_4 = \left[H_3^T P, 0, 0, 0 \right]^T,
$$
$$
\tilde{\Omega}_5 = \left[H_4^T P, 0, 0, 0 \right]^T,
$$

and

$$
\begin{cases}
\tilde{\Omega}_{11} = PA + A^T P - 2R + Q + l^2 M \\
\qquad\qquad\qquad + \beta P + \varepsilon_1 E_1^T E_1, \\
\tilde{\Omega}_{12} = PB - T, \\
\tilde{\Omega}_{22} = l^2 N - \eta e^{-\beta h} Q + \varepsilon_2 E_2^T E_2, \\
\tilde{\Omega}_{33} = -M + \varepsilon_3 E_3^T E_3, \\
\tilde{\Omega}_{44} = -N + \varepsilon_4 E_4^T E_4, \\
R = PK_1, \\
T = PK_2, \\
\eta = 1 - q,
\end{cases}
\tag{15}
$$

here $*$ denotes the symmetrical block in one symmetrical matrix.

Then the large-scale hyper chaotic systems (3) and (4) are finite-time synchronization under the feedback controller (9).

Proof: Choose the following Lyapunov-Krasovskii type function

$$
V(t) = V_1(t) + V_2(t),
\tag{16}
$$

with

$$
V_1(t) = e^T(t) P e(t),
\tag{17}
$$

$$
V_2(t) = \int_{t-\tau(t)}^{t} e^T(s) Q e^{-\beta(t-s)} e(s) ds.
\tag{18}
$$

Differentiating $V_1(t)$ along the error system, we obtain

$$
\begin{aligned}
\dot{V}_1(t) = {} & 2 e^T P \dot{e} \\
= {} & 2 e^T P \{ \bar{A} e(t) + \bar{B} e(t - \tau(t)) + \bar{C}[f(x(t)) \\
& - f(y(t))] + \bar{D}[f(x(t - \tau(t))) \\
& - f(y(t - \tau(t)))] + \Delta \tilde{A} y(t) + \Delta \tilde{B} y(t - \tau(t)) \\
& + \Delta \tilde{C} f(y(t)) + \Delta \tilde{D} f(y(t - \tau(t))) - u(t) \}.
\end{aligned}
$$

Submitting (9) into (8) and using the fact

$$
e^T(t) P \frac{P e(t)}{\|P e(t)\|^2} = 1,
$$

we get

$$
\begin{aligned}
\dot{V}_1(t) = {} & 2 e^T(t) P(\bar{A} - K_1) e(t) + 2 e^T(t) P(\bar{B} - K_2) \\
& \times e(t - \tau(t)) + 2 e^T P \bar{C}[f(x(t)) - f(y(t))] \\
& + 2 e^T(t) P \bar{D}[f(x(t - \tau(t))) \\
& - f(y(t - \tau(t)))] + 2 e^T(t) P \\
& \times [\Delta \tilde{A} y(t) + \Delta \tilde{B} y(t - \tau(t)) + \Delta \tilde{C} \\
& \qquad f(y(t)) + \Delta \tilde{D} f(y(t - \tau(t)))] \\
& - 2 e^T(t) P \operatorname{sgn} e(t)[\alpha_1 Y(t) + \alpha_2 Y(t - \tau(t)) \\
& + \alpha_3 F(y(t)) + \alpha_4 F(y(t - \tau(t)))] \\
& - \gamma V^{\eta}(t).
\end{aligned}
\tag{19}
$$

