RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2017.43.186

ScienceAsia 43 (2017): **186–194**

C^1 rational interpolation schemes for modelling positive and/or monotonic 3D data

Xiangbin Qin[a], **Lei Qin**[b], **Qingsong Xu**[a,*]

[a] School of Mathematics and Statistics, Central South University, Changsha 410083, China
[b] College of Environmental Science and Engineering, Hunan University, Changsha 410082, China

*Corresponding author, e-mail: qsxu@csu.edu.cn

Received 13 Oct 2015
Accepted 26 Jun 2017

ABSTRACT: By using the boolean sum of cubic interpolating operators to blend together kinds of rational quartic/linear interpolation splines as the boundary functions, a class of C^1 bi-cubic partially blended rational quartic/linear interpolation splines with four families of local control parameters is constructed. By developing new constraints on the boundary functions, simple sufficient data-dependent conditions are derived for the local control parameters to generate C^1 positivity- and/or monotonicity-preserving interpolation surfaces for positive and/or monotonic data on rectangular grids.

KEYWORDS: data visualization, interpolation spline, positivity-preserving, monotonicity-preserving, local control parameter

MSC2010: 65D07 65D17

INTRODUCTION

Constructing shape-preserving bivariate interpolation splines for visualizing 3D positive and/or monotonic data on rectangular grids is an essential problem in many computer graphics applications and in data visualization. By developing some constraint conditions on the first partial derivatives and first mixed partial (twist) at the mesh points, some algorithms for generating monotonicity-preserving bi-cubic interpolation surfaces were proposed[1,2]. Costantini and Fontanella proposed a tensor product bivariate polynomial spline with variable degree for constructing axially monotone surfaces interpolating arbitrary sets of gridded data[3]. Sufficient data dependent constraints were also derived on the degree to preserve the shape of 3D monotone data on rectangular grids. The given scheme can produce interpolation surfaces of arbitrary continuity class but its disadvantage is that it is not local; any changes to a degree will influence a corresponding row or column of interpolation surface patches, and in some rectangular patches, the degree of interpolant may become too large, leading to the polynomial patches tending to be linear in x and/or y directions. The resulting surfaces are not always visually pleasing.

Some C^1 shape-preserving bivariate interpolation splines with local control parameters have been proposed by using the Coons surface technique[4]. In Refs. 5–7, by exchanging the cubic Hermite blending functions for the classical bi-cubic Coons surface with different kinds of rational cubic or quartic Hermite-type blending functions, several kinds of C^1 rational bi-cubic or bi-quartic functions were presented along with constraints concerning the local control parameters for visualizing 3D positive data and/or 3D monotonic data on rectangular grids. Like the classical bi-cubic Coons surface technique, these schemes need to provide the twists on the grid lines for generating interpolation surfaces. In Refs. 8–11, based on the boolean sum of cubic interpolating operators, by blending different splines such as variable degree interpolation splines[8] and rational cubic interpolation splines[9–11] as the boundary functions, simple schemes without making use of twists for constructing C^1 positive and/or monotonic interpolation of gridded data were developed. These rational bi-cubic partially blended interpolation spline methods are convenient since it is possible to control the shape of the interpolation surfaces by using the boundary functions, although the generated surfaces have zero twist vectors at the data points.

The sufficient conditions for generating positivity and/or monotonicity preserving interpolation surfaces developed in Refs. 9–11 have a common point that the positivity and/or monotonicity of the global interpolation surfaces are determined by the positivity and/or monotonicity of the four boundary curves of each local interpolation surface patch. However, as pointed out in Ref. 7, these methods do not depict the positive or monotonic surfaces because they conserve the shape of data only on the boundaries of patch. Hence one asks whether it is possible to generate positivity and/or monotonicity preserving interpolation surfaces by controlling the four boundary curves of each local interpolation surface patch.

