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Exponential bounds via Stein's method and exchangeable pairs

Patcharee Sumritnorrapong, Kritsana Neammanee, Jiraphan Suntornchost*

Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok 10330 Thailand

*Corresponding author, e-mail: Jiraphan.S@chula.ac.th

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ABSTRACT: We apply Stein's method to obtain a non-uniform exponential bound for normal approximations for certain types of random variables. In particular, we establish the bound for a random variable such that its exchangeable pair coupling exists and the distance between the pair is bounded. Using our result, we obtain better bounds for a wide range of applications, such as sums of bounded independent random variables, the combinatorial central limit theorem, and the numbers of descents and inversions of a permutation.

KEYWORDS: combinatorial central limit theorem, descent, inversion, non-uniform bound

MSC2010: 60F05

INTRODUCTION

The central limit theorem and Berry-Esseen theorem play important roles in probability theory and statistics. The theorems guarantee that a certain distribution can be asymptotically approximated by a normal distribution. Several techniques to deal with different variations of the central limit theorem and the Berry-Esseen theorem have been proposed. For example, the Laplace Fourier method can be used to prove the classical central limit theorem, and many significant results related to the limit theorem and Berry-Esseen bounds arise from this method. However, it is quite difficult to apply the method to the case of dependent random variables. The breakthrough finally came from Stein[1]. He introduced an alternative approach to the Berry-Esseen normal approximation which is known as Stein's method. His idea is to replace the distance between the characteristic functions in the Fourier analytic method with the distance between the target random variable and the normal distribution. In particular, he showed that $Z \sim \mathcal{N}(0,1)$ if and only if

$$E[Zf(Z) - f'(Z)] = 0,$$

for any absolutely continuous function f for which this expectation exists.

There are many advantages of Stein's method over the classical Fourier analytic approach. For instance, Stein's method can directly yield the error bound for the normal approximation even when the random variables are not identically distributed or independent. Consequently, Stein's method and its extensions and variations have opened up many applications in distribution approximations such as with the multivariate normal approximation[2]. Reviews of Stein's method and its extensions can be found Refs. 3, 4.

Generally, there are three widely used approaches in Stein's method which are the concentration inequality approach, the inductive approach, and the coupling approach. The coupling approach is one of the most important tools in Stein's method. The key idea of this approach is to construct a new random variable coupling with the target random variable. There are three different couplings we can make which are exchangeable pair[5,6], zero bias[7,8] and size bias[9,10].

The setting of the exchangeable pair coupling is as follows. For a random variable W, we say that the pair (W, W') is an exchangeable pair if for all measurable sets B and B', $P(W \in B, W' \in B') = P(W \in B', W' \in B)$. The exchangeable pair (W, W') is said to be λ-Stein pair if[6]

$$E^W(W - W') = \lambda W, \tag{1}$$

where $0 < \lambda < 1$. Generalizations of the λ-Stein pair construction can be found in Refs. 11–13.

Rinott and Rotar[11] gave a uniform bound for the error of the normal approximation via the distance

of exchangeable pair random variables satisfying (1) and $|W'-W| \leqslant A$ for some constant A. In particular, for any real number x, they found that

$$|P(W \leqslant x)-\Phi(x)| \leqslant \frac{12}{\lambda}\sqrt{\mathrm{Var}(E^W(W'-W)^2)}+\frac{48A^3}{\lambda}+\frac{8A^2}{\sqrt{\lambda}}. \quad (2)$$

This result was then used to obtain uniform bounds on the normal approximation in various applications. For example, W can be the numbers of descents and inversions of a permutation[14,15].

Shao and Su[16] established a uniform bound similar to (2) as follows.

$$|P(W \leqslant x)-\Phi(x)| \leqslant \frac{1}{2\lambda}\sqrt{\mathrm{Var}(E^W(W'-W)^2)}+\frac{0.41A^3}{\lambda}+1.5A. \quad (3)$$

Much effort has been put into obtaining non-uniform bounds for normal approximations in different settings. Chuntee and Neammanee[15] established non-uniform polynomial bounds for the num-bers of descents and inversions of a permutation, and then improved the non-uniform bounds to non-uniform exponential bounds[17]. In this work, we further improve their technique to obtain a better non-uniform exponential bound for normal approx-imation of certain types of random variables. In particular, we establish the bound for any random variable such that its λ-Stein pair exists and the distance between the pair is bounded.

The rest of this paper is organized as follows. First, we state our main theorem. Then we apply the theorem to obtain non-uniform exponential bounds for normal approximations in four settings. We provide the proof of the main theorem in the last section.

Theorem 1 *Let W be a random variable with zero mean and variance one and W' be an exchangeable pair of W such that $E^W W' = (1-\lambda)W$ with $0 < \lambda < 1$. Assume that $\delta_W |W'-W| \leqslant A$ for some constant A. For*

$z \in \mathbb{R}$ such that $4A < |z| \leqslant (0.623/\lambda)A$,

$$|P(W \leqslant z)-\Phi(z)| \leqslant C_1(z)\frac{1}{2\lambda}\sqrt{\mathrm{Var}(E^W(W'-W)^2)}+C_2(z)\frac{0.41A^3}{\lambda}+1.5AC_3(z)$$

where Φ is the standard normal distribution and

$$C_1(z)=\sqrt{\frac{3}{2\pi}}\frac{\mathrm{e}^{-z^2/2}}{|z|}+\left(2+\mathrm{e}^{(8/3)\lambda^2|z|^3/A^3}\right)^{1/2}\mathrm{e}^{-(z^2/2)\left((\lambda/(A^2+2\lambda))\right)},$$

$$C_2(z)=0.610\left(\frac{1}{|z|}\left(1+\frac{9z^2}{16}\right)\mathrm{e}^{-(7/32)z^2}+\frac{\mathrm{e}^{-(1/2)z^2}}{\sqrt{2\pi}}\right)+0.620(1+|z|)\,\mathrm{e}^{-\lambda z^2/4A^2+\lambda^2|z|^3/3A^3},$$

$$C_3(z)=\tfrac{8}{3}\,\mathrm{e}^{2\lambda|z|/A+4\lambda^2|z|^3/3A^3-\lambda z^2/2A^2}.$$

Furthermore, if $\max(|z|,|z|^3) \leqslant 1/A$, then there exist positive constants a and C such that $C_i(z) \leqslant C\,\mathrm{e}^{-az^2}$ for $i = 1,2,3$. Hence

$$|P(W \leqslant z)-\Phi(z)| \leqslant C\,\mathrm{e}^{-az^2}\left(\frac{1}{2\lambda}\sqrt{\mathrm{Var}(E^W(W'-W)^2)}+\frac{0.41A^3}{\lambda}+1.5A\right).$$

The non-uniform bound obtained in Theorem 1 is only applicable to a certain interval of $|z|$. However, in most applications, $A = O(1/\sqrt{n})$ and $\lambda = O(1/n)$. Then the lower limit of the interval becomes a constant while the upper limit tends to infinity as $n \to \infty$. Hence we can guarantee that, for any given large $|z|$, we can find a suitable sample size n such that the theorem is applicable and the bound obtained is still smaller than any existing bound. For the case of small values of $|z|$, the uniform bound (3) is already available[16].

APPLICATIONS

In this section, four applications of Theorem 1 are discussed. For each application, we first construct an exchangeable pair W' of the target random variable W such that $E^W W' = (1-\lambda)W$ with $0 < \lambda < 1$ and $\delta_W |W'-W| \leqslant A$ for some constant A and then compute a bound for the term $\mathrm{Var}(E^W(W-W')^2)$.

Independent bounded random variables

Let $X_1, X_2, \ldots, X_n$ be independent and not necessarily identically distributed random variables with zero means and finite variances. Define $W = \sum_{i=1}^n X_i$ and assume that $\mathrm{Var}(W) = 1$.

Chen and Shao[18] gave a uniform bound for a normal approximation for the sum of independent random variables with bounded condition $|X_i| \leqslant \delta$ for $i = 1,2,\ldots,n$:

$$\sup_{z\in\mathbb{R}}|P(W \leqslant z)-\Phi(z)| \leqslant 3.3\delta.$$

Chaidee[19] gave a non-uniform exponential bound improving this bound for large z as follows. For every real number z, $|P(W \leqslant z)-\Phi(z)| \leqslant C_\delta\, \mathrm{e}^{-|z|/2}\delta$, where $C_\delta = 4.45 + 2.21\,\mathrm{e}^{2\delta+(\delta^{-2}(\mathrm{e}^{2\delta}-1-2\delta))}$.