Using Assumption 3 and the fact

$$
\begin{aligned}
& y(t) \leqslant Y(t), \quad f(y(t)) \leqslant F(y(t)), \\
& y(t - \tau(t)) \leqslant Y(t - \tau(t)), \\
& f(y(t - \tau(t))) \leqslant F(y(t - \tau(t))), \\
& Y(t) \geqslant 0, \quad Y(t - \tau(t)) \geqslant 0, \\
& F(y(t)) \geqslant 0, \quad F(y(t - \tau(t))) \geqslant 0,
\end{aligned}
$$

$$e^T(t)\operatorname{sgn} e(t) = [|e_1(t)|, \ldots, |e_{\bar{n}}(t)|]^T,$$

we obtain the following inequations

$$e^T(t)P\Delta\tilde{A}y(t) \leqslant \alpha_1 e^T(t)\operatorname{sgn} e(t)PY(t),$$
$$e^T(t)P\Delta\bar{B}y(t-\tau(t))$$
$$\leqslant \alpha_2 e^T(t)\operatorname{sgn} e(t)PY(t-\tau(t)),$$
$$e^T(t)P\Delta\tilde{C}f(y(t)) \leqslant \alpha_3 e^T(t)\operatorname{sgn} e(t)PF(y(t)),$$
$$e^T(t)P\Delta\tilde{D}f(y(t-\tau(t)))$$
$$\leqslant \alpha_4 e^T(t)\operatorname{sgn} e(t)PF(y(t-\tau(t))),$$

which yields

$$\dot{V}_1(t) \leqslant 2e^T(t)P(\bar{A}-K_1)e(t)$$
$$+2e^T(t)P(\bar{B}-K_2)e(t-\tau(t))$$
$$+2e^T(t)P\bar{C}[f(x(t))-f(y(t))]$$
$$+2e^T(t)P\bar{D}[f(x(t-\tau(t)))$$
$$-f(y(t-\tau(t)))]-\gamma V^\eta(t).$$

Applying Lemma 1, we obtain the following relationships

$$2e^T(t)P\bar{C}[f(x(t))-f(y(t))]$$
$$\leqslant e^T(t)P\bar{C}M^{-1}\bar{C}^T Pe(t)+[f(x(t))$$
$$-f(y(t))]^T M[f(x(t))-f(y(t))],$$
$$2e^T(t)P\bar{D}[f(x(t-\tau(t)))-f(y(t-\tau(t)))]$$
$$\leqslant e^T(t)P\bar{D}N^{-1}\bar{D}^T Pe(t)+[f(x(t-\tau(t)))$$
$$-f(y(t-\tau(t)))]^T N[f(x(t-\tau(t)))$$
$$-f(y(t-\tau(t)))].$$

where M and N are positive diagonal matrices. So we have

$$\dot{V}_1(t) \leqslant e^T(t)[P(\bar{A}-K_1)+(\bar{A}-K_1)^T P]e(t)$$
$$+2e^T(t)P(\bar{B}-K_2)e(t-\tau(t))$$
$$+e^T(t)P\bar{C}M^{-1}\bar{C}^T Pe(t)+[f(x(t))$$
$$-f(y(t))]^T M[f(x(t))-f(y(t))]e^T(t)P$$
$$+\bar{D}N^{-1}\bar{D}^T Pe(t)$$
$$+[f(x(t-\tau(t)))-f(y(t-\tau(t)))]^T$$
$$\times N[f(x(t-\tau(t)))-f(y(t-\tau(t)))]$$
$$-\gamma V^\eta(t). \qquad (20)$$

Applying Assumption 5, we obtain

$$[f(x(t))-f(y(t))]^T M[f(x(t))-f(y(t))]$$
$$\leqslant l^2 e^T(t)Me(t),$$

$$[f(x(t-\tau(t)))-f(y(t-\tau(t)))]^T N$$
$$\times [f(x(t-\tau(t)))-f(y(t-\tau(t)))]$$
$$\leqslant l^2 e^T(t-\tau(t))Ne(t-\tau(t)).$$

Then, (20) is reduced to

$$\dot{V}_1(t) \leqslant e^T(t)[P(\bar{A}-K_1)+(\bar{A}-K_1)^T P]e(t)$$
$$+2e^T(t)P(\bar{B}-K_2)e(t-\tau(t))$$
$$+e^T(t)P\bar{C}M^{-1}\bar{C}^T Pe(t)+e^T(t)P$$
$$\times \bar{D}N^{-1}\bar{D}^T Pe(t)+l^2 e^T(t)Me(t)$$
$$+l^2 e^T(t-\tau(t))Ne(t-\tau(t))-\gamma V^\eta(t).$$