In this paper we present a new C^1 bi-cubic partially blended rational quartic/linear interpolation surface which can preserve the shape of 3D positive and/or monotonic data everywhere in the domain by constraining the four boundary curves of each local interpolation surface patch. To achieve this goal, new constraint conditions on the boundary curves of each local interpolation surface patch, differing from those used in Refs. 9, 10, are developed.

C^1 RATIONAL QUARTIC/LINEAR HERMITE INTERPOLATION SPLINES

In this section, we recall the rational quartic/linear Hermite interpolation spline given in Ref. 12.

Let $f_i \in \mathbb{R}$, $i = 1, \dots, n$, be data given at the distinct knots $x_i \in \mathbb{R}$, $i = 1, \dots, n$, with interval spacing $h_i = x_{i+1} - x_i > 0$, and let $d_i \in \mathbb{R}$ denote the first derivative values defined at the knots. For $x \in [x_i, x_{i+1}]$, $t = (x - x_i)/h_i$, $i = 1, 2, \dots, n-1$, a piecewise rational quartic/linear Hermite interpolation spline, with two local control parameters α_i and β_i, is defined as follows:

$$R(x) = B_0(t;\alpha_i)f_i + B_1(t;\alpha_i)\left(f_i + \frac{h_i}{\alpha_i}d_i\right) + B_2(t;\beta_i)\left(f_{i+1} - \frac{h_i}{\beta_i}d_{i+1}\right) + B_3(t;\beta_i)f_{i+1}, \quad (1)$$

where $\alpha_i, \beta_i \in [2, +\infty)$, and the rational quartic/linear Said-Ball-like basis functions $B_j(t;\alpha)$ and $B_{3-j}(t;\beta)$, $j = 0, 1$ are given by

$$B_0(t;\alpha) = \frac{(1-t)^2}{1+(\alpha-2)t},$$
$$B_1(t;\alpha) = \frac{(1-t)^2 t}{1+(\alpha-2)t}[\alpha + 2(\alpha-2)t],$$
$$B_2(t;\beta) = \frac{(1-t)t^2}{1+(\beta-2)(1-t)}[\beta + 2(\beta-2)(1-t)],$$
$$B_3(t;\beta) = \frac{t^2}{1+(\beta-2)(1-t)}.$$

The spline given in (1) is a C^1 Hermite-type interpolant as it satisfies the end point interpolation properties $R(x_i) = f_i$, $R(x_{i+1}) = f_{i+1}$, $R'(x_i) = d_i$, $R'(x_{i+1}) = d_{i+1}$. Here $R'(x)$ denotes the derivative with respect to the variable x. It can be easily checked that for all $\alpha_i = \beta_i = 2$, the interpolant (1) is exactly the classical cubic Hermite interpolation spline. Zhu et al developed some useful results using the interpolant (1) to preserve the shape of 2D positive and/or monotone data[12]. In this paper, the interpolant (1) is extended to a C^1 bi-cubic partially blended rational quartic/linear interpolation surface for the interpolation of 3D data on rectangular grids. The developed bi-cubic partially blended rational quartic/linear interpolation surface is further used to preserve the shape of 3D positive and/or monotonic data on rectangular domain. We will denote the interpolant $R(x)$ as $R(t; f_i, f_{i+1}; d_i, d_{i+1}; \alpha_i, \beta_i)$ for $x \in [x_i, x_{i+1}]$.

BI-CUBIC PARTIALLY BLENDED RATIONAL QUARTIC/LINEAR INTERPOLATION SURFACES

Let $\{(x_i, y_i, F_{ij}), i = 1, 2, \dots, n; j = 1, 2, \dots, m\}$ be a given set of data points defined over the rectangular domain $D = [x_1, x_n] \times [y_1, y_m]$, where $\pi_x : x_1 < x_2 < \dots < x_n$ is the partition of $[x_1, x_n]$ and $\pi_y : y_1 < y_2 < \dots < y_m$ is the partition of $[y_1, y_m]$. For $(x, y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, by using the boolean sum of cubic interpolating operators to blend together the rational quartic/linear Hermite interpolation splines (1) as four boundary functions, a new bi-cubic partially blended rational quartic/linear interpolation surface is given by