Here we show that the application of our main theorem to the sum of independent bounded random variables can improve the non-uniform exponential bound in Ref. 19. The theorem is stated below.

Theorem 2 *Assume that $|X_i| \leqslant \delta$ for $i = 1, 2, \ldots, n$. Then for fixed $z \in \mathbb{R}$ such that $\max(|z|, |z|^3) \leqslant 1/2\delta$ with $8\delta < |z| \leqslant 1.256n\delta$,*

$$\begin{aligned}|P(W \leqslant z)-\Phi(z)| &\leqslant \sqrt{\frac{3}{8\pi}}\sqrt{n}\delta^2\frac{\mathrm{e}^{-z^2/2}}{|z|} \\ &\quad + \frac{\sqrt{n}\delta^2}{2}\left(2+\mathrm{e}^{1/6n^2\delta^4}\right)^{1/2}\mathrm{e}^{-z^2/(8n\delta^2+4)} \\ &\quad + 2.001n\delta^3\,\mathrm{e}^{-7z^2/32}\left(\frac{1}{|z|}+\frac{9|z|}{16}\right)+\frac{2.001n\delta^3}{\sqrt{2\pi}}\,\mathrm{e}^{-z^2/2} \\ &\quad + 2.034n\delta^3(1+|z|)\,\mathrm{e}^{-z^2/16n\delta^2+1/48n^2\delta^4} \\ &\quad + 8\delta\,\mathrm{e}^{-z^2/8n\delta^2+1/2n\delta^2+1/12n^2\delta^4}.\end{aligned}$$

Furthermore, if $\delta = K/\sqrt{n}$ where $K \geqslant 0.548$, then for $8K < |z| \leqslant n^{1/6}/(2K)^{1/3}$,

$$\begin{aligned}|P(W \leqslant z)-\Phi(z)| &\leqslant \frac{1}{\sqrt{n}}\,\mathrm{e}^{-3z^2/64K^2}(10.243K^4 \\ &\quad +1.879K^3+1.524K^2+0.719K\big).\end{aligned}$$

Proof: Let X_i' be an independent copy of X_i and let I be a uniformly distributed random variable on $\{1, 2, \ldots, n\}$ which is independent of $\{X_i, X_i', i = 1, 2, \ldots, n\}$. Define $W' = W - X_I + X_I'$. Then (W, W') is an exchangeable pair[3] satisfying (1) with $\lambda = 1/n$ and $\delta_W \leqslant 2\delta$. From Ref. 18, $E^W(W-W')^2 = (1/n)(1+\sum_{i=1}^n E^W(X_i^2))$. Hence

$$\mathrm{Var}(E^W(W-W')^2) \leqslant \frac{\delta^4}{n}. \tag{4}$$

Substituting $A = 2\delta$, $\lambda = 1/n$, and (4) into the bound in Theorem 1, we obtain, for $8\delta < |z| \leqslant 1.256n\delta$,

$$\begin{aligned}C_1(z) &= \sqrt{\frac{3}{2\pi}}\frac{\mathrm{e}^{-z^2/2}}{z} \\ &\quad + \left(2+\mathrm{e}^{(1/3)|z|^3/n^2\delta^3}\right)^{1/2}\mathrm{e}^{-z^2/(8n\delta^2+4)}, \\ C_2(z) &= 0.610\left(\frac{1}{|z|}\left(1+\frac{9z^2}{16}\right)\mathrm{e}^{-7z^2/32}+\frac{\mathrm{e}^{-z^2/2}}{\sqrt{2\pi}}\right) \\ &\quad + 0.620(1+|z|)\,\mathrm{e}^{-z^2/4n\delta^2+|z|^3/24n^2\delta^3}, \\ C_3(z) &= \tfrac{8}{3}\,\mathrm{e}^{|z|/n\delta+|z|^3/6n^2\delta^3-z^2/8n\delta^2}.\end{aligned}$$

Hence

$$\begin{aligned}|P(W \leqslant z)-\Phi(z)| &\leqslant \sqrt{\frac{3}{8\pi}}\sqrt{n}\delta^2\frac{\mathrm{e}^{-z^2/2}}{|z|} \\ &\quad + \frac{\sqrt{n}\delta^2}{2}\left(2+\mathrm{e}^{1/6n^2\delta^4}\right)^{1/2}\mathrm{e}^{-z^2/(8n\delta^2+4)} \\ &\quad + 2.001n\delta^3\,\mathrm{e}^{-7z^2/32}\left(\frac{1}{|z|}+\frac{9|z|}{16}\right)+\frac{2.001n\delta^3}{\sqrt{2\pi}}\,\mathrm{e}^{-z^2/2} \\ &\quad + 2.034n\delta^3(1+|z|)\,\mathrm{e}^{-z^2/16n\delta^2+1/48n^2\delta^4} \\ &\quad + 8\delta\,\mathrm{e}^{-z^2/8n\delta^2+1/2n\delta^2+1/12n^2\delta^4},\end{aligned}$$

where we use the fact that $\max(|z|, |z|^3) \leqslant 1/2\delta$ and direct calculation to obtain the inequality. Furthermore, if $\delta = K/\sqrt{n}$, where $K \geqslant 0.548$, then for $8K < |z| \leqslant n^{1/6}/(2K)^{1/3}$,

$$\begin{aligned}&|P(W \leqslant z)-\Phi(z)| \\ &\quad \leqslant \mathrm{e}^{-3z^2/64K^2}\frac{K^2}{\sqrt{n}}\left(\sqrt{\frac{3}{8\pi}}+\tfrac{1}{2}\left(2+\mathrm{e}^{1/6K^4}\right)^{1/2}\right) \\ &\qquad + \mathrm{e}^{-3z^2/64K^2}\frac{2.001K^3}{\sqrt{n}}\left(\frac{1+36K^2}{8K}\,\mathrm{e}^{3-14K^2}+\frac{1}{\sqrt{2\pi}}\right) \\ &\qquad + \mathrm{e}^{-3z^2/64K^2}\frac{2.034K^3}{\sqrt{n}}(1+8K)\,\mathrm{e}^{1/48K^4-1} \\ &\qquad + \frac{8K}{\sqrt{n}}\,\mathrm{e}^{-3z^2/64K^2+1/2K^2+1/12K^4-5z^2/64K^2} \qquad (5) \\ &\quad \leqslant \frac{1}{\sqrt{n}}\,\mathrm{e}^{-3z^2/64K^2}(10.243K^4+1.879K^3 \\ &\qquad +1.524K^2+0.719K),\end{aligned}$$

where we use the fact that $\mathrm{e}^{-z^2/(8K^2+4)} \leqslant \mathrm{e}^{-3z^2/64K^2}$ for $K \geqslant 0.548$ and the fact that $\mathrm{e}^{3z^2/64K^2-7z^2/32}(1/|z|+9|z|/16)$ and $(1+|z|)\,\mathrm{e}^{-z^2/64K^2}$ are decreasing in z on $[8K, \infty)$ to obtain (5). □

Combinatorial central limit theorem

For $n \geqslant 2$, let a_{ij} be an $n \times n$ matrix of real numbers and π be a random permutation of $\{1, 2, \ldots, n\}$ and

$$\mu = na_{\cdot\cdot}, \quad \sigma^2 = \frac{1}{n-1}\sum_{i,j=1}^n (a_{ij}-a_{i\cdot}-a_{\cdot j}+a_{\cdot\cdot})^2$$

where

$$a_{i\cdot} = \frac{1}{n}\sum_{j=1}^n a_{ij}, \quad a_{\cdot j} = \frac{1}{n}\sum_{i=1}^n a_{ij}, \quad a_{\cdot\cdot} = \frac{1}{n^2}\sum_{i,j=1}^n a_{ij}.$$

