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Two numerical iteration methods for solving absolute value equations

Jun He*, **Yanmin Liu**, **Junkang Tian**

School of Mathematics, Zunyi Normal College, Zunyi, Guizhou, 563006, China

*Corresponding author, e-mail: hejunfan1@163.com

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ABSTRACT: We introduce two kinds of iteration methods based on the generalization of the hermitian and skew-hermitian splitting. Their convergence properties are established. We show the effectiveness of our methods.

KEYWORDS: linear complementarity problem; GHSS-based iteration; Picard-HSS iteration; nonlinear HSS-like iteration

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INTRODUCTION

We consider the following absolute value equation (AVE):

$$Ax - |x| = b, \quad A \in \mathbb{C}^{n\times n}, \quad x, b \in \mathbb{C}^n, \qquad (1)$$

where $|x|$ denotes the componentwise absolute value of the vector x. The system (1), which is generally equivalent to the linear complementarity problem[1,2], arises from linear programming, quadratic programming, and other engineering problems[3,4].

In the last decade, some numerical methods have been developed to solve AVEs. These include the SLP method[5], the semismooth Newton method and its inexact variants[6–8], the sign accord method[9], the hybrid algorithm[10], the Picard-CSCS iteration method, and the nonlinear CSCS-like iteration method[11]. When $A \in \mathbb{C}^{n\times n}$ is a non-Hermitian positive definite matrix, the Hermitian and skew-Hermitian splitting (HSS) iteration was first introduced by Bai et al[12] and extended in Ref. 13 for the solution of a class of non-Hermitian linear systems $Ax = b$. Based on the Hermitian and skew-Hermitian splitting, the Picard-HSS iteration method, nonlinear HSS-like iteration method, and the relaxed nonlinear PHSS-like iteration method are introduced to solve AVEs[14,15].

In this paper, two kinds of GHSS-based iteration methods are established to solve (1) efficiently. These are based on the following generalization of the hermitian and skew-hermitian splitting (GHSS):

$$A = G + K + S,$$

where $H = \frac{1}{2}(A + A^{\mathrm{T}}) = G + K$, G and K are Hermitian positive semidefinite, and $S = \frac{1}{2}(A - A^{\mathrm{T}})$. Firstly, convergence conditions of the Picard-GHSS iterative method will be investigated. Then the convergence condition of the nonlinear GHSS-like iteration method is proved by Zhang's techniques[15].

The reminder of this paper is organized as follows. First, we introduce several preliminary results concerning nonsmooth analysis. Then we introduce the Picard-GHSS and the nonlinear GHSS-like iterative methods to solve (1) and investigate its convergence properties. Finally, numerical experiments are reported to show the effectiveness of our methods.

PRELIMINARIES

Let $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ be a specified function, and let x be a given point in $\mathbb{R}^n$. The function Ψ is supposed to be locally Lipschitzian near x if there exist a scalar $\kappa \in \mathbb{R}$ and $\delta > 0$ such that for all $y, z \in \mathbb{R}^n$, $\|y - x\| < \delta$, $\|z - x\| < \delta$, the following inequality holds:

$$\|\Psi(y) - \Psi(z)\| < \kappa\|y - z\|.$$

Let $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ be a locally Lipschitzian function. From Rademacher's theorem[15], Ψ is differentiable almost everywhere. Denote the set of points at which Ψ is differentiable by D_Ψ. We write $\Psi'(x)$ for the usual $n \times n$ Jacobian matrix of partial derivatives whenever x is a point at which the necessary partial derivatives exist. Then, the Bouligand subdifferential of Ψ at $x \in \mathbb{R}^n$, denoted by $\partial_{\mathrm{B}}\Psi(x)$, is given by

$$\partial_{\mathrm{B}}\Psi(x) := \left\{ \lim_{k\to\infty} \Psi'(x)(x^{(k)}) : x^{(k)} \in D_\Psi, x^{(k)} \to x \right\}.$$

