The long-time dynamics of 3D non-autonomous Navier-Stokes equations with variable viscosity

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ABSTRACT: This paper is concerned with the long-time dynamics of some 3D generalized non-autonomous Navier-Stokes equations in a bounded domain governing the motion of fluid flow which comes from a monograph by Ladyzhenskaya. Under some assumptions on the external force and initial data, we prove the existence and structure of uniform attractors for the continuous process.

KEYWORDS: translation bounded functions, uniform attractors

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INTRODUCTION

The objective of this paper is to study the uniform attractor of the 3D incompressible modified Navier-Stokes equations,

$$
\left.
\begin{aligned}
u_t - (v_0 + v_1\|u\|^2)\Delta u + (u \cdot \nabla)u + \nabla p &= \sigma(t,x), \\
\nabla \cdot u &= 0, \\
u\,|_{\partial\Omega} &= 0, \\
u(x,\tau) &= u_0(x),
\end{aligned}
\right\}
$$
(1)

defined in a bounded domain $\Omega \subset \mathbb{R}^3$, with the smooth boundary $\partial\Omega$, where u is the velocity field with components $u^i = u^i(t,x)$, $i = 1,2,3$, and $p = p(t,x)$ is the pressure. The assumption of nonlinear viscosity $v = v_0 + v_1\|u\|^2$, where v_0, $v_1 > 0$ and was proposed in Ref. 1. If the viscosity $v(t) > 0$ becomes a constant, (1) reduces to the well-known classical 3D Navier-Stokes equations.

The classical Navier-Stokes equations (v is a constant) are an important part of the theory for fluid mechanics[1–5]. A comprehensive description of the theory of the Navier-Stokes equations can be found in Refs. 6–8.

For the variable viscosity problem (1), the background of this model and some existence results can be founded in Ref. 1. The uniqueness of this problem was investigated by Ref. 9 with $N < 3$ spatial dimensions. This equation was also considered in Ref. 10, where the existence of weak solutions was obtained for $N \leqslant 4$ and uniqueness for $N \leqslant 3$, and the well-posedness of (1) in a non-cylindrical domain[11]. For the pullback dynamics of system (1) and its modified form, see Refs. 12, 13. However, as far as we know, there is no result about the forward dynamics of this model with some non-autonomous external forces. We shall use the energy equation method to deal with the uniformly asymptotic compactness of continuous processes with respect to the special external force (which is called a symbol) for achieving the existence of uniform attractors.

After introducing the notation and some preliminaries, the well-posedness of problem (1) will be shown. We then gives the theory of uniform attractors and the main results.

NOTATION AND PRELIMINARIES

Throughout this paper, the Hausdorff distance in X from one set B_1 to another B_2 is defined as

$$
\text{dist}_X(B_1, B_2) = \sup_{b_1 \in B_1} \inf_{b_2 \in B_2} \|b_1 - b_2\|_X.
$$

Setting $E := \{u \mid u \in (C_0^\infty(\Omega))^3, div\,u = 0\}$, H is the closure of E with $(L^2(\Omega))^3$ topology, and V is the closure of E in $(H^1(\Omega))^3$ topology. $|\cdot|$ and $(\cdot, \cdot)$ denote the norm and inner product of H. Here,

$$
(u,v) = \sum_{j=1}^{3} \int_\Omega u_j(x)v_j(x)\,\mathrm{d}x, \quad \forall u, v \in H.
$$

$\|\cdot\|$ and $((\cdot, \cdot))$ denote the norm and inner product in V. Here,

$$
((u,v)) = \sum_{i,j=1}^{3} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i}\,\mathrm{d}x, \quad \forall u, v \in V.
$$

Clearly, $V \hookrightarrow H \equiv H' \hookrightarrow V'$, H' and V' are the dual spaces of H and V, respectively, where the injection is dense and continuous. $\|\cdot\|_*$ denotes the norm of V', and $\langle \cdot \rangle$ is the dual product between V and V'.

Let P be the Helmholtz-Leray orthogonal projection operator from $(L^2(\Omega))^3$ onto H. We define $A := -P\Delta$ as the Stokes operator with domain $D(A) = (H^2(\Omega))^3 \cap (H_0^1(\Omega))^3$ and λ is the first eigenvalue of A, then the operator $A : V \to V'$ has the property $\langle Au, v \rangle = ((u, v))$ for all $u, v \in V$ which is an isomorphism from V into V', and $\{\lambda_j\}_{j=1}^{\infty}$ for $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ are eigenvalues of A corresponding to the eigenfunctions $\{\omega_j\}_{j=1}^{\infty}$, i.e., $A\omega_j = \lambda_j \omega_j$.

We define the bilinear operator and trilinear operator as[6,14]

$$B(u, v) := P((u \cdot \nabla)v), \quad \forall u, v \in E, \qquad (2)$$

$$b(u, v, w) = (B(u, v), w) = \sum_{i,j=1}^{3} \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j \, \mathrm{d}x, \quad (3)$$

which satisfy

$$b(u, v, v) = 0, \ b(u, v, w) = -b(u, w, v),$$
$$\forall u, v, w \in V,$$
$$|b(u, v, w)| \leqslant C|u|^{1/2}\|u\|_V^{1/2}\|v\|_V|w|^{1/2}\|w\|_V^{1/2},$$
$$\forall u, v, w \in V.$$

THE GLOBAL EXISTENCE OF SOLUTIONS

The operator $\mathbb{A} : V \to V'$ is defined by $\mathbb{A}u = -\nu_1\|u\|^2\Delta u$ and

$$\langle \mathbb{A}u, v \rangle = \nu_1\|u\|^2\langle -\Delta u, v \rangle$$
$$= \nu_1\|u\|^2((u, v)), \quad \forall u, v \in V. \qquad (4)$$

Noting that $\mathbb{A}$ maps the bounded sets in V to bounded sets in V', we have

$$\|\mathbb{A}\|_{V'} = \sup_{\|v\|=1, \ v \in V} |\langle \mathbb{A}u, v \rangle|$$
$$= \sup_{\|v\|=1, \ v \in V} \nu_1\|u\|^2|a(u, v)| \quad \leqslant \nu_1\|u\|^3.$$

We call the function $u(t, x) \in L^{\infty}(\tau, T; H) \cap L^4(\tau, T; V)$ a *weak solution* of (1) if

$$\frac{\mathrm{d}}{\mathrm{d}t}(u(t), v) + (\nu_0 + \nu_1\|u(t)\|^2)((u(t), v))$$
$$+ b(u, u, v) = \langle g(t, x), v \rangle, \quad u(\tau) = u_0, \ \tau \in \mathbb{R},$$

for $v \in V$ in distributed sense.