This is equivalent to

$$\dot{V}_1(t) \leqslant e^T(t)[P(\bar{A}-K_1)+(\bar{A}-K_1)^T P$$
$$+P\bar{C}M^{-1}\bar{C}^T P+P\bar{D}N^{-1}\bar{D}^T P$$
$$+l^2 M+\beta P]e(t)+e^T(t-\tau(t))$$
$$\times (l^2 N)e(t-\tau(t))+2e^T(t)P(\bar{B}-K_2)$$
$$\times e(t-\tau(t))-\beta V_1(t)-\gamma V^\eta(t). \quad (21)$$

Notice $V_2(t)$ can be rewritten as

$$V_2(t) = e^{-\beta t}\int_{t-\tau(t)}^t e^T(s)Qe^{\beta s}e(s)ds, \qquad (22)$$

the time derivative of $V_2(t)$ along the trajectory of the error system (8) is obtained as follows

$$\dot{V}_2(t) = -\beta e^{-\beta t}\int_{t-\tau(t)}^t e^T(s)Qe^{\beta s}e(s)ds$$
$$+e^{-\beta t}e^T(t)Qe^{\beta t}e(t)$$
$$-(1-\dot{\tau}(t))e^{-\beta t}e^T(t-\tau(t))Q$$
$$\times e^{\beta(t-\tau(t))}e(t-\tau(t))$$
$$= -\beta V_2(t)+e^T(t)Qe(t)-(1-\dot{\tau}(t))$$
$$\times e^T(t-\tau(t))Qe^{-\beta\tau(t)}e(t-\tau(t)).$$

Since

$$0 \leqslant \tau(t) \leqslant h, \quad 0 \leqslant \dot{\tau}(t) \leqslant q, \quad \eta = 1-q,$$

we have

$$-(1-\dot{\tau}(t)) \leqslant -\eta, \quad -e^{-\beta\tau(t)} \leqslant -e^{-\beta h}.$$

Then it follows by

$$-(1-\dot{\tau}(t))e^{-\beta\tau(t)} \leqslant -\eta e^{-\beta h},$$

which yields

$$\dot{V}_2(t) \leqslant -\beta V_2(t)+e^T(t)Qe(t)$$
$$-\eta e^T(t-\tau(t))Qe^{-\beta h}e(t-\tau(t)). \quad (23)$$

Combining (21) and (23), we have

$$
\begin{aligned}
\dot{V}(t) \leqslant e^T(t)[&P(\bar{A}-K_1)+(\bar{A}-K_1)^T P \\
&+ P\bar{C}M^{-1}\bar{C}^T P + P\bar{D}N^{-1}\bar{D}^T P \\
&+ Q + l^2 M + \beta P]e(t) \\
&+ e^T(t-\tau(t))(l^2 N - \eta e^{-2\beta h}Q)e(t-\tau(t)) \\
&+ 2e^T(t)P(\bar{B}-K_2)e(t-\tau(t)) \\
&- \beta V(t) - \gamma V^\eta(t).
\end{aligned} \quad (24)
$$

For simplicity, we denote

$$
\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ * & \Phi_{22} \end{bmatrix},
$$

with

$$
\begin{aligned}
\Phi_{11} = &P(\bar{A}-K_1)+(\bar{A}-K_1)^T P \\
&+ P\bar{C}M^{-1}\bar{C}^T P + P\bar{D}N^{-1}\bar{D}^T P \\
&+ Q + l^2 M + \beta P, \\
\Phi_{12} = &P(\bar{B}-K_2), \\
\Phi_{22} = &l^2 N - \eta e^{-\beta h}Q.
\end{aligned}
$$

Then (24) is reduced to

$$
\begin{aligned}
\dot{V}(t) \leqslant & \begin{pmatrix} e(t) \\ e(t-\tau(t)) \end{pmatrix}^T \Phi \begin{pmatrix} e(t) \\ e(t-\tau(t)) \end{pmatrix} \\
& - \beta V(t) - \gamma V^\eta(t). \quad (25)
\end{aligned}
$$

It is obvious that, if $\Phi < 0$ then $\dot{V}(t) < -\beta V(t) - \gamma V^\eta(t)$.