Clarke's generalized Jacobian[16] of Ψ at x is the convex hull of $\partial_{\mathrm{B}}\Psi(x)$, i.e.,

$$\partial\Psi(x) = \mathrm{conv}\{\partial_{\mathrm{B}}\Psi(x)\}.$$

Since Ψ is a locally Lipschitzian function, the set $\partial_{\mathrm{B}}\Psi(x)$ and $\partial\Psi(x)$ are bounded. By the definition, $\partial_{\mathrm{B}}\Psi(x)$ is also closed. So $\partial_{\mathrm{B}}\Psi(x)$ and $\partial\Psi(x)$ are compact.

Definition 1 [Ref. 17] Ψ is called *semismooth* at x if Ψ is locally Lipschitzian and for all $h \in \mathbb{R}$. with $h \neq 0$, and

$$\lim_{h' \to h, t \downarrow 0} \{Vh' : V \in \partial\Psi(x + th')\} \tag{2}$$

exists. If Ψ is semismooth at all points in a given set, we say that Ψ is semismooth in this set.

If Ψ is semismooth at x, then Ψ must be directionally differentiable at x.

Proposition 1 (Refs. 17, 18) *Suppose that Ψ is semismooth at x. Then the classic directional derivative*

$$\Psi'(x; h) = \lim_{t \downarrow 0} \frac{\Psi(x + th) - \Psi(x)}{t}$$

exists and is equal to the limit in (2).

THE PICARD-GHSS ITERATIVE METHOD

Motivated by Ref. 13, we give the following two numerical iteration methods to solve (1).

Algorithm 1 (Picard-GHSS iterative method) Let $A \in \mathbb{C}^{n\times n}$ be a sparse and positive definite matrix, and $H = \frac{1}{2}(A + A^{\mathrm{T}}) = G + K$ and $S = \frac{1}{2}(A - A^{\mathrm{T}})$ be its Hermitian and skew-Hermitian parts, respectively. Given an initial guess $x^{(0)} \in \mathbb{C}^n$, compute $x^{(k+1)}$ for $k = 0, 1, 2, \dots$ using the following iteration scheme until $\{x^{(k)}\}$ satisfies the stopping criterion:

$$(\alpha I + G)x^{(k+1/2)} = (\alpha I - S - K)x^{(k)} + |x^{(k)}| + b,$$
$$(\alpha I + S + K)x^{(k+1)} = (\alpha I - G)x^{(k+1/2)} + |x^{(k)}| + b,$$

where α is a given positive constant and I is the identity matrix.

Let $\mathcal{M}(\alpha) = (\alpha I + S + K)^{-1}(\alpha I - G)(\alpha I + G)^{-1}(\alpha I - S - K)$, $\mathcal{G}(\alpha) = 2\alpha(\alpha I + S + K)^{-1}(\alpha I + G)^{-1}$, where α is a positive constant and I is the identity matrix of order n. The following theorem suggests sufficient conditions for the convergence of the Picard-GHSS iteration method for solving (1).

Theorem 1 *Let $A \in \mathbb{C}^{n\times n}$ be a sparse and positive definite matrix, $H = \frac{1}{2}(A + A^{\mathrm{T}}) = G + K$ and $S = \frac{1}{2}(A - A^{\mathrm{T}})$ be its Hermitian and skew-Hermitian parts, respectively. Let also $\eta = \|A^{-1}\|_2 < 1$. Then (1) has a unique solution x^*, and for any initial guess $x^{(0)} \in \mathbb{C}^n$ and any sequence of positive integers ℓ_k, $k = 0, 1, 2, \dots$, the iteration sequence $\{x^{(k)}\}_{k=0}^{\infty}$ produced by the Picard-GHSS iteration method converges to x^* provided that $l = \liminf_{k\to\infty} l_k \geqslant N$, where N is a natural number satisfying*

$$\left\|(\mathcal{M}(\alpha))^s\right\|_2 < \frac{1-\eta}{1+\eta}, \quad \forall s \geqslant N.$$