$$S(x,y) = -\mathbf{b}^{\mathrm{T}}\mathbf{U}\mathbf{b}, \quad (2)$$

where $\mathbf{b}^{\mathrm{T}} = (-1\ b_0(t)\ b_1(t))$,

$$\mathbf{U} = \begin{pmatrix} 0 & R(x,y_j) & R(x,y_{j+1}) \\ R(x_i,y) & F_{i,j} & F_{i,j+1} \\ R(x_{i+1},y) & F_{i+1,j} & F_{i+1,j+1} \end{pmatrix},$$

$h_i^x = x_{i+1} - x_i$, $h_j^y = y_{j+1} - y_j$, $t = (x - x_i)/h_i^x$, $s = (y - y_j)/h_j^y$, and

$$\begin{aligned}
b_0(z) &:= (1-z)^2[1+2z],\\
b_1(z) &:= z^2[1+2(1-z)],\\
R(x, y_j) &:= R(t; F_{i,j}, F_{i+1,j}; D^x_{i,j}, D^x_{i+1,j}; \alpha^x_{i,j}, \beta^x_{i,j}),\\
R(x_i, y) &:= R(s; F_{i,j}, F_{i,j+1}; D^y_{i,j}, D^y_{i,j+1}; \alpha^y_{i,j}, \beta^y_{i,j}).
\end{aligned}$$

$D^x_{i,j}$, $D^y_{i,j}$ are the first partial derivatives at the grid point (x_i, y_j) and $(\alpha^x_{i,j})_{(n-1)\times m}$, $(\beta^x_{i,j})_{(n-1)\times m}$, $(\alpha^y_{i,j})_{n\times(m-1)}$, $(\beta^y_{i,j})_{n\times(m-1)}$ are the four families of local control parameters. From the interpolation surface $S(x, y)$ given in (2), we can see that the change of a local control parameter $\alpha^x_{i,j}$ or $\beta^x_{i,j}$ will affect the shape of two neighbouring patches $S(x, y)$ defined in the domain $(x, y) \in (x_i, x_{i+1}) \times (y_{i-1}, y_{i+1})$. Since the four boundary functions $R(x, y_j)$, $R(x, y_{j+1})$, $R(x_i, y)$, $R(x_{i+1}, y)$ are all C^1 continuous, we can easily conclude that the given bi-cubic partially blended rational quartic/linear interpolation surface $S(x, y)$ is global C^1 continuous over the rectangular domain $[x_1, x_n] \times [y_1, y_m]$.

In most applications, the first partial derivatives $D^x_{i,j}$ and $D^y_{i,j}$ are not given and hence must be determined either from given data or by some other means. Here we use the following arithmetic mean method to compute them:

$$\begin{aligned}
D^x_{1,j} &= \Delta^x_{1,j} + (\Delta^x_{1,j} - \Delta^x_{2,j})\frac{h^x_1}{h^x_1 + h^x_2},\\
D^x_{n,j} &= \Delta^x_{n-1,j} + (\Delta^x_{n-1,j} - \Delta^x_{n-2,j})\frac{h^x_{n-1}}{h^x_{n-2} + h^x_{n-1}},\\
D^x_{i,j} &= \frac{\Delta^x_{i-1,j} + \Delta^x_{i,j}}{2},
\end{aligned}$$

for $i = 2, 3, \dots, n-1$; $j = 1, 2, \dots, m$, and

$$\begin{aligned}
D^y_{i,1} &= \Delta^y_{i,1} + (\Delta^y_{i,1} - \Delta^y_{i,2})\frac{h^y_1}{h^y_1 + h^y_2},\\
D^y_{i,m} &= \Delta^y_{i,m-1} + (\Delta^y_{i,m-1} - \Delta^y_{i,m-2})\frac{h^y_{m-1}}{h^y_{m-2} + h^y_{m-1}},\\
D^y_{i,j} &= \frac{\Delta^y_{i,j-1} + \Delta^y_{i,j}}{2},
\end{aligned}$$