Define $W = \sum_{i=1}^n x_{i\pi(i)}$, where $x_{ij} = (a_{ij} - a_{i\cdot} - a_{\cdot j} + a_{\cdot\cdot})/\sigma$. Then $E(W) = 0$, $\mathrm{Var}(W) = 1$ and W is approximately standard normal distributed. The

asymptotic normality for W under various conditions were shown in Refs. 20, 21. Bahr[22] and Ho and Chen[23] obtained a uniform bound for W with the rate $O(1/\sqrt{n})$ under boundedness conditions $\sup_{i,j}|x_{ij}| = O(1/\sqrt{n})$. Ho and Chen's result is that, for every real number z,

$$|P(W_n \leqslant z) - \Phi(z)| \leqslant C\beta,$$

where $\beta = (1/n)\sum_{i,j=1}^{n}|x_{ij}|^3$. Bolthausen[24] gave a uniform bound for W with finite third moment conditions. His result is as follows. There exists an absolute constant $C > 0$, such that

$$\sup_{z\in\mathbb{R}}|P(W \leqslant z) - \Phi(z)| \leqslant C\beta \tag{6}$$

which has an $O(1/\sqrt{n})$ rate of convergence. Chen and Fang[25] calculated the constant of the uniform bound for W in (6) to be 451, and obtained a bound in the form of

$$\sup_{z\in\mathbb{R}}|P(W \leqslant z) - \Phi(z)| \leqslant 451\delta$$

for the bounded condition $\sup_{i,j}|x_{ij}| \leqslant \delta$. Chaidee[19] gave a non-uniform bound for the normal approximation for W with the same bounded condition where $\delta = O(1/\sqrt{n})$. The bound is, for any real number z,

$$|P(W \leqslant z) - \Phi(z)| \leqslant \frac{C\delta}{(1+|z|^3)}, \tag{7}$$

where C is a positive constant.

We now apply our main theorem to improve the non-uniform polynomial bound to an exponential bound in the form of e^{-cz^2}. The obtained bound is much smaller than the polynomial bound given in (7) for sufficiently large z.

Theorem 3 *Assume that* $\sup_{i,j}|x_{ij}| \leqslant \delta$. *Then for fixed* $z \in \mathbb{R}$ *such that* $\max(|z|, |z|^3) \leqslant 1/4\delta$ *with* $16\delta < |z| \leqslant 1.256(n-1)\delta$,

$$\begin{aligned}|P(W \leqslant z) - \Phi(z)| &\leqslant \sqrt{\frac{33}{4\pi}}\sqrt{n}\delta^2\frac{\mathrm{e}^{-z^2/2}}{|z|} \\ &+ \sqrt{\tfrac{11}{2}}\sqrt{n}\delta^2\left(2+\mathrm{e}^{1/6n^2\delta^4}\right)^{1/2}\mathrm{e}^{-z^2/(16n\delta^2+8)} \\ &+ 8.004n\delta^3\,\mathrm{e}^{-7z^2/32}\left(\frac{1}{|z|}+\frac{9|z|}{16}\right) + \frac{8.004n\delta^3}{\sqrt{2\pi}}\,\mathrm{e}^{-z^2/2} \\ &+ 8.135n\delta^3(1+|z|)\,\mathrm{e}^{-z^2/32n\delta^2+1/48n^2\delta^4} \\ &+ 16\delta\,\mathrm{e}^{-z^2/16n\delta^2+1/2n\delta^2+1/12n^2\delta^4}.\end{aligned}$$

Furthermore, if $\delta = K/\sqrt{n}$ *where* $K \geqslant 0.548$, *then for* $16K < |z| \leqslant n^{1/6}/(4K)^{1/3}$,

$$\begin{aligned}|P(W \leqslant z) - \Phi(z)| \leqslant \frac{1}{\sqrt{n}}\,\mathrm{e}^{-3z^2/128K^2}&(22.194K^4 \\ &+ 4.582K^3 + 6.961K^2 + 0.012K).\end{aligned}$$

Proof: For a random permutation π on the symmetric group S_n, let a pair (I, J) be a random variable independent of π with $P(I = i \text{ and } J = j) = 1/n(n-1)$ for every $i, j \in \{1, 2, \ldots, n\}$ such that $i \neq j$. Define π' and W' by

$$\pi'(i) = \begin{cases}\pi(i), & i \notin \{I, J\}, \\ \pi(J), & i = I, \\ \pi(I), & i = J,\end{cases}$$

and $W' = W'(\pi) = W(\pi') = \sum_{i=1}^{n} x_{i\pi'(i)}$. From Ref. 6, (π, π') is an exchangeable pair. Hence (W, W') is also an exchangeable pair. Since $W' - W = x_{I\pi(J)} + x_{J\pi(I)} - x_{I\pi(I)} - x_{J\pi(J)}$, $\sup_{i,j}|x_{ij}| \leqslant \delta$, and $\delta_W \leqslant 4\delta$. Furthermore, the exchangeable pair (W, W') satisfies (1) with $\lambda = 2/(n-1)$. Following the proof in Ref. 6, we can show that

$$\begin{aligned}\mathrm{Var}(E^W(W'-W)^2) &\leqslant \mathrm{Var}(E^\pi(W'-W)^2) \\ &\leqslant \frac{4}{n^2(n-1)^2}\Bigg[\sum_{i,j=1}^{n}x_{ij}^4 + n\sum_{i,j=1}^{n}x_{ij}^4 \\ &\quad + \sum_{i,j=1}^{n}\mathrm{Var}(x_{i\pi(j)}x_{j\pi(i)} + x_{i\pi(i)}x_{j\pi(j)})\Bigg].\end{aligned}$$

Applying the inequality $(a+b)^2 \leqslant 2(a^2+b^2)$ and the Hölder inequality, we can show that

$$\sum_{i,j=1}^{n}\mathrm{Var}(x_{i\pi(j)}x_{j\pi(i)} + x_{i\pi(i)}x_{j\pi(j)}) \leqslant 8\sum_{i,j=1}^{n}x_{ij}^4.$$

Hence

$$\mathrm{Var}(E^W(W'-W)^2) \leqslant \frac{88\delta^4}{n}. \tag{8}$$

Substituting (8), $A = 4\delta$, and $\lambda = 2/(n-1)$ into the bound in Theorem 1 we obtain, for $16\delta < z \leqslant 1.256(n-1)\delta$,

$$\begin{aligned}C_1(z) &\leqslant \sqrt{\frac{3}{2\pi}}\frac{\mathrm{e}^{-z^2/2}}{|z|} \\ &\quad + \left(2+\mathrm{e}^{(2/3)|z|^3/n^2\delta^3}\right)^{1/2}\mathrm{e}^{-z^2/(16n\delta^2+8)} \\ C_2(z) &\leqslant 0.610\left(\frac{1}{|z|}\left(1+\frac{9z^2}{16}\right)\mathrm{e}^{-7z^2/32} + \frac{\mathrm{e}^{-z^2/2}}{\sqrt{2\pi}}\right) \\ &\quad + 0.620(1+|z|)\,\mathrm{e}^{-z^2/32n\delta^2+|z|^3/12n^2\delta^3} \\ C_3(z) &\leqslant \tfrac{8}{3}\,\mathrm{e}^{2|z|/n\delta+|z|^3/3n^2\delta^3-z^2/16n\delta^2}.\end{aligned}$$

Hence

$$\begin{aligned}|P(W\leqslant z)-\Phi(z)| &\leqslant \sqrt{\frac{33}{4\pi}}\sqrt{n}\delta^2\frac{\mathrm{e}^{-z^2/2}}{|z|}\\ &\quad+\sqrt{\tfrac{11}{2}}\sqrt{n}\delta^2\left(2+\mathrm{e}^{1/6n^2\delta^4}\right)^{1/2}\mathrm{e}^{-z^2/(16n\delta^2+8)}\\ &\quad+8.004n\delta^3\,\mathrm{e}^{-7z^2/32}\left(\frac{1}{|z|}+\frac{9|z|}{16}\right)+\frac{8.004n\delta^3}{\sqrt{2\pi}}\,\mathrm{e}^{-z^2/2}\\ &\quad+8.135n\delta^3(1+|z|)\,\mathrm{e}^{-z^2/32n\delta^2+1/48n^2\delta^4}\\ &\quad+16\delta\,\mathrm{e}^{-z^2/16n\delta^2+1/2n\delta^2+1/12n^2\delta^4},\end{aligned}$$

where we use the fact that $\max(|z|,|z|^3)\leqslant 1/4\delta$ and a direct calculation to obtain the inequality. Furthermore, if $\delta=K/\sqrt{n}$ where $K\geqslant 0.548$, then for $16K<|z|\leqslant n^{1/6}/(4K)^{1/3}$,