Proof: Since $\eta < 1$ and from the conclusion of Ref. 19, AVE (1) has a unique solution $x^* \in \mathbb{C}^n$. As with the proof of the sufficient condition for the convergence of the Picard-HSS iteration method[15], for $k = 0, 1, 2, \dots$, the $(k+1)$th iterate of the Picard-GHSS iteration is

$$x^{(k+1)} = (\mathcal{M}(\alpha))^{l_k} x^{(k)} + \sum_{j=0}^{l_k - 1} (\mathcal{M}(\alpha))^j \mathcal{G}(\alpha)(|x^{(k)}| + b). \tag{3}$$

On the other hand, since x^* is the solution of (1), we can obtain

$$x^* = (\mathcal{M}(\alpha))^{l_k} x^* + \sum_{j=0}^{l_k - 1} (\mathcal{M}(\alpha))^j \mathcal{G}(\alpha)(|x^*| + b). \tag{4}$$

Subtracting (4) from (3) yields

$$x^{(k+1)} - x^* = (\mathcal{M}(\alpha))^{l_k}(x^{(k)} - x^*) + \sum_{j=0}^{l_k - 1} (\mathcal{M}(\alpha))^j \mathcal{G}(\alpha)(|x^{(k)}| - |x^*|). \tag{5}$$

Furthermore, since $\rho(\mathcal{M}(\alpha)) < 1$, we obtain

$$\begin{aligned}
&\sum_{j=0}^{l_k - 1} (\mathcal{M}(\alpha))^j \mathcal{G}(\alpha) \\
&\quad = (I - (\mathcal{M}(\alpha))^{l_k})(I - \mathcal{M}(\alpha))^{-1}\mathcal{G}(\alpha) \\
&\quad = (I - (\mathcal{M}(\alpha))^{l_k})A^{-1}.
\end{aligned}$$

Substituting the above identity in (5) yields

$$\begin{aligned}
&x^{(k+1)} - x^* \\
&\quad = (\mathcal{M}(\alpha))^{l_k}(x^{(k)} - x^*) \\
&\qquad + (I - (\mathcal{M}(\alpha))^{l_k})A^{-1}(|x^{(k)}| - |x^*|) \\
&\quad = (\mathcal{M}(\alpha))^{l_k}\left[(x^{(k)} - x^*) - A^{-1}(|x^{(k)}| - |x^*|)\right] \\
&\qquad + A^{-1}(|x^{(k)}| - |x^*|).
\end{aligned}$$

Hence

$$\|x^{(k+1)}-x^*\|_2 \leqslant \left(\|(\mathscr{M}(\alpha))^{l_k}\|_2(1+\eta)+\eta\right)\|x^{(k)}-x^*\|_2.$$

The above inequality is true due to the fact that for any $x, y \in \mathbb{C}^n$. It follows that $\||x|-|y|\|_2 \leqslant \|x-y\|_2$. Since $\rho(\mathscr{M}(\alpha)) < 1$, we have $\lim_{s\to\infty}(\mathscr{M}(\alpha))^s = 0$. Thus there exists a natural number N such that

$$\|(\mathscr{M}(\alpha))^s\|_2 < \frac{1-\eta}{1+\eta}, \quad \forall s \geqslant N.$$

If we suppose that $l = \liminf_{k\to\infty} l_k \geqslant N$. □

THE NONLINEAR GHSS-LIKE ITERATIVE METHOD

Algorithm 2 (Nonlinear GHSS-like iterative method) Let $A \in C^{n\times n}$ be a sparse and positive definite matrix, and $H = \frac{1}{2}(A+A^{\mathrm{T}}) = G+K$ and $S = \frac{1}{2}(A-A^{\mathrm{T}})$ be its Hermitian and skew-Hermitian parts, respectively. Given an initial guess $x^{(0)} \in C^n$, compute $x^{(k+1)}$ for $k = 0, 1, 2, \dots$ using the following iteration scheme until $\{x^{(k)}\}$ satisfies the stopping criterion:

$$\begin{aligned}(\alpha I+G)x^{(k+1/2)} &= (\alpha I-S-K)x^{(k)}+|x^{(k)}|+b,\\ (\alpha I+S+K)x^{(k+1)} &= (\alpha I-G)x^{(k+1/2)}\\ &\quad +|x^{(k+1/2)}|+b,\end{aligned}$$

where α is a given positive constant and I is the identity matrix.