for $i = 1, 2, \dots, n$, $j = 2, 3, \dots, m-1$, where $\Delta^x_{i,j} = (F_{i+1,j} - F_{i,j})/h^x_i$ and $\Delta^y_{i,j} = (F_{i,j+1} - F_{i,j})/h^y_j$. This arithmetic mean method is computationally economical and suitable for the visualization of shaped data[9].

C^1 SHAPE-PRESERVING INTERPOLATION SURFACES

In this section, we develop simple schemes so that the C^1 interpolation surface $S(x, y)$ can preserve the shape of 3D positive and/or monotonic data on rectangular grids. For any $\alpha, \beta \in [2, +\infty)$, the four rational quartic/linear Said-Ball-like basis functions $B_j(t; \alpha)$ and $B_{3-j}(t; \beta)$, $j = 1, 2$ are nonnegative and satisfy

$$\begin{aligned}
B_0(t; \alpha) + B_1(t; \alpha) &= b_0(t),\\
B_2(t; \beta) + B_3(t; \beta) &= b_1(t).
\end{aligned} \tag{3}$$

For $(x, y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, we rewrite the expression of the interpolation surface $S(x, y)$ given in (2) as

$$\begin{aligned}
S(x, y) &= b_0(s)R(x, y_j) + b_1(s)R(x, y_{j+1})\\
&\quad + b_0(t)R(x_i, y) + b_1(t)R(x_{i+1}, y)\\
&\quad - b_0(t)b_0(s)F_{i,j} - b_0(t)b_1(s)F_{i,j+1}\\
&\quad - b_1(t)b_0(s)F_{i+1,j} - b_1(t)b_1(s)F_{i+1,j+1}\\
&= b_0(s)[R(x, y_j) - \tfrac{1}{2}b_0(t)F_{i,j} - \tfrac{1}{2}b_1(t)F_{i+1,j}]\\
&\quad + b_1(s)[R(x, y_{j+1}) - \tfrac{1}{2}b_0(t)F_{i,j+1} - \tfrac{1}{2}b_1(t)F_{i+1,j+1}]\\
&\quad + b_0(t)[R(x_i, y) - \tfrac{1}{2}b_0(s)F_{i,j} - \tfrac{1}{2}b_1(s)F_{i,j+1}]\\
&\quad + b_1(t)[R(x_{i+1}, y) - \tfrac{1}{2}b_0(s)F_{i+1,j} - \tfrac{1}{2}b_1(s)F_{i+1,j+1}].
\end{aligned} \tag{4}$$

C^1 positivity-preserving interpolation surfaces

Let $\{(x_i, y_i, F_{i,j})\}$ be a positive data set defined over the rectangular grid $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $i = 1, 2, \dots, n-1$, $j = 1, 2, \dots, m-1$ such that $F_{i,j} > 0$, $\forall i, j$. The interpolation surface $S(x, y)$ given in (2) preserves the shape of positive data if $S(x, y) > 0$ for all $(x, y) \in [x_1, x_n] \times [y_1, y_m]$. Without loss of generality, for any $(x, y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, since the two blending functions $b_0(z)$ and $b_1(z)$ are nonnegative on $[0, 1]$ and strictly positive in $(0, 1)$, we can see from (4) that the interpolation surface $S(x, y)$ is positive if the following constraints hold:

$$\left.\begin{aligned}
R(x, y_j) - \tfrac{1}{2}b_0(t)F_{i,j} - \tfrac{1}{2}b_1(t)F_{i+1,j} > 0,\\
R(x, y_{j+1}) - \tfrac{1}{2}b_0(t)F_{i,j+1} - \tfrac{1}{2}b_1(t)F_{i+1,j+1} > 0,\\
R(x_i, y) - \tfrac{1}{2}b_0(s)F_{i,j} - \tfrac{1}{2}b_1(s)F_{i,j+1} > 0,\\
R(x_{i+1}, y) - \tfrac{1}{2}b_0(s)F_{i+1,j} - \tfrac{1}{2}b_1(s)F_{i+1,j+1} > 0.
\end{aligned}\right\} \tag{5}$$