Based on the above definition and notation, (1) can be rewritten as an equivalent abstract functional equation:

$$u_t + \nu_0 Au + \mathbb{A}u + B(u, u) = f(t, x), \quad \nabla \cdot u = 0. \quad (5)$$

The existence, uniqueness and regularity of a global solution for (5) can be derived by the Galerkin approximation method and some energy estimates. Here we omit the details; the technique can be found in Refs. 9, 10.

Theorem 1 *(a) Assume the external force $f(t, x) \in L_{\mathrm{loc}}^{4/3}(\mathbb{R}, V')$ and initial data $u_0 \in H$. Then the system (5) has a unique weak solution*

$$u(x, t) \in L^{\infty}(\tau, T; H) \cap L^4(\tau, T; V)$$

in dimension $N \leqslant 3$. (b) Furthermore, since the solution has continuous dependence on the initial data and $(\mathrm{d}u/\mathrm{d}t) \in L^2(\tau, T; V')$, from the Aubin-Lions Lemma, we derive that $u(t, x) \in C(\tau, T; H)$ which generates a continuous process $U(t, \tau) : H \to H$.

THE THEORY OF UNIFORM ATTRACTORS

We firstly recall some theory of uniform attractors[15,16]. Let X be a Banach space with a spatial variable. The space $L_{\mathrm{loc}}^p(\mathbb{R}; X)$ denotes all functions with spatial values in a Banach space X and time variable locally p-power integrable in the Bochner sense with the norm $\int_{t_1}^{t_2} \|\cdot\|_X^p \, \mathrm{d}s < \infty$ for any time interval $[t_1, t_2] \subseteq \mathbb{R}$. We choose $\Sigma = \mathscr{H}(\sigma_0)$ as a symbol space of the system,

$$\mathscr{H}(\sigma_0) = \overline{\{\sigma_0(\cdot + h) \mid h \in \mathbb{R}^+\}}^{\mathscr{E}}$$

for every fixed $\sigma_0 \in \mathscr{E}$. Let $\{T(\cdot)\}$ be the translation semigroup (also the shift operator) defined on Σ (the symbol space): $T(\cdot)\sigma(s) = \sigma(s + \cdot)$ and the symbol space is invariant in the sense $T(\cdot)\Sigma = \Sigma$. The space $L_b^2(\mathbb{R}; X)$ denotes the union of all translation bounded functions satisfying

$$\|\sigma\|_{L_b^2(\mathbb{R};X)}^2 = \sup_{t \in \mathbb{R}} \int_t^{t+1} \|\sigma(s)\|_X^2 \, \mathrm{d}s < \infty,$$

for $\sigma \in L_b^2(\mathbb{R}; X)$. We say $\sigma \in L_{\mathrm{loc}}^2(\mathbb{R}; X)$ is translation compact in $L_{\mathrm{loc}}^2(\mathbb{R}; X)$ if $\mathscr{H}(\sigma)$ is compact in $L_{\mathrm{loc}}^2(\mathbb{R}; X)$ which is denoted by $L_c^2(\mathbb{R}; X)$. $L_c^{2,w}(\mathbb{R}; X)$ is the weak translation compact space. We define the normal functional space $L_n^2(\mathbb{R}; X)$ as there exists a $\eta > 0$, such that for any $\varepsilon > 0$

$$L_n^2(\mathbb{R}; X) = \left\{ \phi \in L_{\mathrm{loc}}^2(\mathbb{R}; X) : \sup_{t \in \mathbb{R}} \int_t^{t+\eta} \|\phi(s)\|_X^2 \, \mathrm{d}s \leqslant \varepsilon \right\}.$$

Remark 1 If we choose $\sigma_0 \in L_{\mathrm{loc}}^2(\mathbb{R}; X)$ and fixed, then for every $\sigma \in \mathscr{H}(\sigma_0)$, we have

$$\|\sigma\|_{L_b^2(\mathbb{R};X)}^2 \leqslant C\|\sigma_0\|_{L_b^2(\mathbb{R};X)}^2,$$
$$L_c^2(\mathbb{R}; X) \subset L_n^2(\mathbb{R}; X) \subset L_b^2(\mathbb{R}; X) \subset L_{\mathrm{loc}}^2(\mathbb{R}; X).$$

Denoting the phase space of global solutions as E, we can define $\{U_\sigma(t,\tau)\} : E \to E$ as evolutionary processes if

$$U_\sigma(t,s)U_\sigma(s,\tau) = U_\sigma(t,\tau), \quad \forall t \geqslant s \geqslant \tau, \ \tau \in \mathbb{R},$$
$$U_\sigma(\tau,\tau) = \mathrm{Id} \ (\text{identity}), \quad \forall \tau \in \mathbb{R}$$

for any function $\sigma \in \Sigma$, $\Sigma \subset \mathscr{E}$. Furthermore, we assume that the family of processes $\{U_\sigma(t,\tau)\}$ satisfies the translation identity for invariance

$$U_{\sigma(t)}(t+h, \tau+h) = U_{T(h)\sigma(t)}(t,\tau),$$
$$\forall \sigma \in \Sigma, \ t \geqslant \tau, \ \tau, s \in \mathbb{R}. \quad (6)$$

We define the *skew product flow* as $S(t) = (U_\sigma(t,\tau), T(t)) \in E \times \Sigma$, i.e.,

$$S(t)(u(s),\sigma(s)) = (U_{\sigma(s)}(t,0)u(s), T(t)\sigma(s))$$
$$= (u(t+s), \sigma(t+s))$$

for every $(u(s)\sigma(s)) \in E \times \Sigma$.