Define

$$\begin{aligned}\mathscr{U}(x) &= (\alpha I+G)^{-1}[(\alpha I-S-K)x+|x|+b],\\ \mathscr{V}(x) &= (\alpha I+S+K)^{-1}[(\alpha I-G)x+|x|+b],\end{aligned} \tag{6}$$

and

$$\Theta(x) = \mathscr{V}\circ\mathscr{U}(x) := \mathscr{V}(\mathscr{U}(x)). \tag{7}$$

Then the nonlinear GHSS-like iterative scheme can be equivalently expressed as

$$x^{(k+1)} = \Theta(x^{(k)}). \tag{8}$$

Using Zhang's techniques [15], we will analyse the convergence of the nonlinear GHSS-like iteration method.

Definition 2 [Ref. 20] Let $\Theta: \mathbb{D}\subset\mathbb{R}^n \to \mathbb{R}^n$. Then x^* is a point of attraction of the iteration (8) if there is an open neighbourhood S of the point x^* such that $S\subset\mathbb{D}$ and, for any $x^{(0)} \in S$, the iterates $\{x^{(k)}\}$ all lie in $\mathbb{D}$ and converge to x^*.

Proposition 2 (Ref. 15) *Suppose that $\Theta: \mathbb{R}^n \to \mathbb{R}^n$ has a fixed point $x^* \in \mathbb{R}^n$ and is semismooth at x^*. If for all $V \in \partial_{\mathrm{B}}\Theta(x^*)$, we have $\rho(V) < 1$, where $\rho(V)$ denotes the spectral radius of V. Then x^* is a point of attraction of the iteration scheme (8).*

From statements in Ref. 15, let x^* satisfy $Ax^* - |x^*| = b$. We compute the Bouligand subdifferential of $\Theta(x^*)$ defined by (7)–(8) at x^*. Because of the special form of $\mathscr{U}$ and $\mathscr{V}$, it is easy to verify that, $x^* = \mathscr{U}(x^*)$, $x^* = \mathscr{V}(x^*)$, and $x^* = \Theta(x^*)$. Observe the special form of Θ. We have that

$$\begin{aligned}\partial_{\mathrm{B}}\Theta(x^*) &= \left\{\lim_{k\to\infty}\Theta'(x^{(k)}) : x^{(k)} \in D_\Theta, x^{(k)} \to x^*\right\}\\ &= \Big\{\lim_{k\to\infty}\mathscr{V}'(y^{(k)})\mathscr{U}'(x^{(k)}) : x^{(k)} \in D_{\mathscr{U}},\\ &\qquad y^{(k)} = \mathscr{V}(x^{(k)}) \in D_{\mathscr{V}}, x^{(k)} \to x^*\Big\}\\ &\subset \left\{\lim_{y^{(k)}\to x^*}\mathscr{V}'(y^{(k)}) : y^{(k)} \in D_{\mathscr{V}}\right\},\end{aligned}$$

and

$$\left\{\lim_{x^{(k)}\to x^*}\mathscr{U}'(x^{(k)}) : x^{(k)} \in D_{\mathscr{U}}\right\} \subset \partial_{\mathrm{B}}\mathscr{V}(x^*)\partial_{\mathrm{B}}\mathscr{U}(x^*),$$

where

$$\begin{aligned}\partial_{\mathrm{B}}\mathscr{V}(x^*)\partial_{\mathrm{B}}\mathscr{U}(x^*) := \{W : W = EF,\\ E \in \partial_{\mathrm{B}}\mathscr{V}(x^*),\ F \in \partial_{\mathrm{B}}\mathscr{U}(x^*)\}.\end{aligned}$$

According to the above discussion and Proposition 2, we immediately obtain the following conclusion about the convergence of the nonlinear GHSS-like iteration method.