$$\begin{aligned}&|P(W\leqslant z)-\Phi(z)|\\ &\quad\leqslant \mathrm{e}^{-3z^2/128K^2}\frac{K^2}{\sqrt{n}}\left(\sqrt{\frac{33}{4\pi}}+\sqrt{\tfrac{11}{2}}\left(2+\mathrm{e}^{2/6K^4}\right)^{1/2}\right)\\ &\quad+\mathrm{e}^{-3z^2/128K^2}\frac{8.004K^3}{\sqrt{n}}\left(\frac{1+144K^2}{16K}\,\mathrm{e}^{6-56K^2}+\frac{1}{\sqrt{2\pi}}\right)\\ &\quad+\mathrm{e}^{-3z^2/128K^2}\frac{8.135K^3}{\sqrt{n}}(1+16K)\,\mathrm{e}^{-2+1/48K^2}\\ &\quad+\frac{16K}{\sqrt{n}}\,\mathrm{e}^{-3z^2/128K^2+1/2K^2+1/12K^4-5z^2/128K^2}\qquad(9)\\ &\quad\leqslant \frac{1}{\sqrt{n}}\,\mathrm{e}^{-3z^2/128K^2}(22.194K^4+4.582K^3\\ &\qquad+6.961K^2+0.012K),\end{aligned}$$

where we use the fact that $\mathrm{e}^{-z^2/(16K^2+8)}\leqslant \mathrm{e}^{-3z^2/128K^2}$ for $K\geqslant 0.548$ and the fact that $\mathrm{e}^{3z^2/128K^2-7z^2/32}(1/|z|+9|z|/16)$ and $(1+|z|)\,\mathrm{e}^{-z^2/128K^2}$ are decreasing in z on $[16K,\infty)$ to obtain (9). □

The numbers of descents and inversions of a permutation

For a random permutation π on the symmetric group S_n, let $\mathrm{Des}(\pi)$ be the number of descents of permutation π. Then $\mathrm{Des}(\pi)$ is the number of pairs $(i,i+1)$ with $1\leqslant i\leqslant n-1$ such that $\pi(i)>\pi(i+1)$. Similarly, let $\mathrm{Inv}(\pi)$ be the number of inversions of permutation π. Then $\mathrm{Inv}(\pi)$ is the number of pairs (i,j) with $1\leqslant i<j\leqslant n$ such that $\pi(i)>\pi(j)$. The expectations and variances of the numbers of descents and inversions are $E(\mathrm{Des}(\pi))=\frac{1}{2}(n-1)$, $\mathrm{Var}(\mathrm{Des}(\pi))=\frac{1}{12}(n+1)$, $E(\mathrm{Inv}(\pi))=\frac{1}{4}n(n-1)$ and $\mathrm{Var}(\mathrm{Inv}(\pi))=\frac{1}{72}n(n-1)(2n+5)$, respectively[14]. The asymptotic normality for the number of descents is proved in Ref. 26 and for the number of inversions is proved in Ref. 27.

Fulman[14] gave uniform bounds of the normal approximation to the numbers of descents and inversions. The exchangeable pair techniques of Stein's method and the theorem in Ref. 11 were used to prove the results[14]

$$\sup_{z\in\mathbb{R}}\left|P\left(\frac{\mathrm{Des}(\pi)-E(\mathrm{Des}(\pi))}{\sqrt{\mathrm{Var}(\mathrm{Des}(\pi))}}\leqslant z\right)-\Phi(z)\right|\leqslant\frac{C_1}{\sqrt{n}},$$

$$\sup_{z\in\mathbb{R}}\left|P\left(\frac{\mathrm{Inv}(\pi)-E(\mathrm{Inv}(\pi))}{\sqrt{\mathrm{Var}(\mathrm{Inv}(\pi))}}\leqslant z\right)-\Phi(z)\right|\leqslant\frac{C_2}{\sqrt{n}},$$

where C_1 and C_2 are positive constants.

Chuntee and Neammanee[15] calculated the constants in Fulman's results which are 1096 and 5421, respectively. In their work, they also proposed a method to reduce the constants to be 13.42 and 14.24 for the number of descents and the number of inversions, respectively. Furthermore, they generalized the two bounds to be non-uniform bounds in polynomial form. Chuntee and Neammanee[17] improved the rate of convergences for descents and inversions from the polynomial bounds to exponential bounds. Their results are as follows. For sufficiently large n,

$$\left|P\left(\frac{\mathrm{Des}(\pi)-E(\mathrm{Des}(\pi))}{\sqrt{\mathrm{Var}(\mathrm{Des}(\pi))}}\leqslant z\right)-\Phi(z)\right|\leqslant\frac{51.25}{\sqrt{n}}\,\mathrm{e}^{-|z|/4}\qquad(10)$$

and

$$\left|P\left(\frac{\mathrm{Inv}(\pi)-E(\mathrm{Inv}(\pi))}{\sqrt{\mathrm{Var}(\mathrm{Inv}(\pi))}}\leqslant z\right)-\Phi(z)\right|\leqslant\frac{792.71}{\sqrt{n}}\,\mathrm{e}^{-|z|/4}.\qquad(11)$$

Here we provide smaller exponential bounds for normal approximations to the numbers of descents and inversions.

Theorem 4 *For fixed $z\in\mathbb{R}$ and $n>z^6$. If $|z|>8\sqrt{3}$, then*

$$\left|P\left(\frac{\mathrm{Des}(\pi)-E(\mathrm{Des}(\pi))}{\sqrt{\mathrm{Var}(\mathrm{Des}(\pi))}}\leqslant z\right)-\Phi(z)\right|\leqslant\frac{10.980}{\sqrt{n}}\,\mathrm{e}^{-z^2/32}.\qquad(12)$$

Table 1 Comparisons of the constant C for the number of descents and inversions in the form of $C/\sqrt{n}$.

bound	$z=25$	$z=100$	$z=1000$
(10)	0.3454	7.2×10^{-10}	1.4×10^{-107}
(12)	2.0×10^{-7}	2.2×10^{-135}	2.2×10^{-13571}
(11)	1.5303	1.2×10^{-8}	2.2×10^{-106}
(13)	0.1391	5.4×10^{-44}	1.2×10^{-4522}

Theorem 5 *For fixed $z\in\mathbb{R}$ and $n>z^6$, if $|z|>24$, then*

$$\left|P\left(\frac{\mathrm{Inv}(\pi)-E(\mathrm{Inv}(\pi))}{\sqrt{\mathrm{Var}(\mathrm{Inv}(\pi))}}\leqslant z\right)-\Phi(z)\right| \leqslant \frac{93.467}{\sqrt{n}}\,\mathrm{e}^{-z^2/96}. \quad (13)$$

Remark 1 For sufficiently large n, the bounds given in Theorems 4 and 5 are smaller than the ones given in Ref. 17 when $z>8\sqrt{3}$ and $z>24$ for the cases of descents and inversions, respectively.

From Table 1, we can see that our bounds (12, 13) are respectively smaller than the exponential bounds (10, 11) in Ref. 17 when n satisfies the condition that $n>z^6$. The improvements are clearly seen for large values of z.

Proof of Theorem 4: For $\pi\in S_n$, define

$$W:=\frac{\mathrm{Des}(\pi)-E(\mathrm{Des}(\pi))}{\sqrt{\mathrm{Var}(\mathrm{Des}(\pi))}},$$

and

$$\pi'(i)=\begin{cases}\pi(i), & i\notin\{I,I+1,\ldots,n\},\\ \pi(i+1), & i\in\{I,I+1,\ldots,n-1\},\\ \pi(I), & i=n,\end{cases} \quad (14)$$

where I is a uniform random variable taking values on $\{1,2,\ldots,n\}$. The permutation π' was shown by Fulman[14] to be an exchangeable pair of π. Consequently,

$$W':=\frac{\mathrm{Des}(\pi')-E(\mathrm{Des}(\pi'))}{\sqrt{\mathrm{Var}(\mathrm{Des}(\pi'))}}$$

is an exchangeable pair of W. Moreover, the exchangeable pair (W,W') satisfies (1) with $\lambda=2/n$ and $\delta_W\leqslant 2\sqrt{3}/\sqrt{n}$.