For $R(x, y_j) - \frac{1}{2}b_0(t)F_{i,j} - \frac{1}{2}b_1(t)F_{i+1,j}$, from (3), we have

$$
\begin{aligned}
&R(x,y_j)-\tfrac{1}{2}b_0(t)F_{i,j}-\tfrac{1}{2}b_1(t)F_{i+1,j}\\
&\quad=\tfrac{1}{2}B_0(t;\alpha^x_{i,j})F_{i,j}+B_1(t;\alpha^x_{i,j})\left(\frac{F_{i,j}}{2}+\frac{h^x_i}{\alpha^x_{i,j}}D^x_{i,j}\right)\\
&\qquad+B_2(t;\beta^x_{i,j})\left(\frac{F_{i+1,j}}{2}-\frac{h^x_i}{\beta^x_{i,j}}D^x_{i+1,j}\right)\\
&\qquad+\tfrac{1}{2}B_3(t;\beta^x_{i,j})F_{i+1,j}.
\end{aligned}
$$

Since the four rational quartic/linear Said-Ball-like basis functions $B_j(t;\alpha^x_{i,j})$ and $B_{3-j}(t;\beta^x_{i,j})$, $j=1,\,2$ are nonnegative for any $\alpha^x_{i,j},\beta^x_{i,j}\in[2,+\infty)$, we can see that the following constraints are sufficient to ensure $R(x,y_j)-\frac{1}{2}b_0(t)F_{i,j}-\frac{1}{2}b_1(t)F_{i+1,j}>0$:

$$
\begin{gathered}
\alpha^x_{i,j}\geqslant 2,\ \beta^x_{i,j}\geqslant 2,\\
\frac{F_{i,j}}{2}+\frac{h^x_i}{\alpha^x_{i,j}}D^x_{i,j}\geqslant 0,\ \frac{F_{i+1,j}}{2}-\frac{h^x_i}{\beta^x_{i,j}}D^x_{i+1,j}\geqslant 0,
\end{gathered}
$$

which bring forth the following sufficient conditions:

$$
\begin{aligned}
\alpha^x_{i,j}&\geqslant\max\{-2h^x_iD^x_{i,j}/F_{i,j},2\},\\
\beta^x_{i,j}&\geqslant\max\{2h^x_iD^x_{i+1,j}/F_{i+1,j},2\}.
\end{aligned}\tag{6}
$$

In the same fashion, we can derive similar sufficient conditions for $R(x,y_{j+1})-\frac{1}{2}b_0(t)F_{i,j+1}-\frac{1}{2}b_1(t)F_{i+1,j+1}>0$, $R(x_i,y)-\frac{1}{2}b_0(s)F_{i,j}-\frac{1}{2}b_1(s)F_{i,j+1}>0$, and $R(x_{i+1},y)-\frac{1}{2}b_0(s)F_{i+1,j}-\frac{1}{2}b_1(s)F_{i+1,j+1}>0$. In conclusion, for a positive data set, the following constraint conditions are sufficient to ensure $S(x,y)>0$, $\forall(x,y)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]$:

$$
\left.
\begin{aligned}
\alpha^x_{i,j}&=\max\{-2h^x_iD^x_{i,j}/F_{i,j},2\}+a^x_{i,j},\\
\alpha^x_{i,j+1}&=\max\{-2h^x_iD^x_{i,j+1}/F_{i,j+1},2\}+a^x_{i,j+1},\\
\beta^x_{i,j}&=\max\{2h^x_iD^x_{i+1,j}/F_{i+1,j},2\}+b^x_{i,j},\\
\beta^x_{i,j+1}&=\max\{2h^x_iD^x_{i,j+1}/F_{i+1,j+1},2\}+b^x_{i,j+1},\\
\alpha^y_{i,j}&=\max\{-2h^y_jD^y_{i,j}/F_{i,j},2\}+a^y_{i,j},\\
\alpha^y_{i+1,j}&=\max\{-2h^y_jD^y_{i+1,j}/F_{i+1,j},2\}+a^y_{i+1,j},\\
\beta^y_{i,j}&=\max\{2h^y_jD^y_{i,j+1}/F_{i,j+1},2\}+b^y_{i,j},\\
\beta^y_{i+1,j}&=\max\{2h^y_jD^y_{i+1,j+1}/F_{i+1,j+1},2\}+b^y_{i+1,j},
\end{aligned}
\right\}\tag{7}
$$

where $i=1,2,\dots,n-1$, $j=1,2,\dots,m-1$, and $a^x_{i,j}$, $a^y_{i,j}$, $b^x_{i,j}$, and $b^y_{i,j}$ are arbitrary nonnegative real numbers and serve as free shape parameters.

C^1 monotonicity-preserving interpolation surfaces

Let $\{(x_i,y_i,F_{i,j})\}$ be a monotonic increasing data set defined over the rectangular grid $D=[x_i,x_{i+1}]\times[y_j,y_{j+1}]$, $i=1,2,\dots,n-1$, $j=1,2,\dots,m-1$ such that $F_{i+1,j}>F_{i,j}$, $F_{i,j+1}>F_{i,j}$, $\forall i,\,j$, or equivalently $\Delta^x_{i,j}>0$, $\Delta^y_{i,j}>0$, $\forall i,\,j$. For a monotonic increasing preserving interpolation surface $S(x,y)$, it is necessary that the corresponding partial derivatives $D^x_{i,j}$ and $D^y_{i,j}$ should satisfy

$$
D^x_{i,j}>0,\quad D^y_{i,j}>0,\quad\forall i,j.\tag{8}
$$

The interpolation surface $S(x,y)$ preserves the shape of monotonic increasing data if

$$
\frac{\partial S(x,y)}{\partial x}>0,\quad\frac{\partial S(x,y)}{\partial y}>0,\tag{9}
$$

for any $(x,y)\in[x_1,x_n]\times[y_1,y_m]$.

Without loss of generality, for any $(x,y)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]$, from (4), direct computation gives that

$$
\begin{aligned}
\frac{\partial S(x,y)}{\partial x}&=b_0(s)[R'(x,y_j)-3t(1-t)\Delta^x_{i,j}]\\
&\quad+b_1(s)[R'(x,y_{j+1})-3t(1-t)\Delta^x_{i,j+1}]\\
&+\frac{6t(1-t)}{h^x_i}\{[R(x_{i+1},y)-\tfrac{1}{2}b_0(s)F_{i+1,j}-\tfrac{1}{2}b_1(s)F_{i+1,j+1}]\\
&\quad-[R(x_i,y)-\tfrac{1}{2}b_0(s)F_{i,j}-\tfrac{1}{2}b_1(s)F_{i,j+1}]\}.
\end{aligned}\tag{10}
$$

Thus we can see that the constraints

$$
\left.
\begin{aligned}
&R'(x,y_j)-3t(1-t)\Delta^x_{i,j}>0,\\
&R'(x,y_{j+1})-3t(1-t)\Delta^x_{i,j+1}>0,\\
&[R(x_{i+1},y)-\tfrac{1}{2}b_0(s)F_{i+1,j}-\tfrac{1}{2}b_1(s)F_{i+1,j+1}]\\
&\qquad-[R(x_i,y)-\tfrac{1}{2}b_0(s)F_{i,j}-\tfrac{1}{2}b_1(s)F_{i,j+1}]>0
\end{aligned}
\right\}\tag{11}
$$

are sufficient to ensure $\partial S(x,y)/\partial x>0$, $\forall(x,y)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]$.