Theorem 2 *Let the point x^* satisfy $Ax^* = |x^*| + b$. Let $A \in C^{n\times n}$ be a sparse and positive definite matrix, and $H = \frac{1}{2}(A+A^{\mathrm{T}}) = G+K$ and $S = \frac{1}{2}(A-A^{\mathrm{T}})$ be its Hermitian and skew-Hermitian parts, respectively. Furthermore, $F, \tilde{F} \in \partial_{\mathrm{B}}|x^*|$. We write*

$$\mathscr{M}(\alpha; F, \tilde{F}) = \mathscr{T}_1(\alpha; F)\mathscr{T}_2(\alpha; \tilde{F}),$$

where

$$\begin{aligned}\mathscr{T}_1(\alpha; F) &= (\alpha I+S+K)^{-1}[(\alpha I-G)+F],\\ \mathscr{T}_2(\alpha; \tilde{F}) &= (\alpha I+G)^{-1}[(\alpha I-S-K)+\tilde{F}].\end{aligned}$$

If for all $F,\ \tilde{F} \in \partial_{\mathrm{B}}|x^*|$, $\rho(\mathscr{M}(\alpha; F, \tilde{F})) < 1$, then x^* is a point of attraction of the nonlinear GHSS-like iteration method.

Proof: It is clear that $\mathscr{U}$ and $\mathscr{V}$ are semismooth and so Θ is semismooth. Let $E \in \partial_{\mathrm{B}}|x^*|$. Then it is not hard to show that E is a diagonal matrix:

$$E = \operatorname{diag}(E_{11}, E_{22}, \dots, E_{nn}).$$

We have $E_{ii} = 1$, if $x_i^* > 0$; $E_{ii} = -1$, if $x_i^* < 0$, and $E_{ii} \in \{1, -1\}$, if $x_i^* = 0$. If $W \in \partial_{\mathrm{B}}\mathscr{F}(x^*)$, then $W = (\alpha I + S + K)^{-1}[(\alpha I - G) + F]$, where $F \in \partial_{\mathrm{B}}|x^*|$. If $\tilde{W} \in \partial_{\mathrm{B}}\mathscr{U}(x^*)$, then $\tilde{W} = (\alpha I + G)^{-1}[(\alpha I - S - H) + \tilde{F}]$, where $\tilde{F} \in \partial_{\mathrm{B}}|x^*|$. Since $\partial_{\mathrm{B}}\Theta(x^*) \subset \partial_{\mathrm{B}}\mathscr{V}(x^*)\partial_{\mathrm{B}}\mathscr{U}(x^*)$. If for all $F, \tilde{F} \in \partial_{\mathrm{B}}|x^*|$, $\rho(\mathscr{M}(\alpha; F, \tilde{F})) < 1$, then for all $W \in \partial_{\mathrm{B}}\Theta(x^*)$, we have $\rho(W) < 1$. We can complete the proof using Proposition 2. □

Corollary 1 *Let the point x^* satisfy $Ax^* = |x^*| + b$. Let $A \in C^{n\times n}$ be a sparse and positive definite matrix, and $H = \frac{1}{2}(A + A^{\mathrm{T}}) = G + K$ and $S = \frac{1}{2}(A - A^{\mathrm{T}})$ be its Hermitian and skew-Hermitian parts, respectively. Furthermore, $F, \tilde{F} \in \partial_{\mathrm{B}}|x^*|$. We write*

$$t_1(\alpha) = \|(\alpha I + S + K)^{-1}(\alpha I - G)\|,$$
$$t_2(\alpha) = \|(\alpha I + G)^{-1}(\alpha I - S - K)\|,$$

and

$$\delta = \max\{\|(\alpha I + G)^{-1}\tilde{F}\|, \|(\alpha I + S + K)^{-1}F\|\}.$$