Applying Shure complement, we obtain that $\Phi < 0$ if and only if

$$
\bar{\Omega} = \begin{bmatrix} \bar{\Omega}_{11} & \bar{\Omega}_{12} & P\bar{C} & P\bar{D} \\ * & \bar{\Omega}_{22} & 0 & 0 \\ * & * & -M & 0 \\ * & * & * & -N \end{bmatrix} < 0, \quad (26)
$$

where

$$
\begin{aligned}
\bar{\Omega}_{11} = &P(\bar{A}-K_1)+(\bar{A}-K_1)^T P + Q + l^2 M \\
&+ \beta P, \\
\bar{\Omega}_{12} = &\Phi_{12} = P(\bar{B}-K_2), \\
\bar{\Omega}_{22} = &\Phi_{22} = l^2 N - \eta e^{-\beta h}Q.
\end{aligned}
$$

Combined with $(F_j(t))^T F_j(t) \leqslant I$, $j = 1,2,3,4$, based on the Lemma 1 and Shure complement, we obtain a sufficient condition of $\bar{\Omega} < 0$ as follows

$$
\tilde{\Omega} = \begin{bmatrix} \tilde{\Omega}_1 & \tilde{\Omega}_2 & \tilde{\Omega}_3 & \tilde{\Omega}_4 & \tilde{\Omega}_5 \\ * & -\varepsilon_1 I & 0 & 0 & 0 \\ * & * & -\varepsilon_2 I & 0 & 0 \\ * & * & * & -\varepsilon_3 I & 0 \\ * & * & * & * & -\varepsilon_4 I \end{bmatrix} \quad (27)
$$

where

$$
\tilde{\Omega}_1 = \begin{bmatrix} \tilde{\Omega}_{11} & \tilde{\Omega}_{12} & PC & PD \\ * & \tilde{\Omega}_{22} & 0 & 0 \\ * & * & \tilde{\Omega}_{33} & 0 \\ * & * & * & \tilde{\Omega}_{44} \end{bmatrix}
$$

$$
\begin{aligned}
\tilde{\Omega}_2 &= \begin{bmatrix} H_1^T P, 0, 0, 0 \end{bmatrix}^T, \\
\tilde{\Omega}_3 &= \begin{bmatrix} H_2^T P, 0, 0, 0 \end{bmatrix}^T, \\
\tilde{\Omega}_4 &= \begin{bmatrix} H_3^T P, 0, 0, 0 \end{bmatrix}^T, \\
\tilde{\Omega}_5 &= \begin{bmatrix} H_4^T P, 0, 0, 0 \end{bmatrix}^T,
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\Omega}_{11} = &P(A-K_1)+(A-K_1)^T P + Q + l^2 M \\
&+ \beta P + \varepsilon_1 E_1{}^T E_1, \\
\tilde{\Omega}_{12} = &P(B-K_2), \\
\tilde{\Omega}_{22} = &l^2 N - \eta e^{-\beta h}Q + \varepsilon_2 E_2{}^T E_2, \\
\tilde{\Omega}_{33} = &-M + \varepsilon_3 E_3{}^T E_3, \\
\tilde{\Omega}_{44} = &-N + \varepsilon_4 E_4{}^T E_4.
\end{aligned}
$$