For $R'(x,y_j)-3t(1-t)\Delta^x_{i,j}$, we have

$$
\begin{aligned}
&R'(x,y_j)-3t(1-t)\Delta^x_{i,j}\\
&\quad=\frac{(1-t)[\alpha^x_{i,j}+(\alpha^x_{i,j}-2)t]}{\alpha^x_{i,j}[1+(\alpha^x_{i,j}-2)t]^2}D^x_{i,j}\\
&\qquad+\frac{t[\alpha^x_{i,j}+(\alpha^x_{i,j}-2)(1-t)]}{\alpha^x_{i,j}[1+(\alpha^x_{i,j}-2)(1-t)]^2}D^x_{i+1,j}\\
&\qquad+6t(1-t)\left[\left(\frac{\Delta^x_{i,j}}{4}-\frac{D^x_{i,j}}{\alpha^x_{i,j}}\right)+\left(\frac{\Delta^x_{i,j}}{4}-\frac{D^x_{i+1,j}}{\beta^x_{i,j}}\right)\right].
\end{aligned}
$$

It follows that the following conditions are sufficient for $R'(x,y_j)-3t(1-t)\Delta^x_{i,j}>0$:

$$\begin{aligned}\alpha^x_{i,j}&\geqslant \max\{4D^x_{i,j}/\Delta^x_{i,j},2\},\\ \beta^x_{i,j}&\geqslant \max\{4D^x_{i+1,j}/\Delta^x_{i,j},2\}.\end{aligned}\tag{12}$$

Similarly, we can conclude that the following conditions are sufficient for $R'(x,y_{j+1})-3t(1-t)\Delta^x_{i,j+1}>0$:

$$\begin{aligned}\alpha^x_{i,j+1}&\geqslant \max\{4D^x_{i,j+1}/\Delta^x_{i,j+1},2\},\\ \beta^x_{i,j+1}&\geqslant \max\{4D^x_{i+1,j+1}/\Delta^x_{i,j+1},2\}.\end{aligned}\tag{13}$$

For $[R(x_{i+1},y)-\frac{1}{2}b_0(s)F_{i+1,j}-\frac{1}{2}b_1(s)F_{i+1,j+1}]-[R(x_i,y)-\frac{1}{2}b_0(s)F_{i,j}-\frac{1}{2}b_1(s)F_{i,j+1}]$, after some manipulation, we have

$$\begin{aligned}&[R(x_{i+1},y)-\tfrac{1}{2}b_0(s)F_{i+1,j}-\tfrac{1}{2}b_1(s)F_{i+1,j+1}]\\ &\quad-[R(x_i,y)-\tfrac{1}{2}b_0(s)F_{i,j}-\tfrac{1}{2}b_1(s)F_{i,j+1}]\\ &=b_0(s)\left[\frac{h^y_j}{\alpha^y_{i+1,j}}D^y_{i+1,j}+\left(\frac{F_{i+1,j}-F_{i,j}}{2}-\frac{h^y_j}{\alpha^y_{i,j}}D^y_{i,j}\right)\right]\\ &\quad+b_1(s)\left[\frac{h^y_j}{\beta^y_{i,j}}D^y_{i,j+1}+\left(\frac{F_{i+1,j+1}-F_{i,j+1}}{2}\right.\right.\\ &\quad\left.\left.-\frac{h^y_j}{\beta^y_{i+1,j}}D^y_{i+1,j+1}\right)\right]+\frac{(1-s)^2h^y_j}{\phi^y_{i,j}}\{[\alpha^y_{i+1,j}D^y_{i,j}\\ &\quad-\alpha^y_{i,j}D^y_{i+1,j}](1-s)+[\alpha^y_{i+1,j}(\alpha^y_{i+1,j}-1)D^y_{i,j}\\ &\quad-\alpha^y_{i,j}(\alpha^y_{i,j}-1)D^y_{i+1,j}]s\}+\frac{s^2h^y_j}{\varphi^y_{i,j}}\{[\beta^y_{i,j}D^y_{i+1,j+1}\\ &\quad-\beta^y_{i+1,j}D^y_{i,j+1}]s+[\beta^y_{i,j}(\beta^y_{i,j}-1)D^y_{i+1,j+1}\\ &\quad-\beta^y_{i+1,j}(\beta^y_{i+1,j}-1)D^y_{i,j+1}(1-s)]\},\end{aligned}$$