Remark 2 If the family of processes satisfies the translation identity (6) and it is $(E \times \Sigma, \Sigma)$-continuous, then the skew product flow is a semigroup.

Theorem 2 (Ref. 15) *Assume that (i) Σ is the symbol space and it is invariant under the continuous translation semigroup $\{T(h)\}$ defined above; (ii) the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ acting on the phase space E which satisfies the translation identity (6) is uniformly asymptotically compact and $(E \times \Sigma, E)$-continuous; (iii) there exists a uniformly bounded absorbing set B with respect to every symbol $\sigma \in \Sigma$.*

Let Π_1 and Π_2 be two projectors from $E \times \Sigma$ onto E and Σ, respectively: $\Pi_1(u,\sigma) = u$ and $\Pi_2(u,\sigma) = \sigma$. Then the skew product flow possesses a global attractor $\mathscr{A}$.

A curve $u(s)$, $s \in \mathbb{R}$ is called the *complete trajectory* for the process $\{U(t,\tau)\}$ if $U(t,\tau)u(\tau) = u(t)$ for all $t \geqslant \tau$, $\tau \in \mathbb{R}$. The *kernel* $\mathscr{K}$ of the process $\{U(t,\tau)\}$ is defined as all bounded complete trajectories: $\mathscr{K} = \{u(\cdot) \mid u(\cdot)$ are complete trajectories and $\|u(\cdot)\|_E \leqslant C\}$. The *kernel section* $\mathscr{K}(s) = \{u(s) \mid u(\cdot) \in \mathscr{K}\}$ is the kernel at time $t = s$.

Theorem 3 (Ref. 15) *The compact global attractor has the structure $\mathscr{A} = \bigcup_{\sigma \in \Sigma} \mathscr{K}_\sigma(\tau) \times \{\sigma\}$ which is (a) strictly invariant ($S(t)\mathscr{A} = \mathscr{A}$) and (b) satisfies $\Pi_1\mathscr{A} = \mathscr{A}_\Sigma = \bigcup_{\sigma \in \Sigma} \mathscr{K}_\sigma(\tau)$ is the uniform attractor for $\{U_\sigma(t,\tau)\}$, $\Pi_2\mathscr{A} = \Sigma$, where $\mathscr{K}_\sigma(\tau)$ is the section at $t = \tau$ of the kernel $\mathscr{K}_\sigma$ for the process $\{U_\sigma(t,\tau)\}$ with symbol $\sigma \in \Sigma$.*

THE UNIFORM ATTRACTOR FOR 3D NAVIER-STOKES EQUATIONS WITH VARIABLE VISCOSITY

Symbol space

We denote $E_1 = L^2_{\mathrm{loc}}(\mathbb{R}; (L^2(\Omega))^2)$ as the 2-power locally integrable functional space, and define

$$\hat{E}_1 = L^2_b(\mathbb{R}; (L^2(\Omega))^2), \quad \bar{E}_1 = L^2_c(\mathbb{R}; (L^2(\Omega))^2),$$
$$\tilde{E}_1 = L^2_n(\mathbb{R}; (L^2(\Omega))^2)$$

as translation bounded, translation compact, and normal functional spaces, respectively.

Choosing an arbitrary function $\sigma_0 \in \bar{E}_1$, $\hat{E}_1$, or $\tilde{E}_1$, then we can define the symbol space $\mathscr{H}(\sigma_0)$ which is called the hull of σ_0 by

$$\bar{\Sigma} = \mathscr{H}(\sigma_0) = [\sigma_0(t+h) = T(h)\sigma_0(t) \mid h \in \mathbb{R}^+]_{E_1}, \quad (7)$$

where $[\cdot]_{E_1}$ denotes the closure in the strong topology of E_1, and $T(\cdot)$ denotes the translation semigroup.

Proposition 1 (Ref. 15) *For every fixed function $\sigma_0(\cdot)$ and $\sigma(\cdot)$ belonging to its corresponding symbol space, we have $\|\sigma\|^2_{L^2_b(\mathbb{R}; L^2(\Omega))} \leqslant C\|\sigma_0\|^2_{L^2_b(\mathbb{R}; L^2(\Omega))}.$*

The skew product flow and its global attractors

The problem (1) possesses a unique global weak solution in H. Thus for any symbol $\sigma(\cdot) \in \Sigma_0$ and $u_\tau \in H$, the unique solution generates the process $U_{\sigma(\cdot)}(t,\tau) : H \to H$ as

$$U_{\sigma(\cdot)}(t,\tau)u_\tau = u(t,\tau; u_\tau) \in H \quad (8)$$

which satisfies the translation identity.

Choosing the fixed symbol space Σ which can be any one of $\bar{E}_1$, $\hat{E}_1$ or $\tilde{E}_1$, we can derive the skew product flow $S(t) = (U_{f(\cdot)}(t,\tau), T(\cdot))$ in phase space $H \times \Sigma$.

The uniformly absorbing set

Lemma 1 (Ref. 15) *For every $\tau \in \mathbb{R}$, any nonnegative locally summable function ϕ on R_τ and for every $\beta > 0$, we have*

$$\int_\tau^t \phi(s)e^{-\beta(t-s)}\,\mathrm{d}s \leqslant \frac{1}{1-e^{-\beta}} \sup_{\theta \geqslant \tau} \int_\theta^{\theta+1} \phi(s)\,\mathrm{d}s,$$

for all $t \geqslant \tau$.

Lemma 2 *Assume $f \in \Sigma \subset L^2_b(\mathbb{R}; H)$, $u_\tau \in H$. Then there exists a time*

$$T_0 = \tau + \frac{1}{\lambda_1 \nu_0} \ln\left(\frac{|u_\tau|^2}{\frac{4}{\lambda_1 \nu_0} + \frac{4}{\lambda_1^2 \nu_0^2}} \|f_0\|^2_{L^2_b(\mathbb{R}; H)} \right)$$

such that for any $t \geqslant T_0$, the process $U(t, \tau)$ possesses a bounded uniformly (w.r.t. $f \in \Sigma$) absorbing set $\mathfrak{B}$ in H, where $\mathfrak{B} = \{u \in H : |u| \leqslant C\|f_0\|_{L_b^2(R;H)} \doteq R_0\}$ is a bounded set in H which attracts all bounded subsets in H, where $R_0^2 = (8/\lambda_1 \nu_0)(1 + 1/\lambda_1 \nu_0)\|f_0\|_{L_b^2(\mathbb{R};H)}^2$.