If $t_1(\alpha)t_2(\alpha) < 1$ and for all $F, \tilde{F} \in \partial_{\mathrm{B}}|x^*|$,

$$\delta < \frac{2 - 2t_1(\alpha)t_2(\alpha)}{\sqrt{(t_1(\alpha) - t_2(\alpha))^2 + 4} + t_1(\alpha) + t_2(\alpha)}, \quad (9)$$

then x^* is a point of attraction of the nonlinear GHSS-like iterative method.

Proof: By simple calculations we obtain $\mathscr{M}(\alpha; F, \tilde{F}) = (\alpha I + S + K)^{-1}[(\alpha I - G) + F](\alpha I + G)^{-1}[(\alpha I - S - K) + \tilde{F}] = (\alpha I + S + K)^{-1}(\alpha I - G)(\alpha I + G)^{-1}(\alpha I - S - K) + (\alpha I + S + K)^{-1}(\alpha I - G)(\alpha I + G)^{-1}\tilde{F} + (\alpha I + S + K)^{-1}F(\alpha I + G)^{-1}(\alpha I - S - K) + (\alpha I + S + K)^{-1}F(\alpha I + G)^{-1}\tilde{F}$. Hence

$$\begin{aligned}
&\|\mathscr{M}(\alpha; F, \tilde{F})\| \\
&\quad \leqslant \|(\alpha I + S + K)^{-1}(\alpha I - G)\| \\
&\qquad \cdot \|(\alpha I + G)^{-1}(\alpha I - S - K)\| \\
&\qquad + \|(\alpha I + S + K)^{-1}(\alpha I - G)\|\|(\alpha I + G)^{-1}\tilde{F}\| \\
&\qquad + \|(\alpha I + S + K)^{-1}F\|\|(\alpha I + G)^{-1}(\alpha I - S - K)\| \\
&\qquad + \|(\alpha I + S + K)^{-1}F\|\|(\alpha I + G)^{-1}\tilde{F}\| \\
&\quad \leqslant t_1(\alpha)t_2(\alpha) + \delta(t_1(\alpha) + t_2(\alpha)) + \delta^2.
\end{aligned}$$

With the help of (9) we obtain

$$t_1(\alpha)t_2(\alpha) + \delta(t_1(\alpha) + t_2(\alpha)) + \delta^2 < 1.$$

Hence we have

$$\rho(\mathscr{M}(\alpha; F, \tilde{F})) \leqslant \|\mathscr{M}(\alpha; F, \tilde{F})\| < 1,$$

which follows from Theorem 2. □

NUMERICAL EXAMPLES

In this section we present a sample of numerical experiments conducted in order to assess the effectiveness of the Picard-GHSS and nonlinear GHSS-like iterative methods. All experiments were performed in MATLAB R2010a, on an Intel Core i5-3210CPU at 2.50 GHz with 4.00 GB RAM, and terminated when the current residual satisfied

$$\frac{\|Ax^{(k)} - |x^{(k)}| - b\|_2}{\|b\|} < 10^{-6}.$$

The stopping criterion for the inner iterations of the Picard-GHSS and nonlinear GHSS-like iterative methods was set to be

$$\frac{\|b^{(k)} - As^{(k,l_k)}\|_2}{\|b^{(k)}\|_2} \leqslant \eta_k,$$

where $b^{(k)} = \|x^{(k)}\| + b - Ax^{(k)}$, $s^{(k,l_k)} = x^{(k,l_k)} - x^{(k,l_k-1)}$, l_k is the number of the inner iteration steps and η_k is the prescribed tolerance for controlling the accuracy of the inner iterations at the kth outer iteration. If η_k is fixed for all k, then it is simply denoted by η_k. Here, we take $\eta = 0.1$.