Chuntee and Neammanee[15] showed that

$$\mathrm{Var}(E^W(W'-W)^2)\leqslant\frac{25.6}{n^3}. \quad (15)$$

Substituting (15), $A=2\sqrt{3}/\sqrt{n}$ and $\lambda=2/n$ into Theorem 1, we have, for $|z|>8\sqrt{3}$,

$$C_1(z)=\sqrt{\frac{3}{2\pi}}\frac{\mathrm{e}^{-z^2/2}}{|z|}+\left(2+\mathrm{e}^{(4/9\sqrt{3})|z|^3/\sqrt{n}}\right)^{1/2}\mathrm{e}^{-z^2/16}$$

$$C_2(z)=0.610\left(\frac{1}{|z|}\left(1+\frac{9z^2}{16}\right)\mathrm{e}^{-(7/32)z^2}+\frac{\mathrm{e}^{-z^2/2}}{\sqrt{2\pi}}\right)+0.620(1+|z|)\,\mathrm{e}^{-z^2/24+(1/18\sqrt{3})|z|^3/\sqrt{n}}$$

$$C_3(z)=\tfrac{8}{3}\,\mathrm{e}^{(2/\sqrt{3})|z|/\sqrt{n}+(2/9\sqrt{3})|z|^3/\sqrt{n}-z^2/12}.$$

Hence

$$\begin{aligned}&|P(W\leqslant z)-\Phi(z)|\\ &\leqslant\frac{\sqrt{25.6}}{4\sqrt{n}}\,\mathrm{e}^{-z^2/32}\left[\sqrt{\frac{3}{2\pi}}\frac{\mathrm{e}^{-15z^2/32}}{|z|}+\sqrt{2+\mathrm{e}^{4/9\sqrt{3}}}\,\mathrm{e}^{-z^2/32}\right]\\ &+\frac{\mathrm{e}^{-z^2/32}}{\sqrt{n}}\left[5.199\left(\frac{16+9z^2}{16|z|}\,\mathrm{e}^{-(6/32)z^2}+\frac{\mathrm{e}^{-(15/32)z^2}}{\sqrt{2\pi}}\right)+5.284(1+|z|)\,\mathrm{e}^{-z^2/96+1/18\sqrt{3}}\right]\\ &+\mathrm{e}^{-z^2/32}\frac{8\sqrt{3}}{\sqrt{n}}\,\mathrm{e}^{2/\sqrt{3}+2/9\sqrt{3}-5z^2/96} \quad (16)\\ &\leqslant\frac{10.980}{\sqrt{n}}\,\mathrm{e}^{-z^2/32},\end{aligned}$$

where we use the fact that $n>z^6$ to obtain (16) and $\sqrt{3/2\pi}\,\mathrm{e}^{-15z^2/32}/|z|+(2+\mathrm{e}^{4/9\sqrt{3}})^{1/2}\mathrm{e}^{-z^2/32}\leqslant 0.0045$, $((16+9z^2)/16|z|)\,\mathrm{e}^{-6z^2/32}+(\mathrm{e}^{-(15/32)z^2}/\sqrt{2\pi})\leqslant 1.83\times10^{-15}$, $(1+|z|)\,\mathrm{e}^{-z^2/96+1/18\sqrt{3}}\leqslant 2.0762$, $\mathrm{e}^{2/\sqrt{3}+2/9\sqrt{3}-(5/96)z^2}\leqslant 0.0002$, for $|z|>8\sqrt{3}$ to obtain the last inequality. □

Proof of Theorem 5: Similarly, for

$$W:=\frac{\mathrm{Inv}(\pi)-E(\mathrm{Inv}(\pi))}{\sqrt{\mathrm{Var}(\mathrm{Inv}(\pi))}},$$

define the exchangeable pair random variable W' by

$$W':=\frac{\mathrm{Inv}(\pi')-E(\mathrm{Inv}(\pi'))}{\sqrt{\mathrm{Var}(\mathrm{Inv}(\pi'))}}.$$

Fulman[14] showed that the exchangeable pair (W,W') satisfies (1) with $\lambda=2/n$ and $\delta_W\leqslant 6/\sqrt{n}$. Hence[15]

$$\mathrm{Var}(E^W(W'-W)^2)\leqslant\frac{30.8}{n^3}. \quad (17)$$

Substituting (17), $A = 6/\sqrt{n}$, and $\lambda = 2/n$ into Theorem 1, we obtain, for $|z| > 24$,

$$C_1(z) = \sqrt{\frac{3}{2\pi}}\frac{\mathrm{e}^{-z^2/2}}{|z|} + \left(2 + \mathrm{e}^{(4/81)|z|^3/\sqrt{n}}\right)^{1/2}\mathrm{e}^{-z^2/20}$$

$$C_2(z) = 0.610\left(\frac{1}{|z|}\left(1+\frac{9z^2}{16}\right)\mathrm{e}^{-(7/32)z^2} + \frac{\mathrm{e}^{-z^2/2}}{\sqrt{2\pi}}\right) + 0.620(1+|z|)\,\mathrm{e}^{-z^2/72+|z|^3/162\sqrt{n}}$$

$$C_3(z) = \tfrac{8}{3}\,\mathrm{e}^{(2/3)|z|/\sqrt{n}+(2/81)|z|^3/\sqrt{n}-z^2/36}.$$

Hence

$$\begin{aligned}
&|P(W \leqslant z) - \Phi(z)| \\
&\quad\leqslant \frac{\sqrt{30.8}}{4\sqrt{n}}\,\mathrm{e}^{-z^2/96}\Bigg[\sqrt{\frac{3}{2\pi}}\frac{\mathrm{e}^{-47z^2/96}}{|z|} \\
&\qquad + \left(2+\mathrm{e}^{4/81}\right)^{1/2}\mathrm{e}^{-19z^2/480}\Bigg] \\
&\qquad + \frac{1}{\sqrt{n}}\,\mathrm{e}^{-z^2/96}\Bigg[27.011\Bigg(\frac{16+9z^2}{16|z|}\,\mathrm{e}^{-(5/24)z^2} \\
&\qquad + \frac{\mathrm{e}^{-(47/96)z^2}}{\sqrt{2\pi}}\Bigg) + 27.454(1+|z|)\,\mathrm{e}^{-z^2/288+1/162}\Bigg] \\
&\qquad\qquad + \mathrm{e}^{-z^2/96}\frac{24}{\sqrt{n}}\,\mathrm{e}^{2/3+2/81-5z^2/288} \\
&\quad\leqslant \frac{93.467}{\sqrt{n}}\,\mathrm{e}^{-z^2/96}, \qquad (18)
\end{aligned}$$

where we use the fact that $n > z^6$ to obtain (18) and $\sqrt{3/2\pi}\,\mathrm{e}^{-47z^2/96}/|z| + (2 + \mathrm{e}^{4/81})^{1/2}\mathrm{e}^{-19z^2/480} \leqslant 2.19 \times 10^{-10}$, $((16 + 9z^2)/16|z|)\mathrm{e}^{-5z^2/24} + (\mathrm{e}^{-47z^2/96}/\sqrt{2\pi}) \leqslant 1.04 \times 10^{-51}$, $(1 + |z|)\mathrm{e}^{-z^2/288+1/162} \leqslant 3.4044$, $\mathrm{e}^{2/3+2/81-5z^2/288} \leqslant 9.07\times10^{-5}$, for $|z| > 24$ to obtain the last inequality. □

PROOF OF MAIN RESULT

Recall that the exchangeable pair (W, W') satisfies $E^W(W - W') = \lambda W$ and $\delta_W \leqslant A$. Chen and Shao[18] proved that

$$EWg(W) = E\int_{-\infty}^{\infty} g'(W+t)\hat{K}(t)\,\mathrm{d}t, \qquad (19)$$

where

$$\hat{K}(t) = \frac{1}{2\lambda}(W'-W)[\mathbb{I}(0 \leqslant t \leqslant W'-W) - \mathbb{I}(W'-W \leqslant t \leqslant 0)],$$

and g is a continuous and piecewise continuously differentiable function. We can see that $\hat{K}(t)$ is always non-negative and $\hat{K}(t) = 0$ for $|t| > \delta_W$ and

$$\int_{-\infty}^{\infty}\hat{K}(t)\,\mathrm{d}t = \frac{1}{2\lambda}(W'-W)^2 \leqslant \frac{A^2}{2\lambda}. \qquad (20)$$

To prove the main result, we first obtain an exponential bound for the moment of W.