Proof: Taking the inner product of (1) with u, we derive the energy equation

$$\frac{d}{dt}|u(t)|^2 + 2\nu_0\|u(t)\|^2 + 2\nu_1\|u(t)\|^4 = 2\langle f(t), u(t)\rangle, \tag{9}$$

and thus

$$\frac{d}{dt}|u(t)|^2 + \lambda_1 \nu_0|u(t)|^2 \leqslant \frac{|f(t)|^2}{\nu_0 \lambda_1}.$$

From the Gronwall inequality, one can easily deduce

$$|u(t)|^2 \leqslant |u_\tau|^2 e^{-\lambda_1 \nu_0(t-\tau)} + 4 e^{-\lambda_1 \nu_0 t} \int_\tau^t \frac{e^{\lambda_1 \nu_0 \xi}}{\nu_0 \lambda_1}|f(\xi)|^2 \, d\xi. \tag{10}$$

By [Lemma 1](#), we derive

$$|u(t)|^2$$
$$\leqslant |u_\tau|^2 e^{-\lambda_1 \nu_0(t-\tau)}$$
$$+ \frac{4}{\lambda_1 \nu_0} \int_\tau^t e^{\lambda_1 \nu_0(\xi-t)}\|f(\xi)\|^2 \, d\xi$$
$$\leqslant |u_\tau|^2 e^{-\lambda_1 \nu_0(t-\tau)} + \frac{4}{\lambda_1 \nu_0}\Big(\int_{t-1}^t |f(s)|^2 \, ds$$
$$+ e^{-\lambda_1 \nu_0} \int_{t-2}^{t-1} \|f(s)\|^2 \, ds$$
$$+ e^{-2\lambda_1 \nu_0} \int_{t-3}^{t-2} |f(s)|^2 \, ds + \cdots \Big)$$
$$\leqslant |u_\tau|^2 e^{-\lambda_1 \nu_0(t-\tau)}$$
$$+ \frac{4}{\lambda_1 \nu_0}\Big(1 + e^{-\lambda_1 \nu_0} + e^{-2\lambda_1 \nu_0} + \cdots \Big)$$
$$\times \|f(s)\|_{L_b^2(\mathbb{R};H)}^2$$
$$\leqslant |u_\tau|^2 e^{-\lambda_1 \nu_0(t-\tau)}$$
$$+ \frac{4}{\lambda_1 \nu_0}\Big(1 + \frac{1}{\lambda_1 \nu_0}\Big)\|f(s)\|_{L_b^2(\mathbb{R};H)}^2$$
$$\leqslant |u_\tau|^2 e^{-\lambda_1 \nu_0(t-\tau)}$$
$$+ \frac{4}{\lambda_1 \nu_0}\Big(1 + \frac{1}{\lambda_1 \nu_0}\Big)\|f_0\|_{L_b^2(\mathbb{R};H)}^2,$$

and choosing

$$|u_\tau|^2 e^{-\lambda_1 \nu_0(t-\tau)} \leqslant \frac{4}{\lambda_1 \nu_0}\Big(1 + \frac{1}{\lambda_1 \nu_0}\Big)\|f_0\|_{L_b^2(\mathbb{R};H)}^2,$$

we deduce that there exists a time $T_0 = \tau + (1/\lambda_1 \nu_0)\ln((|u_\tau|^2/((4/\lambda_1 \nu_0) + 4/\lambda_1^2 \nu_0^2))\|f_0\|_{L_b^2(\mathbb{R};H)}^2)$, such that for $t \geqslant T_0$, the bounded set $\mathfrak{B} = \{u : |u|^2 \leqslant R_0^2\}$ is a uniformly (w.r.t. $f \in \Sigma$) absorbing set for the process $\{U_f(t, \tau)\}$ in H, where

$$R_0^2 = \frac{8}{\lambda_1 \nu_0}\Big(1 + \frac{1}{\lambda_1 \nu_0}\Big)\|f_0\|_{L_b^2(\mathbb{R};H)}^2, \quad t \in \mathbb{R}.$$

$\square$

The continuity of skew product flow

From the Aubin-Lions compact argument and definition of symbol spaces, we can derive the continuity of process and skew product flow.

Lemma 3 *For any $f \in \Sigma = \mathscr{H}(f_0)$, the family of processes $\{U_f(t, \tau)\}$ defined on H, corresponding to our problem is $(H \times \Sigma, H)$ continuous.*

Proof: This is true by the property of a global solution which is continuous. $\square$

Lemma 4 *Let $\{u_\tau^n\}$ be a sequence in H converging to $u_\tau \in H$ in H-topology and $\{f^n\} \subset \Sigma$ be a sequence converging to $f \in \Sigma$. Then for any fixed $\tau \in \mathbb{R}$, it yields*
(i) $U_{f^n}(\cdot, \tau)u_\tau^n \to U_f(\cdot, \tau)u_\tau$ weakly in H.
(ii) $U_{f^n}(t, \tau)u_\tau^n \rightharpoonup U_f(t, \tau)u_\tau$ weakly in $L^2(\tau, T; V)$.
(iii) $U_{f^n}(t, \tau)u_\tau^n \rightharpoonup U_f(t, \tau)u_\tau$ weakly in $L^4(\tau, T; V)$.