The generated test problems are the two-dimensional convection-diffusion equation[15]:

$$-u_{xx} - u_{yy} + q(u_x + u_y) + pu = f(x, y), \ (x, y) \in \Omega,$$
$$u(x, y) = 0, \ (x, y) \in \partial\Omega,$$

where $\Omega = (0, 1) \times (0, 1)$, $\partial\Omega$ is its boundary, q is a positive constant used to measure the magnitude of the diffusive term, and p is a real number. We use the five-point finite difference scheme for the diffusive terms and the central difference scheme for the convective terms. Let $h = 1/(m + 1)$ and $\mathrm{Re} = \frac{1}{2}qh$ denote the equidistant step size and the mesh Reynolds number, respectively. Then we obtain a system of linear equations $Ax = d$, where A is a matrix of order $n = m^2$ of the form

$$A = T_x \otimes I_m + I_m \otimes T_y + pI_n = A_1 + A_2 + pI_n, \quad (10)$$

with

$$T_x = \begin{pmatrix} t_1 & t_3 & & \\ t_2 & t_1 & \ddots & \\ & \ddots & \ddots & t_3 \\ & & t_2 & t_1 \end{pmatrix}_{m\times m},$$

Table 1 Numerical results for test problems with different values of m and q ($p = 2$).

q	Method		$m = 10$	$m = 20$	$m = 40$
0	PG	IT	9	10	9
		CPU	0.0119	0.2686	7.5407
		RES	9.2×10^{-6}	9.6×10^{-6}	8.3×10^{-6}
	G	IT	**7**	**7**	**7**
		CPU	0.0079	0.1044	3.1403
		RES	3.7×10^{-6}	7.3×10^{-6}	5.7×10^{-6}
	PH	IT	13	13	13
		CPU	0.0089	0.2271	9.6186
		RES	7.9×10^{-6}	8.3×10^{-6}	4.4×10^{-6}
	H	IT	10	10	10
		CPU	0.0093	0.1157	3.8794
		RES	5.3×10^{-6}	9.5×10^{-6}	9.6×10^{-6}
1	PG	IT	10	10	10
		CPU	0.0051	0.1486	4.5015
		RES	9.8×10^{-6}	8.2×10^{-6}	6.0×10^{-6}
	G	IT	**7**	**7**	**7**
		CPU	0.0050	0.1201	3.2571
		RES	6.9×10^{-6}	8.5×10^{-6}	6.1×10^{-6}
	PH	IT	13	13	13
		CPU	0.0107	0.1709	5.7426
		RES	7.9×10^{-6}	7.0×10^{-6}	7.4×10^{-6}
	H	IT	10	11	11
		CPU	0.0058	0.1413	4.7607
		RES	9.9×10^{-6}	8.1×10^{-6}	5.3×10^{-6}
10	PG	IT	10	10	10
		CPU	0.0083	0.1382	4.4619
		RES	9.6×10^{-6}	8.4×10^{-6}	9.1×10^{-6}
	G	IT	**7**	**7**	**7**
		CPU	0.0081	0.1169	3.2828
		RES	6.9×10^{-6}	6.2×10^{-6}	5.1×10^{-6}
	PH	IT	13	14	13
		CPU	0.0171	0.2006	5.6773
		RES	9.6×10^{-6}	4.7×10^{-6}	4.7×10^{-6}
	H	IT	11	11	11
		CPU	0.0083	0.1570	4.8900
		RES	8.5×10^{-6}	8.4×10^{-6}	5.4×10^{-6}

PG = Picard-GHSS; G = GHSS-like; PH = Picard-HSS; H = HSS-like

Table 2 Numerical results for test problems with different values of m and q ($p = 2.5$).