Lemma 1 *Let $s > 0$ and $sA \leqslant 1.256$. Then*

$$E\,\mathrm{e}^{sW} \leqslant \mathrm{e}^{A^2s^2/4\lambda + A^3s^3/3\lambda}.$$

Proof: We first prove the inequality $\mathrm{e}^x - 1 \leqslant 2x$ for $0 \leqslant x \leqslant 1.256$. Note that $\mathrm{e}^x - 1 = 2x$ when $x = 0$. Let $g(x) = (\mathrm{e}^x - 1)/2x$, for $x > 0$. Then g is an increasing function on $(0, \infty)$ and $g(1.256) = 0.9997$. This implies that $g(x) \leqslant 1$ for $0 < x \leqslant 1.256$. Hence $\mathrm{e}^x - 1 \leqslant 2x$ for $0 \leqslant x \leqslant 1.256$.

For $s \in (0, \infty)$, let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(w) = \mathrm{e}^{sw}$ and $h_W : (0, \infty) \to \mathbb{R}$ be defined by $h_W(t) = E\,\mathrm{e}^{tW}$. Following Ref. 17, we can show that

$$h_W'(s) \leqslant \frac{sA^2}{2\lambda}E\,\mathrm{e}^{sW} + \frac{A^3s^2}{\lambda}E\,\mathrm{e}^{sW},$$

where we use the fact that $0 \leqslant s\delta_W \leqslant sA \leqslant 1.256$, the property of exchangeable pair (20), and the inequality $\mathrm{e}^x - 1 \leqslant 2x$ for $0 \leqslant x \leqslant 1.256$ to obtain the inequality. Dividing the inequality by $h_W(s)$ and then integrating we obtain

$$h_W(s) \leqslant \mathrm{e}^{s^2A^2/4\lambda + A^3s^3/3\lambda}.$$

□

Proof of Theorem 1: By the symmetry of the standard normal distribution, $\Phi(z) = 1 - \Phi(-z)$, it suffices to prove the main result for the case $4A < z \leqslant 0.623A/\lambda$. Stein's equation for the standard normal approximation is

$$g'(w) - wg(w) = \mathbb{I}(w \leqslant z) - \Phi(z), \qquad (21)$$

where Φ is the standard normal distribution, $g : \mathbb{R} \to \mathbb{R}$ is a continuous and piecewise differentiable function. The solution g_z of Stein's equation (21) is given by[6]

$$g_z(w) = \begin{cases} \sqrt{2\pi}\,\mathrm{e}^{w^2/2}\Phi(w)[1-\Phi(z)], & w \leqslant z, \\ \sqrt{2\pi}\,\mathrm{e}^{w^2/2}\Phi(z)[1-\Phi(w)], & w > z, \end{cases}$$

and

$$g_z'(w) = \begin{cases} (1-\Phi(z))(1+\sqrt{2\pi}w\,\mathrm{e}^{w^2/2}\Phi(w)), & w < z, \\ \Phi(z)(-1+\sqrt{2\pi}w\,\mathrm{e}^{w^2/2}(1-\Phi(w))), & w > z, \end{cases} \qquad (22)$$

satisfying the following properties: for all $s,t,w\in\mathbb{R}$,

$$0<g_z(w)\leqslant\min\left(\frac{\sqrt{2\pi}}{4},\frac{1}{|z|}\right),\quad \text{for } z\neq 0,$$

$$|g_z'(w)|\leqslant 1,\ |g_z'(s)-g_z'(t)|\leqslant 1,$$ [28]

$$\left|g_z'(w+s)-g_z'(w+t)-\int_t^s h(w+u)\,\mathrm{d}u\right| \leqslant \mathbb{I}(z-\max(s,t)<w<z-\min(s,t)). \quad (23)$$

For $\varepsilon>0$, using (22) and the fact that

$$1-\Phi(z)\leqslant\frac{\mathrm{e}^{-z^2/2}}{\sqrt{2\pi}z}\quad \text{for } z>0,$$
$$\Phi(z)\leqslant\frac{\mathrm{e}^{-z^2/2}}{\sqrt{2\pi}|z|}\quad \text{for } z<0$$

we have

$$0\leqslant g_z'(w)=(1-\Phi(z))\left(1+\sqrt{2\pi}w\,\mathrm{e}^{w^2/2}\Phi(w)\right)$$
$$\leqslant\frac{\mathrm{e}^{-z^2/2}}{\sqrt{2\pi}z},\quad \text{for } w\leqslant 0,$$
$$|g_z'(w)|<1,\quad \text{for } w>\frac{z}{1+\varepsilon},$$

and

$$0<g_z'(w)$$
$$\leqslant(1-\Phi(z))\left(1+\sqrt{2\pi}\left(\frac{z}{1+\varepsilon}\right)\mathrm{e}^{z^2/2(1+\varepsilon)^2}\right)$$
$$\leqslant\frac{\mathrm{e}^{-z^2/2}}{\sqrt{2\pi}z}+\mathrm{e}^{(z^2/2)\left(1/(1+\varepsilon)^2-1\right)},\quad \text{for } 0<w\leqslant\frac{z}{1+\varepsilon}.$$

By using the inequality $(a+b)^2\leqslant 2(a^2+b^2)$ for $a,b\in\mathbb{R}$, we obtain

$$|g_z'(w)|^2\leqslant\begin{cases}\frac{\mathrm{e}^{-z^2}}{2\pi z^2}, & w\leqslant 0,\\ \frac{\mathrm{e}^{-z^2}}{\pi z}+2\,\mathrm{e}^{z^2\left(1/(1+\varepsilon)^2-1\right)}, & 0<w\leqslant\frac{z}{1+\varepsilon},\\ 1, & w>\frac{z}{1+\varepsilon}.\end{cases} \quad (24)$$

Following Ref. 17,

$$|P(W\leqslant z)-\Phi(z)|\leqslant|T_1|+|T_2|, \quad (25)$$

where

$$T_1=Eg_z'(W)-Eg_z'(W)\int_{-\infty}^{\infty}\hat{K}(t)\,\mathrm{d}t,$$
$$T_2=E\int_{-\infty}^{\infty}(g_z'(W)-g_z'(W+t))\hat{K}(t)\,\mathrm{d}t.$$

To bound $|T_1|$, we follow the technique used in Ref. 17 to obtain

$$|T_1|\leqslant\sqrt{E(g_z'(W))^2}\sqrt{E\left(1-\frac{1}{2\lambda}E^W(W'-W)^2\right)^2}. \quad (26)$$

By using (1), we obtain

$$E(W-W')^2=E(E^W(W-W')^2)=2\lambda$$

and therefore

$$E\left(1-\frac{1}{2\lambda}E^W(W'-W)^2\right)^2=\frac{\mathrm{Var}(E^W(W'-W)^2)}{4\lambda^2}. \quad (27)$$

To give a bound for $E|g_z'(W)|^2$, we apply the properties of Stein's solution g_z stated in (24). Hence

$$E|g_z'(W)|^2=E|g_z'(W)|^2\mathbb{I}(W\leqslant 0)+E|g_z'(W)|^2\mathbb{I}\left(0<W\leqslant\frac{z}{1+\varepsilon}\right)+E|g_z'(W)|^2\mathbb{I}\left(W>\frac{z}{1+\varepsilon}\right)$$
$$\leqslant\frac{\mathrm{e}^{-z^2}}{2\pi z^2}P(W\leqslant 0)+\left(\frac{\mathrm{e}^{-z^2}}{\pi z^2}+2\,\mathrm{e}^{z^2\left(1/(1+\varepsilon)^2-1\right)}\right)P\left(0<W\leqslant\frac{z}{1+\varepsilon}\right)+P\left(W>\frac{z}{1+\varepsilon}\right).$$

Applying Markov's inequality and Lemma 1, we obtain

$$P\left(W>\frac{z}{1+\varepsilon}\right)=P\left(\frac{2\lambda zW}{A^2}>\frac{2\lambda z^2}{A^2(1+\varepsilon)}\right)\leqslant\mathrm{e}^{(-2\lambda z^2)/A^2(1+\varepsilon)}E\,\mathrm{e}^{2\lambda zW/A^2}\leqslant\mathrm{e}^{(-2\lambda z^2)/A^2(1+\varepsilon)+\lambda z^2/A^2+8\lambda^2z^3/3A^3}.$$