Proof: Let $u_n(t) = U_{f^n}(t, \tau)u_\tau^n$ and $u(t) = U_f(t, \tau)u_\tau$ for $t \geqslant \tau$. From the existence of a global weak solution, we deduce that

$$\{u_n\}_n \text{ is bounded in } L^\infty([\tau, \infty); H) \cap L^2(\tau, T; V),$$
$$\forall T > \tau. \tag{11}$$

Hence since $u_n' = f(t) - \nu_0 A u_n - \mathbb{A} u_n + B(u_n)$, it follows that $\{u_n'\}_n$ is bounded in $L^2(\tau, T; V')$ for all $T > 0$. Then for all $v \in V$ and $\tau \leqslant t \leqslant t + a \leqslant T$, by the Cauchy-Schwarz inequality, we obtain

$$(u_n(t+a) - u_n(t), v) = \int_t^{t+a} \langle u_n'(s), v\rangle \, ds$$
$$\leqslant c_T\|v\|a^{1/2}, \tag{12}$$

where $c_T > 0$ is independent of n. Choosing $v = u_n(t+a) - u_n(t) \in V$ for almost every t, we have

$$|u_n(t+a) - u_n(t)|^2 \leqslant c_T a^{1/2}\|u_n(t+a) - u_n(t)\|,$$

and hence

$$\int_t^{t+a} |u_n(t+a) - u_n(t)|^2 \, dt$$

$$\leqslant c_T a^{1/2} \int_t^{t+a} \|u_n(t+a) - u_n(t)\| \, dt. \quad (13)$$

Using (11) and (13) we obtain

$$\int_t^{t+a} |u_n(t+a) - u_n(t)|^2 \, dt \leqslant \bar{c}_T a^{1/2}$$

for another positive constant $\bar{c}_T$ independent of n. Hence

$$\limsup_{a \to 0} \int_t^{t+a} |u_n(t+a) - u_n(t)|^2 \, dt = 0, \quad \forall r > 0. \quad (14)$$

Then from (11) and by a diagonal process, we can extract a subsequence $\{u_{n'}\}_{n'}$ such that

$$u_{n'} \rightharpoonup \tilde{u} \text{ weakly in } L^\infty([\tau, \infty); H), \text{ as } n' \to \infty,$$

$$u_{n'} \rightharpoonup \tilde{u} \text{ weakly in } L^2_{\text{loc}}([\tau, \infty); V), \text{ as } n' \to \infty,$$

and $u_{n'} \to \tilde{u}$ strongly in $L^2_{\text{loc}}([\tau, \infty); H)$ as $n' \to \infty$ for all $r > 0$, for some $\tilde{u} \in L^\infty([\tau, \infty); H) \cap L^2_{\text{loc}}([\tau, \infty); V)$. By the uniqueness of solutions, it yields $\tilde{u} = u$, i.e., $\{u_n\}_n$ converges to u weakly in $L^2_{\text{loc}}([\tau, \infty); V)$. Hence for all $v \in E$, it follows that

$$(u_n(t), v) \to (u(t), v), \text{ for a.e. } t \geqslant \tau.$$

Furthermore, from (11) and (12), we see that $\{(u_n(t), v)\}_n$ is equibounded and equicontinuous on $[\tau, T]$ for all $T > \tau$. Hence

$$(u_n(t), v) \to (u(t), v), \text{ for all } t \geqslant \tau. \quad (15)$$

Since E is dense in H, the lemma holds. By the energy inequality, we can prove (iii) by a similar technique. □

The uniformly asymptotic compactness of a process

Lemma 5 *For any $f \in \Sigma$, the family of processes $\{U_f(t, \tau)\}, f \in \Sigma$ defined on H to problem (1) is asymptotically compact in H.*

Proof: Let $\{u_\tau^n\}$ be a bounded sequence in H and $\{f^n\}$ in Σ converge to u_τ and f, respectively, as $n \to \infty$.

From Lemma 2, we see that for any fixed $u_\tau \in H$ and $\tau \in \mathbb{R}$, there exists a time $T_0 = T_0(\tau)$ such that

for all $t_n \geqslant T_0$, $\{U_{f^n}(t_n, \tau) u_\tau^n\} \subseteq \mathfrak{B}$, $\{U_{f^n}(t_n, \tau) u_\tau^n\}$ is weakly relatively compact in H and

$$U_{f^n}(t_n, \tau) u_\tau^n \rightharpoonup u \text{ weakly in } H \text{ as } n \to \infty \quad (16)$$

for some $u \in H$ and convergent subsequence. Similarly, for each $T > 0$ and $t_n \geqslant T_0 + T$, we also have

$$u_T^n \doteq U_{f^n}(t_n - T, \tau) u_\tau^n \rightharpoonup u_T \text{ weakly in } H \text{ as } n \to \infty \quad (17)$$

for some $u_T \in H$.

Noting the property of translation semigroup $\{T(t) : t \geqslant \tau\}$ which satisfies

$$U_{T(h)f}(t, \tau) = U_f(t+h, \tau+h), \quad \forall h \geqslant 0, \ f \in \Sigma, \quad (18)$$

then for any $t_n - T \geqslant \tau$, we derive that

$$U_{f^n}(t_n, \tau) = U_{f^n}(t_n, t_n - T) U_{f^n}(t_n - T, \tau)$$
$$= U_{T(t_n - T)f^n}(T, 0) U_{f^n}(t_n - T, \tau). \quad (19)$$

Denoting $f_T^n = T(t_n - T)f^n$, by (18) and (19),

$$U_{f^n}(t_n, \tau) u_\tau^n = U_{f_T^n}(T, 0) U_{f^n}(t_n - T, \tau) u_\tau^n$$
$$= U_{f_T^n}(T, 0) u_T^n, \quad t_n - T \geqslant \tau. \quad (20)$$

Since $\{f_T^n\} \subset \Sigma$ and Σ is compact in $L^2_{\text{loc}}(\mathbb{R}; H)$, there exists a subsequence of $\{f_T^n\}$ (also denoted by $\{f_T^n\}$) and some $f_T \in \Sigma$ such that

$$f_T^n \to f_T \text{ strongly in } L^2_{\text{loc}}(\mathbb{R}; H) \text{ as } n \to \infty, \forall T > 0. \quad (21)$$

By Lemma 4, (16), (17), (20), and (21) we obtain $u = U_{f_T}(T, 0) u_T$ for all $T > 0$.