q	Method		$m = 10$	$m = 20$	$m = 40$
0	PG	IT	8	9	9
		CPU	0.0049	0.2091	5.8690
		RES	9.3×10^{-6}	2.0×10^{-6}	2.2×10^{-6}
	G	IT	**5**	**5**	**5**
		CPU	0.0032	0.1580	3.8387
		RES	9.9×10^{-6}	7.4×10^{-6}	6.7×10^{-6}
	PH	IT	11	11	12
		CPU	0.0161	0.2055	7.4478
		RES	9.0×10^{-6}	9.4×10^{-6}	4.0×10^{-6}
	H	IT	9	10	10
		CPU	0.0083	0.1237	6.0485
		RES	9.2×10^{-6}	8.1×10^{-6}	5.9×10^{-6}
1	PG	IT	8	9	9
		CPU	0.0096	0.2115	5.8399
		RES	9.2×10^{-6}	1.8×10^{-6}	2.0×10^{-6}
	G	IT	**5**	**5**	**5**
		CPU	0.0040	0.1420	3.7202
		RES	6.9×10^{-6}	8.5×10^{-6}	6.1×10^{-6}
	PH	IT	11	12	12
		CPU	0.0074	0.3014	7.9885
		RES	9.2×10^{-6}	4.1×10^{-6}	4.7×10^{-6}
	H	IT	10	10	10
		CPU	0.0070	0.2296	6.6400
		RES	9.1×10^{-6}	6.0×10^{-6}	4.1×10^{-6}
10	PG	IT	9	9	9
		CPU	0.0112	0.2203	6.1671
		RES	7.5×10^{-6}	8.0×10^{-6}	7.5×10^{-6}
	G	IT	**6**	**6**	**6**
		CPU	0.0068	0.1450	4.1590
		RES	8.7×10^{-6}	5.9×10^{-6}	3.0×10^{-6}
	PH	IT	12	12	12
		CPU	0.0150	0.2429	8.0871
		RES	7.6×10^{-6}	5.5×10^{-6}	5.9×10^{-6}
	H	IT	10	10	11
		CPU	0.0093	0.2298	6.2374
		RES	8.7×10^{-6}	5.7×10^{-6}	5.6×10^{-6}

and

$$T_y = \begin{pmatrix} 0 & t_3 & & \\ t_2 & 0 & \ddots & \\ & \ddots & \ddots & t_3 \\ & & t_2 & 0 \end{pmatrix}_{m\times m},$$

where $t_1 = 4$, $t_2 = -1 - \mathrm{Re}$, $t_3 = -1 + \mathrm{Re}$, I_m and I_n are the identity matrices of order m and n, respectively, and $\otimes$ means the Kronecker product. In our numerical experiments, the matrix A in (1) is defined by (10) with different values of q and $G = \frac{1}{2}(A_1 + A_1^{\mathrm{T}})$. In order to make the hermitian part of $A_2 + pI_n$ positive definite, we take $p = 2$ (or 2.5), and we take the zero vector as the initial guess, and the right-hand side vector b of (1) is taken in such a way that the vector $x = (x_1, x_2, \dots, x_n)$ with $x_k = (-1)^k i$ ($k = 1, 2, \dots, n$) is the exact solution.

The optimal parameter α employed in each method is experimentally determined such that it results in the least number of iterations. In Table 1 and Table 2, we report the numerical results from the Picard-HSS, the nonlinear HSS-like, the Picard-GHSS and the nonlinear GHSS-like iterations for different p and q. We also present the elapsed CPU time in seconds for the convergence (denoted by CPU) and the number of total iteration steps for the convergence. We find that all the methods give approximate solutions of AVEs for all the different matrix dimensions tried, and the nonlinear GHSS-like method is the most efficient method.

CONCLUSIONS

In this paper, two kinds of GHSS-based iteration methods based on the generalization of the hermitian and skew-hermitian splitting (GHSS) have been described, and the local convergence of the Picard-GHSS and nonlinear GHSS-like iterative methods have been given. Numerical tests show that Picard-GHSS and nonlinear GHSS-like iterative methods perform better than Picard-HSS and nonlinear HSS-like methods.

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