To obtain the LMI-based conditions, we set

$$
R = PK_1, \quad T = PK_2,
$$

it is clear that if the matrix P is invertible, the control gains K_1, K_2 can be calculated by

$$
K_1 = P^{-1}R, \quad K_2 = P^{-1}T.
$$

Based on the above discussion, we conclude that

$$
\begin{aligned}
\tilde{\Omega} < 0 \quad &\Rightarrow \quad \bar{\Omega} < 0 \\
&\Rightarrow \quad \Phi < 0 \\
&\Rightarrow \quad \dot{V}(t) < -\beta V(t) - \gamma V^\eta(t). \quad (28)
\end{aligned}
$$

Applying Lemma 2, the large-scale chaotic systems (3) and (4) are finite-time synchronization when the LMI (14) holds, and T is given by

$$
T = \frac{1}{\beta(1-\eta)} \ln\left(1 + \frac{\beta V^{1-\eta}(0)}{\gamma}\right). \quad (29)
$$

Hence, the proof is completed. □

NUMERICAL SIMULATION

In our simulation, Lorenz system and Chen system with time-varying delay are chosen as the drive systems, which is described by

$$
\begin{aligned}
\dot{x}(t) = &(A+\Delta A)x(t)+(B+\Delta B)x(t-\tau(t)) \\
&+ (C+\Delta C)f(x(t)) \\
&+ (D+\Delta D)f(x(t-\tau(t))), \quad (30)
\end{aligned}
$$
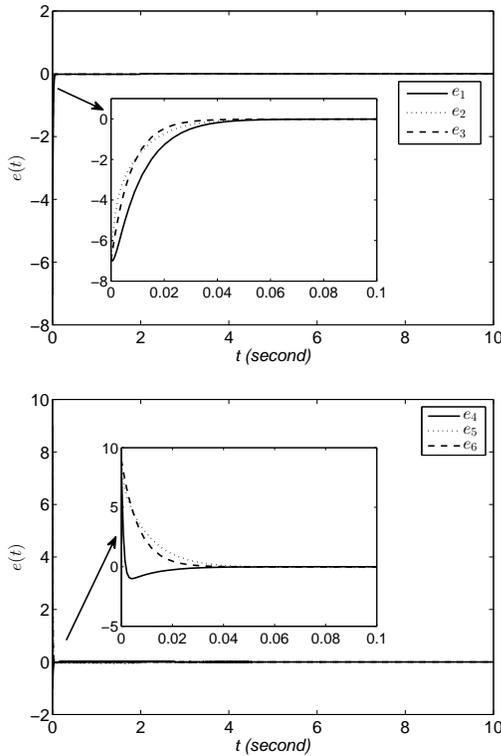
Fig. 1 Time response of the synchronization errors $e_i(t)$ with $\beta = \gamma = 2$, (a) $i = 1,2,3$, (b) $i = 4,5,6$.
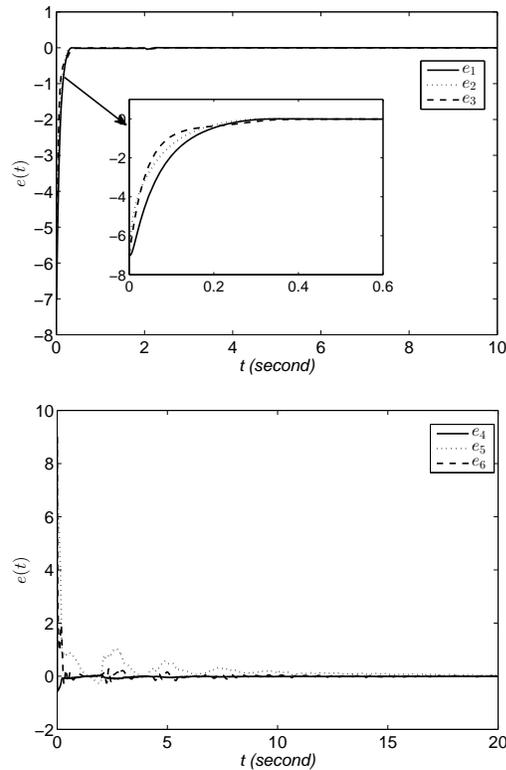
Fig. 2 Time response of the synchronization errors $e_i(t)$ with $\beta = \gamma = 0$, (a) $i = 1,2,3$, (b) $i = 4,5,6$.