In the following, we shall prove the asymptotic compactness in H via the energy equation method, i.e., $|U_{f_T^n}(t_n, \tau) u_\tau^n - u| \to 0$ as $n \to \infty$. To achieve this goal, we need to prove the upper and lower continuity of the processes

$$\liminf_n |U_{f^n}(t_n, \tau) u_\tau^n| = \liminf_n |U_{f_T^n}(T, 0) u_T^n| \geqslant |u|, \quad (22)$$

$$\limsup_n |U_{f^n}(t_n, \tau) u_\tau^n| = \limsup_n |U_{f_T^n}(T, 0) u_T^n| \leqslant |u|. \quad (23)$$

The weak convergence for the process ensures that (22) is true. What we need next is to prove (23) from the weak and norm convergence of the corresponding sequences.

Taking the inner product of (5) with $e^{2\lambda_1 \nu_0 t} u(t)$ in H, we derive that

$$\frac{1}{2} \frac{d}{dt} e^{2\lambda_1 \nu_0 t}(u, u) + \nu_0 e^{2\lambda_1 \nu_0 t}(Au(t), u(t))$$
$$+ \nu_1 e^{2\lambda_1 \nu_0 t}(\mathbb{A}u, u)$$
$$= \lambda_1 \nu_0 e^{2\lambda_1 \nu_0 t}(u, u) + e^{2\lambda_1 \nu_0 t}(f, u(t)), \quad (24)$$

integrating (24) over $[\tau, t]$ yields

$$
(u(t), u(t)) = (u(\tau), u(\tau)) e^{-2\lambda_1 v_0 t}
$$
$$
+ 2 \int_\tau^t e^{-2\lambda_1 v_0(t-s)}(f(s), u(s)) \, ds
$$
$$
- 2 \int_\tau^t v_0 \, e^{-2\lambda_1 v_0(t-s)}(Au(s), u(s)) \, ds
$$
$$
- 2 \int_\tau^t v_1 \, e^{-2\lambda_1 v_0(t-s)}(\mathbb{A}(s), u(s)) \, ds
$$
$$
+ 2\lambda_1 v_0 \int_\tau^t e^{-2\lambda_1 v_0(t-s)}(u(s), u(s)) \, ds, \quad (25)
$$

for all $t \geqslant \tau$, $\tau \in \mathbb{R}$.

Applying (25) to $U_{f_T^n}(T, 0)u_T^n$, we derive that

$$
(U_{f_T^n}(T, 0)u_T^n, U_{f_T^n}(T, 0)u_T^n) = (u_T^n, u_T^n) e^{-2\lambda_1 v_0 T}
$$
$$
+ 2 \int_\tau^T e^{-2\lambda_1 v_0(T-s)}
$$
$$
\times (f_s^n(s), U_{f_s^n}(s, 0)u_s^n) \, ds
$$
$$
- \left(2 \int_\tau^T v_0 \, e^{-2\lambda_1 v_0(T-s)} \right.
$$
$$
\times (AU_{f_s^n}(s, 0)u_s^n, U_{f_s^n}(s, 0)u_s^n) \, ds
$$
$$
+ 2 \int_\tau^T v_1 \, e^{-2\lambda_1 v_0(T-s)}
$$
$$
\left. \times (\mathbb{A}U_{f_s^n}(s, 0)u_s^n, U_{f_s^n}(s, 0)u_s^n) \, ds \right)
$$
$$
+ 2\lambda_1 v_0 \int_\tau^T e^{-2\lambda_1 v_0(T-s)}
$$
$$
\times (U_{f_s^n}(s, 0)u_s^n, U_{f_s^n}(s, 0)u_s^n) \, ds
$$
$$
= I_1 + I_2 + I_3 + I_4. \quad (26)
$$

From Lemma 2, we deduce that

$$
\limsup_n I_1 = \limsup_n (u_T^n, u_T^n) e^{-2\lambda_1 v_0 T} \leqslant R_0^2 \, e^{-2\lambda_1 v_0 T}.
$$
$$
(27)
$$

By Lemma 4,

$$
U_{f_s^n}(s, 0)u_s^n \rightharpoonup U_{f_s}(s, 0)u_s \text{ weakly in } H, \ L^2(\tau, T; V),
$$
$$
\forall T > \tau. \quad (28)
$$

Hence we deduce from (26) and (28) that

$$
\lim_{n \to \infty} I_2 = \lim_{n \to \infty} 2 \int_\tau^T e^{-2\lambda_1 v_0(T-s)}(f_s^n(s), U_{f_s^n}(s, 0)u_s^n) \, ds
$$
$$
= 2 \int_\tau^T e^{-2\lambda_1 v_0(T-s)}(f_s(s), U_{f_s}(s, 0)u_s) \, ds \quad (29)
$$

and

$$
\lim_{n \to \infty} I_4 = \lim_{n \to \infty} 2\lambda_1 v_0 \int_\tau^T e^{-2\lambda_1 v_0(T-s)}
$$
$$
\times (U_{f_s^n}(s, 0)u_s^n, U_{f_s^n}(s, 0)u_s^n) \, ds
$$
$$
= 2\lambda_1 v_0 \int_\tau^T e^{-2\lambda_1 v_0(T-s)}
$$
$$
\times (U_{f_s}(s, 0)u_s, U_{f_s}(s, 0)u_s) \, ds. \quad (30)
$$