Choosing ε such that $1/(1+\varepsilon)=(-\lambda+\sqrt{A^4+A^2\lambda+\lambda^2})/A^2$ gives

$$E|g_z'(W)|^2\leqslant\frac{3\,\mathrm{e}^{-z^2}}{2\pi z^2}+\left(2+\mathrm{e}^{(8/3)\lambda^2z^3/A^3}\right)\mathrm{e}^{-z^2(\lambda/A^2)\left((-2\lambda+2\sqrt{A^4+A^2\lambda+\lambda^2})/A^2-1\right)}\leqslant\frac{3\,\mathrm{e}^{-z^2}}{2\pi z^2}+\left(2+\mathrm{e}^{(8/3)\lambda^2z^3/A^3}\right)\mathrm{e}^{-z^2\left((\lambda/(A^2+2\lambda))\right)}. \quad (28)$$

Combining (26)–(28),

$$\begin{aligned}|T_1| &\leqslant \left[\frac{3\,\mathrm{e}^{-z^2}}{2\pi z^2}+\left(2+\mathrm{e}^{8\lambda^2z^3/3A^3}\right)\mathrm{e}^{-z^2\lambda/(A^2+2\lambda)}\right]^{1/2} \\ &\quad\times\frac{1}{2\lambda}\sqrt{\mathrm{Var}(E^W(W'-W)^2)} \\ &\leqslant \left[\sqrt{\frac{3}{2\pi}}\frac{\mathrm{e}^{-z^2/2}}{z}\right. \\ &\quad\left.+\left(2+\mathrm{e}^{8\lambda^2z^3/3A^3}\right)^{1/2}\mathrm{e}^{-z^2\lambda/2(A^2+2\lambda)}\right] \\ &\quad\times\frac{1}{2\lambda}\sqrt{\mathrm{Var}(E^W(W'-W)^2)}. \end{aligned} \tag{29}$$

To bound $|T_2|$, we notice that

$$\begin{aligned}|T_2| &\leqslant E\int_{-\infty}^{\infty}\left|g_z'(W)-g_z'(W+t)\right|\hat{K}(t)\,\mathrm{d}t \\ &\leqslant T_{21}+T_{22}+T_{23}, \end{aligned} \tag{30}$$

where

$$T_{21}=E\mathbb{I}\left(\delta_W\geqslant\frac{z}{4}\right)\int_{-\infty}^{\infty}|g_z'(W)-g_z'(W+t)|\hat{K}(t)\,\mathrm{d}t$$

$$T_{22}=E\mathbb{I}\left(\delta_W<\frac{z}{4}\right)\times\int_{-\infty}^{\infty}\mathbb{I}(z-\max(0,t)<W<z-\min(0,t))\hat{K}(t)\,\mathrm{d}t$$

$$T_{23}=E\mathbb{I}\left(\delta_W<\frac{z}{4}\right)\int_{-\infty}^{\infty}\int_{t}^{0}h(W+u)\hat{K}(t)\,\mathrm{d}u\,\mathrm{d}t,$$

and $h(w)=(wg_z(w))'$. The term T_{21} vanishes by the assumption that $z>4A$. Hence the bound for T_2 only depends on T_{22} and T_{23}. A bound for T_{22} is obtained as follows. For $|t|\leqslant\delta_W$ and $z-\delta_W<W<z+\delta_W$, we have $z-2A<z-2\delta_W<W+t$ which implies that

$$\mathrm{e}^{sz(z-2A)}\leqslant\mathrm{e}^{sz(W+t)}\quad\text{for any } s>0. \tag{31}$$

Using (31), we can see that

$$\begin{aligned}|T_{22}| &\leqslant E\int_{|t|\leqslant\delta_W}\mathbb{I}(z-|t|<W<z+|t|)\hat{K}(t)\,\mathrm{d}t \\ &\leqslant E\int_{|t|\leqslant\delta_W}\mathbb{I}(z-\delta_W<W<z+\delta_W)\hat{K}(t)\,\mathrm{d}t \\ &\leqslant \frac{\mathrm{e}^{2szA}}{\mathrm{e}^{sz^2}} \\ &\quad\times E\int_{|t|\leqslant\delta_W}\mathrm{e}^{sz(W+t)}\mathbb{I}(z-\delta_W<W<z+\delta_W)\hat{K}(t)\,\mathrm{d}t. \end{aligned} \tag{32}$$

Let $s=\lambda/A^2$, $\delta>0$ and $f_\delta:\mathbb{R}\to\mathbb{R}$ be defined as

$$f_\delta(t)=\begin{cases}0, & t<z-2\delta,\\ \mathrm{e}^{szt}(t-z+2\delta), & z-2\delta\leqslant t\leqslant z+2\delta,\\ 4\delta\,\mathrm{e}^{szt}, & t>z+2\delta.\end{cases}$$

Then

$$f_\delta'(t)\geqslant\begin{cases}\mathrm{e}^{szt}, & z-2\delta<t<z+2\delta,\\ 0, & t<z-2\delta \text{ or } t>z+2\delta,\end{cases} \tag{33}$$

$$|f_\delta(t)|\leqslant 4\delta\,\mathrm{e}^{szt},\qquad\text{for all } t\in\mathbb{R}. \tag{34}$$

From (19), (33), and (34),

$$\begin{aligned}&E\int_{|t|\leqslant\delta_W}\mathrm{e}^{sz(W+t)}\mathbb{I}(z-\delta_W<W<z+\delta_W)\hat{K}(t)\,\mathrm{d}t \\ &\quad\leqslant E\int_{|t|\leqslant\delta_W}f'(W+t)\hat{K}(t)\,\mathrm{d}t \\ &\quad\leqslant EWf_{\delta_W}(W)\leqslant 4E|W|\delta_W\,\mathrm{e}^{szW}. \end{aligned} \tag{35}$$

From (32) and (35) and using the Hölder inequality and Lemma 1, we obtain

$$\begin{aligned}|T_{22}| &\leqslant \frac{4A\,\mathrm{e}^{2szA}}{\mathrm{e}^{sz^2}}(E|W|^2)^{1/2}(E\,\mathrm{e}^{2szW})^{1/2} \\ &\leqslant \frac{4A\,\mathrm{e}^{2szA}}{\mathrm{e}^{sz^2}}(E\,\mathrm{e}^{2szW})^{1/2} \\ &\leqslant \frac{4A\,\mathrm{e}^{2szA}}{\mathrm{e}^{sz^2}}\left(\mathrm{e}^{(A^2/4\lambda)(2sz)^2+(A^3/3\lambda)(2sz)^3}\right)^{1/2} \\ &= 4A\,\mathrm{e}^{2\lambda z/A+4\lambda^2z^3/3A^3-\lambda z^2/2A^2}. \end{aligned}$$

Hence

$$|T_{22}|\leqslant 4A\,\mathrm{e}^{2\lambda z/A+4\lambda^2z^3/3A^3-\lambda z^2/2A^2}. \tag{36}$$

For the bound for T_{23}, we notice that

$$\begin{aligned}T_{23} &\leqslant E\mathbb{I}\left(\delta_W<\frac{z}{4}\right) \\ &\quad\times\left[\int_{-\infty}^{\infty}\int_{t}^{0}h(W+u)\hat{K}(t)\mathbb{I}\left(W+u\leqslant\frac{3z}{4}\right)\mathrm{d}u\,\mathrm{d}t\right. \\ &\quad\left.+\int_{-\infty}^{\infty}\int_{t}^{0}h(W+u)\hat{K}(t)\mathbb{I}\left(W+u>\frac{3z}{4}\right)\mathrm{d}u\,\mathrm{d}t\right]. \end{aligned} \tag{37}$$