Meanwhile, the identical system with different parametric uncertainties and different initial value are considered as the response systems, which is described by

$$\dot{y}(t) = (A + \Delta\hat{A})y(t) + (B + \Delta\hat{B})y(t - \tau(t))$$
$$+ (C + \Delta\hat{C})f(y(t))$$
$$+ (D + \Delta\hat{D})f(y(t - \tau(t))) + u(t), \quad (31)$$

where

$$A = \begin{bmatrix} -10 & 10 & 0 & 0 & 0 & 0 \\ 28 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -8/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -35 & 35 & 0 \\ 0 & 0 & 0 & -7 & 28 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$$f(x(\cdot)) = \begin{bmatrix} 0, -x_1(\cdot)x_3(\cdot), x_1(\cdot)x_2(\cdot), 0, \\ -x_4(\cdot)x_6(\cdot), x_4(\cdot)x_5(\cdot) \end{bmatrix}^T,$$

and the parametric uncertainties are given as

$$\Delta A = \text{diag}\{0.1\sin t, \ -0.1\sin 3t, \ 0.1\sin t,$$
$$0.1\cos 2t, \ 0.1\cos 4t, \ 0.1\sin t\},$$

$$\Delta B = \text{diag}\{0.01\cos 2t, \ 0.01\sin t, \ -0.01$$
$$\sin 2t, \ 0.01\cos t, \ 0.01\cos 2t, \ -0.01\cos t\},$$

$$\Delta C = \text{diag}\{-0.01\cos 3t, \ -0.01\cos 3t, \ 0.01\sin t,$$
$$-0.01\cos 3t, \ 0.01\cos t, \ 0.01\sin 3t\},$$

$$\Delta D = \text{diag}\{0.04\sin t, \ -0.04\sin t, \ 0.04\sin 3t,$$
$$0.04\cos t, \ 0.04\sin t, \ 0.04\cos 2t\},$$

and

$$\Delta\hat{A} = \text{diag}\{-0.05\sin t, -0.06\sin 3t, -0.03$$
$$\cos 2t, 0.08\sin 2t, -0.12\sin 4t, 0.2\sin 2t\},$$

$$\Delta\hat{B} = \text{diag}\{0.02\cos t, 0.01\sin 2t, 0.01\cos 2t,$$
$$-0.02\cos t, 0.01\sin 2t, -0.02\cos t\},$$

$$\Delta\hat{C} = \text{diag}\{0.02\sin 3t, -0.01\cos t, 0.02\sin t,$$
$$-0.04\cos 2t, 0.01\cos t, -0.01\cos 2t\},$$

$$\Delta\hat{D} = \text{diag}\{0.04\sin t, 0.02\cos 2t, 0.04\sin 2t,$$
$$0.02\sin 2t, 0.04\cos t, -0.01\cos 2t\},$$

then, according to Assumption 2 we have

$$H_1 = 0.1I, \ H_2 = 0.1I, \ H_3 = 0.1I, \ H_4 = 0.2I,$$

$E_1 = I,\ E_2 = 0.1I,\ E_3 = 0.1I,\ E_4 = 0.2I.$

and

$$F_1(t) = \text{diag}\{\sin t, -\sin 3t, \sin t, \cos 2t,$$
$$\cos 4t, \sin t\},$$
$$F_2(t) = \text{diag}\{\cos 2t, \sin t, -\sin 2t, \cos t,$$
$$\cos 2t, -\cos t\},$$
$$F_3(t) = \text{diag}\{-\cos 3t, -\cos 3t, \sin t, -\cos 3t,$$
$$\cos t, \sin 3t\},$$
$$F_4(t) = \text{diag}\{\sin t, -\sin t, \sin 3t, \cos t,$$
$$\sin t, \cos 2t\}.$$