Next, we want to deal with the third term on the right-hand side of (26). In fact, since A is a bounded linear operator from V into V', $\mathbb{A} : V \to V'$ is bounded. From Lemma 4 we obtain

$$
\limsup_n \left(-2 \int_\tau^T v_0 \, e^{-2\lambda_1 v_0(T-s)} \right.
$$
$$
\left. \times (AU_{f_s^n}(s, 0)u_s^n, U_{f_s^n}(s, 0)u_s^n) \, ds \right)
$$
$$
= -\liminf_n \left(2 \int_\tau^T v_0 \, e^{-2\lambda_1 v_0(T-s)} \right.
$$
$$
\left. \times \|U_{f_s^n}(s, 0)u_s^n\|^2 \, ds \right)
$$
$$
\leqslant - \left(2 \int_\tau^T v_0 \, e^{-2\lambda_1 v_0(T-s)} \|U_{f_s}(s, 0)u_s\|^2 \, ds \right)
$$
$$
= -2 \int_\tau^T v_0 \, e^{-2\lambda_1 v_0(T-s)}
$$
$$
\times (AU_{f_s}(s, 0)u_s, U_{f_s}(s, 0)u_s) \, ds
$$

and

$$
\limsup_n \left(-2 \int_\tau^T v_1 \, e^{-2\lambda_1 v_0(T-s)} \right.
$$
$$
\left. \times (\mathbb{A}U_{f_s^n}(s, 0)u_s^n, U_{f_s^n}(s, 0)u_s^n) \, ds \right)
$$
$$
= -\liminf_n \left(2 \int_\tau^T v_1 \, e^{-2\lambda_1 v_0(T-s)} \right.
$$
$$
\left. \times \|U_{f_s^n}(s, 0)u_s^n\|^4 \, ds \right)
$$
$$
\leqslant -2 \int_\tau^T v_1 \, e^{-2\lambda_1 v_0(T-s)} \|U_{f_s}(s, 0)u_s\|^4 \, ds
$$
$$
= -2 \int_\tau^T v_1 \, e^{-2\lambda_1 v_0(T-s)}
$$
$$
\times (\mathbb{A}U_{f_s}(s, 0)u_s, U_{f_s}(s, 0)u_s) \, ds.
$$

Hence

$$
\limsup_n I_3 = \limsup_n \left(-2 \int_\tau^T v_0 \, e^{-2\lambda_1 v_0(T-s)} \right.
$$

$$\times (AU_{f_s^n}(s,0)u_s^n, U_{f_s^n}(s,0)u_s^n)\,ds$$

$$-2\int_\tau^T \nu_1 e^{-2\lambda_1\nu_0(T-s)}$$

$$\times (\mathbb{A}U_{f_s^n}(s,0)u_s^n, U_{f_s^n}(s,0)u_s^n)\,ds\Bigg)$$

$$= -\liminf_n \Bigg(2\int_\tau^T \nu_0 e^{-2\lambda_1\nu_0(T-s)}$$

$$\times (AU_{f_s^n}(s,0)u_s^n, U_{f_s^n}(s,0)u_s^n)\,ds$$

$$+ 2\int_\tau^T \nu_1 e^{-2\lambda_1\nu_0(T-s)}$$

$$\times (\mathbb{A}U_{f_s^n}(s,0)u_s^n, U_{f_s^n}(s,0)u_s^n)\,ds\Bigg)$$

$$\leqslant -\Bigg(2\int_\tau^T \nu_0 e^{-2\lambda_1\nu_0(T-s)}$$

$$\times (AU_{f_s}(s,0)u_s, U_{f_s}(s,0)u_s)\,ds$$

$$+ 2\int_\tau^T \nu_1 e^{-2\lambda_1\nu_0(T-s)}$$

$$\times (\mathbb{A}U_{f_s}(s,0)u_s, U_{f_s}(s,0)u_s)\,ds\Bigg). \tag{31}$$

Combining (26) and (27)–(30), we conclude that

$$\limsup_n (U_{f_T^n}(T,0)u_T^n, U_{f_T^n}(T,0)u_T^n) \leqslant R_0^2 e^{-2\lambda_1\nu_0 T}$$

$$+ 2\int_\tau^T e^{-2\lambda_1\nu_0(T-s)}(f_s(s), U_{f_s}(s,0)u_s)\,ds$$

$$-\Bigg(2\int_\tau^T \nu_0 e^{-2\lambda_1\nu_0(T-s)}(AU_{f_s}(s,0)u_s, U_{f_s}(s,0)u_s)\,ds$$

$$+ 2\int_\tau^T \nu_1 e^{-2\lambda_1\nu_0(T-s)}(\mathbb{A}U_{f_s}(s,0)u_s, U_{f_s}(s,0)u_s)\,ds\Bigg)$$

$$+ 2\lambda_1\nu_0\int_\tau^T e^{-2\lambda_1\nu_0(T-s)}(U_{f_s}(s,0)u_s, U_{f_s}(s,0)u_s)\,ds. \tag{32}$$

By the same technique as above, applying (25) to $u = U_{f_T}(T,0)u_T$ yields

$$(U_{f_T}(T,0)u_T, U_{f_T}(T,0)u_T) = (u_T, u_T)e^{-2\lambda_1\nu_0 T}$$

$$+ 2\int_\tau^T e^{-2\lambda_1\nu_0(T-s)}(f_s(s), U_{f_s}(s,0)u_s)\,ds$$

$$-\Bigg(2\int_\tau^T \nu_0 e^{-2\lambda_1\nu_0(T-s)}(AU_{f_s}(s,0)u_s, U_{f_s}(s,0)u_s)\,ds$$

$$+ 2\int_\tau^T \nu_1 e^{-2\lambda_1\nu_0(T-s)}(\mathbb{A}U_{f_s}(s,0)u_s, U_{f_s}(s,0)u_s)\,ds\Bigg)$$

$$+ 2\lambda_1\nu_0\int_\tau^T e^{-2\lambda_1\nu_0(T-s)}(U_{f_s}(T,0)u_s, U_{f_s}(s,0)u_s)\,ds. \tag{33}$$

It follows from (32), (33), and (17) that for any $T > \tau$,

$$\limsup_n (U_{f_T^n}(T,0)u_T^n, U_{f_T^n}(T,0)u_T^n)$$

$$\leqslant (u_T, u_T) + R_0^2 e^{-2\lambda_1\nu_0 T} - (u_T^n, u_T^n)e^{-2\lambda_1\nu_0 T}$$

$$\leqslant (U_{f_T}(T,0)u_T, U_{f_T}(T,0)u_T) + R_0^2 e^{-2\lambda_1\nu_0 T}.$$

Hence letting T tend to ∞, we derive that

$$\lim_{n\to\infty}|U_{f^n}(t_n,\tau)u_\tau^n|^2$$

$$= \limsup_{n,T\to\infty}|U_{f_T^n}(T,0)u_T^n|^2$$

$$= \limsup_{n,T\to\infty}(U_{f_T^n}(T,0)u_T^n, U_{f_T^n}(T,0)u_T^n)$$

$$\leqslant \lim_{T\to\infty}(U_{f_T}(T,0)u_T, U_{f_T}(T,0)u_T) = |u|^2.$$

From (22),

$$\limsup_{n,T\to\infty}|U_{f_T^n}(T,0)u_T^n|^2 = \lim_{n\to\infty}|U_{f^n}(t_n,\tau)u_\tau^n|^2 = |u|^2. \tag{34}$$