From the definition $h(w)=(wg_z(w))'$ written as

$$h(w)=\begin{cases}(\sqrt{2\pi}(1+w^2)\,\mathrm{e}^{w^2/2}\Phi(w)+w)[1-\Phi(z)], & w\leqslant z,\\ (\sqrt{2\pi}(1+w^2)\,\mathrm{e}^{w^2/2}(1-\Phi(w))-w)\Phi(z), & w>z,\end{cases}$$

we can obtain a bound for the function h as follows. If $w \leqslant 0$, then $h(w) \leqslant 2(1-\Phi(z)) \leqslant 2\,\mathrm{e}^{-z^2/2}/\sqrt{2\pi}z$. If $0 < w \leqslant 3z/4$, then

$$\begin{aligned} h(w) &\leqslant \left(\sqrt{2\pi}\left(1+\frac{9z^2}{16}\right)\mathrm{e}^{(9/32)z^2}+z\right)\frac{\mathrm{e}^{-z^2/2}}{\sqrt{2\pi}z} \\ &\leqslant \frac{1}{z}\left(1+\frac{9z^2}{16}\right)\mathrm{e}^{-(7/32)z^2}+\frac{\mathrm{e}^{-z^2/2}}{\sqrt{2\pi}}. \end{aligned}$$

Hence, for $w \leqslant 3z/4$,

$$h(w) \leqslant \frac{1}{z}\left(1+\frac{9z^2}{16}\right)\mathrm{e}^{-(7/32)z^2}+\frac{\mathrm{e}^{-z^2/2}}{\sqrt{2\pi}}. \tag{38}$$

For $w > 3z/4$, following Ref. 17 we have

$$h(w) \leqslant 1.001(1+z) \quad \text{for } w > \frac{3z}{4}. \tag{39}$$

Using (38) and the fact that

$$\int_{-\infty}^{\infty}\int_{t}^{0}\hat{K}(t)\,\mathrm{d}u\,\mathrm{d}t \leqslant \frac{1}{4\lambda}|W'-W|^3 \leqslant \frac{A^3}{4\lambda}, \tag{40}$$

we obtain

$$\begin{aligned} &E\mathbb{I}\left(\delta_W < \frac{z}{4}\right) \\ &\quad\times\int_{-\infty}^{\infty}\int_{t}^{0}h(W+u)\hat{K}(t)\mathbb{I}\left(W+u \leqslant \frac{3z}{4}\right)\mathrm{d}u\,\mathrm{d}t \\ &\quad\leqslant \frac{A^3}{4\lambda}\left(\frac{1}{z}\left(1+\frac{9z^2}{16}\right)\mathrm{e}^{-(7/32)z^2}+\frac{\mathrm{e}^{-z^2/2}}{\sqrt{2\pi}}\right). \end{aligned} \tag{41}$$

Applying Markov's inequality and Lemma 1, we obtain

$$P\left(W > \frac{z}{2}\right) = P\left(\frac{\lambda z W}{A^2} > \frac{\lambda z^2}{2A^2}\right) \leqslant \mathrm{e}^{(-\lambda z^2)/4A^2+\lambda^2 z^3/3A^3}. \tag{42}$$

Using (39), (40) and (42), we can show that

$$\begin{aligned} &E\mathbb{I}\left(\delta_W < \frac{z}{4}\right) \\ &\quad\times\int_{-\infty}^{\infty}\int_{t}^{0}h(W+u)\hat{K}(t)\mathbb{I}\left(W+u > \frac{3z}{4}\right)\mathrm{d}u\,\mathrm{d}t \\ &\quad\leqslant 1.001(1+z)E\mathbb{I}\left(\delta_W < \frac{z}{4}\right) \\ &\times\int_{-\infty}^{\infty}\int_{t}^{0}\hat{K}(t)\mathbb{I}\left(W+\delta_W > \frac{3z}{4}\right)\mathrm{d}u\,\mathrm{d}t \\ &\quad\leqslant 1.001(1+z)E\int_{-\infty}^{\infty}\int_{t}^{0}\hat{K}(t)\mathbb{I}\left(W > \frac{z}{2}\right)\mathrm{d}u\,\mathrm{d}t \\ &\quad\leqslant 1.001(1+z)\frac{A^3}{4\lambda}P\left(W > \frac{z}{2}\right) \\ &\quad\leqslant \frac{1.001A^3}{4\lambda}(1+z)\,\mathrm{e}^{-\lambda z^2/4A^2+\lambda^2 z^3/3A^3}. \end{aligned} \tag{43}$$

From (37), (41), and (43), we have

$$\begin{aligned} |T_{23}| &\leqslant \frac{A^3}{4\lambda}\left(\frac{1}{z}\left(1+\frac{9z^2}{16}\right)\mathrm{e}^{-(7/32)z^2}+\frac{\mathrm{e}^{-(1/2)z^2}}{\sqrt{2\pi}}\right) \\ &\quad+\frac{1.001A^3}{4\lambda}(1+z)\,\mathrm{e}^{-\lambda z^2/4A^2+\lambda^2 z^3/3A^3}. \end{aligned} \tag{44}$$

From (30), (36), and (44),

$$\begin{aligned} |T_2| &\leqslant 4A\,\mathrm{e}^{2\lambda z/A+4\lambda^2 z^3/3A^3-\lambda z^2/2A^2}+\frac{A^3}{4\lambda} \\ &\quad\times\left(\frac{1}{z}\left(1+\frac{9z^2}{16}\right)\mathrm{e}^{-(7/32)z^2}+\frac{\mathrm{e}^{-(1/2)z^2}}{\sqrt{2\pi}}\right) \\ &\quad+\frac{1.001A^3}{4\lambda}(1+z)\,\mathrm{e}^{-\lambda z^2/4A^2+\lambda^2 z^3/3A^3}. \end{aligned} \tag{45}$$

By (25), (29), and (45), we have

$$\begin{aligned} &|P(W \leqslant z)-\Phi(z)| \\ &\quad\leqslant\Bigg[\sqrt{\frac{3}{2\pi}}\frac{\mathrm{e}^{-z^2/2}}{z} \\ &\qquad+\left(2+\mathrm{e}^{(8/3)\lambda^2 z^3/A^3}\right)^{1/2}\mathrm{e}^{-(z^2/2)\left((\lambda/(A^2+2\lambda))\right)}\Bigg] \\ &\qquad\times\frac{1}{2\lambda}\sqrt{\mathrm{Var}(E^W(W'-W)^2)} \\ &\qquad+\frac{A^3}{4\lambda}\left(\frac{1}{z}\left(1+\frac{9z^2}{16}\right)\mathrm{e}^{-(7/32)z^2}+\frac{\mathrm{e}^{-(1/2)z^2}}{\sqrt{2\pi}}\right) \\ &\qquad+\frac{1.001A^3}{4\lambda}(1+z)\,\mathrm{e}^{-\lambda z^2/4A^2+\lambda^2 z^3/3A^3} \\ &\qquad+4A\,\mathrm{e}^{2\lambda z/A+4\lambda^2 z^3/3A^3-\lambda z^2/2A^2} \\ &\quad\leqslant C_1(z)\frac{1}{2\lambda}\sqrt{\mathrm{Var}(E^W(W'-W)^2)} \\ &\qquad+C_2(z)\frac{0.41A^3}{\lambda}+1.5AC_3(z) \end{aligned}$$

where

$$\begin{aligned} C_1(z) &= \sqrt{\frac{3}{2\pi}}\frac{\mathrm{e}^{-z^2/2}}{z} \\ &\quad+\left(2+\mathrm{e}^{(8/3)\lambda^2 z^3/A^3}\right)^{1/2}\mathrm{e}^{-(z^2/2)\left((\lambda/(A^2+2\lambda))\right)} \\ C_2(z) &= 0.610\left(\frac{1}{z}\left(1+\frac{9z^2}{16}\right)\mathrm{e}^{-(7/32)z^2}+\frac{\mathrm{e}^{-(1/2)z^2}}{\sqrt{2\pi}}\right) \\ &\quad+0.620(1+z)\,\mathrm{e}^{-\lambda z^2/4A^2+\lambda^2 z^3/3A^3} \\ C_3(z) &= \tfrac{8}{3}\,\mathrm{e}^{2\lambda z/A+4\lambda^2 z^3/3A^3-\lambda z^2/2A^2}. \end{aligned}$$

Furthermore, if $\max(|z|,|z|^3) \leqslant 1/A$, then we can see that

$$\frac{\lambda^2|z|^3}{A^3} \leqslant \frac{\lambda^2}{A^4}, \quad \frac{\lambda|z|}{A} \leqslant \frac{\lambda}{A^2}.$$

Hence for $i = 1, 2, 3$, there exist positive constants a and C such that $C_i(z) \leqslant C\,\mathrm{e}^{-az^2}$. □

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