Choosing $\beta = 2, \gamma = 2, \eta = 0.5$, using the MATLAB LMI Toolbox, we obtain

$$K_1 = \begin{bmatrix} K_{11} & 0 \\ 0 & K_{12} \end{bmatrix}, \quad K_2 = \begin{bmatrix} K_{21} & 0 \\ 0 & K_{22} \end{bmatrix},$$

where

$$K_{11} = \begin{bmatrix} 152.1597 & -248.2540 & 0 \\ -229.9479 & 591.3069 & 0 \\ 0 & 0 & 86.5369 \end{bmatrix},$$

$$K_{12} = \begin{bmatrix} 875.8940 & 214.3962 & 0 \\ 151.3313 & 117.3855 & 0 \\ 0 & 0 & 93.1439 \end{bmatrix},$$

$$K_{21} = \begin{bmatrix} 0.6639 & -1.0658 & 0 \\ -0.7290 & 2.7139 & 0 \\ 0 & 0 & 0.3488 \end{bmatrix},$$

$$K_{22} = \begin{bmatrix} 3.0586 & 4.8151 & 0 \\ 0.4773 & 3.0556 & 0 \\ 0 & 0 & 0.3683 \end{bmatrix}.$$

In the simulation, the drive system is started with $x(0) = (1, 1, 1, 1, 1, 1)$ and the response system is initialized with $y(0) = (8, 8, 8, -8, -8, -8)$, the delay time $\tau(t) = 2 + (\cos^3 t)/6$. Using the controller (13) with $\xi = 0.01$, the synchronization error is revealed in Fig. 1. It is obvious that the synchronization error converges to zero in a short time.

When we choose the parameters $\beta = \gamma = 0$, the controller (13) will reduce as an ordinary linear controller

$$u(t) = K_1 e(t) + K_2 e(t - \tau(t))$$
$$+ \text{sgn}\, e(t)[\alpha_1 Y(t) + Y(t - \tau(t))$$
$$+ \alpha_3 F(y(t)) + \alpha_4 F(y(t - \tau(t)))],$$

then, the inequality

$$\dot{V}(t) \leqslant \begin{pmatrix} e(t) \\ e(t - \tau(t)) \end{pmatrix}^T \Phi \begin{pmatrix} e(t) \\ e(t - \tau(t)) \end{pmatrix}$$
$$- \beta V(t) - \gamma V^\eta(t)$$

reduces to

$$\dot{V}(t) \leqslant \begin{pmatrix} e(t) \\ e(t - \tau(t)) \end{pmatrix}^T \Phi \begin{pmatrix} e(t) \\ e(t - \tau(t)) \end{pmatrix}.$$

Once the LMI (14) in Theorem 1 holds, we have $\Phi < 0$, which yield $\dot{V}(t) < 0$, applying Lyapunov stability theory, we derive the error system (8) is asymptotically stable. The simulation result is show in Fig. 2. In addition, the comparison between Fig. 1 and 2 shows that the control technique in this paper can effectively shorten the synchronization time, which verifies the superiority of the of the proposed method.

CONCLUSION

In this paper, we dealt with the finite-time synchronization of two large-scale hyper chaotic systems with parametric uncertainties and time-varying delay. Using the finite-time control technique and Lyapunov stability theory, we designed a suitable time-delay state feedback controller and put forward a sufficient finite-time synchronization condition in the form of LMI. The numerical simulation verified the correctness and effectiveness of the proposed scheme. In the further work, we will introduce the event-triggering mechanism based on the result of this paper to improve the ability of data filtering in the process of network synchronization and save the burden of the network.

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[index heading!index subheading]:
finite-time control technique; combined chaotic system; uncertain parameter; nonlinear control