Combining the norm convergence (34) and weak convergence (16) we conclude that

$$\lim_{n\to\infty}|U_{f_T^n}(t_n,\tau)u_\tau^n - u|^2 = 0,$$

which implies the lemma is true. $\qquad\square$

Lemma 6 *The family of processes* $\{U_f(t,\tau)\}, f \in \Sigma$ *defined on H for problem* (1) *is uniformly (w.r.t.* $f \in \Sigma$*) asymptotically compact in H-topology.*

Proof: From the theory in Ref. 15, to achieve the uniformly asymptotic compactness of a process, what we need is to verify that $\{U_f(t,\tau)\}$ possesses a compact uniformly (w.r.t. $f \in \Sigma$) attracting set in H.

For all uniformly bounded absorbing sets $\mathfrak{B}$ in H, we can construct the ω-limit set as

$$\omega_{\tau,\Sigma}(\mathfrak{B}) \doteq \bigcap_{t\geqslant\tau}\overline{\bigcup_{f\in\Sigma}\bigcup_{s\geqslant t}U_f(s,\tau)\mathfrak{B}},$$

where $\overline{(\cdot)}$ denotes the closure topology in H. From the definition of ω-limit set, we see that

$$u \in \omega_{\tau,\Sigma}(\mathfrak{B}) \Longleftrightarrow \exists \text{ sequences } \{u^n\} \subset \mathfrak{B},$$

$$\{f^n\} \subset \Sigma, \ \{t_n\} \subset [\tau,\infty), \text{ such that}$$

$$\lim_{n\to\infty}t_n = \infty, \ U_{f^n}(t_n,\tau)u^n \to u$$

$$\text{weakly in } H \text{ as } n \to \infty.$$

The existence of uniformly an absorbing set implies $\omega_{\tau,\Sigma}(\mathfrak{B})$ is independent of τ. We claim that $\omega_{\tau,\Sigma}(\mathfrak{B})$ is a compact uniformly (w.r.t. $f \in \Sigma$) attracting set for the process $\{U_f(t,\tau)\}$ in H. Combining the uniformly (w.r.t. $f \in \Sigma$) absorbing property (Lemma 2), the asymptotic compactness property of the process $\{U_f(t,\tau)\}$, $f \in \Sigma$ in H (Lemma 5), we conclude that $\omega_{\tau,\Sigma}(\mathfrak{B})$ is a nonempty compact set in H.

For any bounded subset $B \in H$, we obtain the following property from Lemma 2:

$$\lim_{t \to \infty} \sup_{f \in \Sigma} \mathrm{dist}_H(U_f(t,\tau)B, \omega_{\tau,\Sigma}(\mathfrak{B})) = 0, \quad (35)$$

which also means that

$$\omega_{\tau,\Sigma}(B) \subseteq \omega_{\tau,\Sigma}(\mathfrak{B}). \quad (36)$$

Hence (35) and (36) imply the uniformly (w.r.t. $f \in \Sigma$) attracting property of $\omega_{\tau,\Sigma}(\mathfrak{B})$ in H, and the uniformly asymptotic compactness of the process holds. □

The global and uniform attractors

From Lemmas 2-6, we summarize the properties of a process as follows: (i) $\{U_f(t,\tau)\}$ is continuous in H; (ii) the process possesses a uniformly absorbing set in H; (iii) the process is uniformly asymptotically compact in H with respect to $f \in \Sigma$. Combining the proposition of translation semigroup and symbol spaces, we can show that the skew product flow is continuous and attracting which satisfies the compact property, so we have the existence of global and uniform attractors for the skew product flow and process, respectively, as in the following.

Theorem 4 *Let $f \in \Sigma$ be the symbols, $u_0 \in H$. Then the global weak solution for problem* (1) *generates a family of processes $\{U_f(t,\tau)\}$ ($f \in \Sigma$) and skew product flow $S(t) = (U_f(t,\tau), T(t)) \in H \times \Sigma$. If the symbol spaces are chosen as translation compact, bounded or normal class functional spaces, the skew product flow possesses a compact global attractor $\mathscr{A}$ in $H \times \Sigma$:*

$$\mathscr{A} = \bigcup_{\sigma \in \Sigma} \mathscr{K}_\sigma(\tau) \times \{\sigma\},$$

where $\mathscr{K}_\sigma(\tau)$ denotes all bounded complete trajectories under the process and translation semigroup. Furthermore, let Π_1 and Π_2 be two projections from $H \times \Sigma$ to H and Σ, respectively. Then

$$\Pi_1 \mathscr{A} = \mathscr{A}_\Sigma = \bigcup_{\sigma \in \Sigma} \mathscr{K}_\sigma(\tau)$$

is the uniform attractor of process in H, and $\Pi_1 \mathscr{A} = \mathfrak{U} = \Sigma$ is the global attractor of the translation semigroup (also as shift operator) $T(\cdot)$.

Proof: Based on the continuity of a process (Lemma 3) and the shift operator, the existence of a uniformly absorbing set, and the uniform asymptotic compactness of a process, by the theory in Ref. 15, we can prove our result. □

Further research

For this Ladyzhenskaya model, as an approximation to the 3D classical incompressible Navier-Stokes equations, which has some similar properties to the non-Newtonian fluid model, some further interesting problems can also be considered. (a) If the domain is unbounded, does the uniform attractor exist? (b) For the 2D Navier-Stokes equation, the weak solution can be regularized to a strong solution with a periodic boundary, and hence a weak attractor has more regularity, but what about this Ladyzhenskaya model? (c) The continuity of the attractor as $\nu_1 \to 0$ may occur, but in which topology?

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