where $\phi^y_{i,j}=\alpha^y_{i,j}\alpha^y_{i+1,j}[1+(\alpha^y_{i,j}-2)s][1+(\alpha^y_{i+1,j}-2)s]$ and $\varphi^y_{i,j}=\beta^y_{i,j}\beta^y_{i+1,j}[1+(\beta^y_{i,j}-2)(1-s)][1+(\beta^y_{i+1,j}-2)(1-s)]$. Thus we can see that the constraints

$$\begin{aligned}\frac{F_{i+1,j}-F_{i,j}}{2}-\frac{h^y_j}{\alpha^y_{i,j}}D^y_{i,j}&\geqslant 0,\\ \frac{F_{i+1,j+1}-F_{i,j+1}}{2}-\frac{h^y_j}{\beta^y_{i+1,j}}D^y_{i+1,j+1}&\geqslant 0,\\ \alpha^y_{i+1,j}D^y_{i,j}-\alpha^y_{i,j}D^y_{i+1,j}&\geqslant 0,\\ \alpha^y_{i+1,j}(\alpha^y_{i+1,j}-1)D^y_{i,j}-\alpha^y_{i,j}(\alpha^y_{i,j}-1)D^y_{i+1,j}&\geqslant 0,\\ \beta^y_{i,j}D^y_{i+1,j+1}-\beta^y_{i+1,j}D^y_{i,j+1}&\geqslant 0,\\ \beta^y_{i,j}(\beta^y_{i,j}-1)D^y_{i+1,j+1}-\beta^y_{i+1,j}(\beta^y_{i+1,j}-1)D^y_{i,j+1}&\geqslant 0\end{aligned}$$

are sufficient for $[R(x_{i+1},y)-\frac{1}{2}b_0(s)F_{i+1,j}-\frac{1}{2}b_1(s)F_{i+1,j+1}]-[R(x_i,y)-\frac{1}{2}b_0(s)F_{i,j}-\frac{1}{2}b_1(s)F_{i,j+1}]>0$, from which we can obtain the sufficient conditions

$$\left.\begin{aligned}\alpha^y_{i,j}&\geqslant \max\{2h^y_jD^y_{i,j}/h^x_i\Delta^x_{i,j},2\},\\ \alpha^y_{i+1,j}&\geqslant \max\{D^y_{i+1,j}\alpha^y_{i,j}/D^y_{i,j},\alpha^y_{i,j}\},\\ \beta^y_{i+1,j}&\geqslant \max\{2h^y_jD^y_{i+1,j+1}/h^x_i\Delta^x_{i,j+1},2\},\\ \beta^y_{i,j}&\geqslant \max\{D^y_{i,j+1}\beta^y_{i+1,j}/D^y_{i+1,j+1},\beta^y_{i+1,j}\}.\end{aligned}\right\}\tag{14}$$

From (12), (13), and (14), we can obtain the following sufficient conditions for $\partial S(x,y)/\partial x>0$, $\forall(x,y)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]$:

$$\left.\begin{aligned}\alpha^x_{i,j}&\geqslant \max\{4D^x_{i,j}/\Delta^x_{i,j},2\},\\ \alpha^x_{i,j+1}&\geqslant \max\{4D^x_{i,j+1}/\Delta^x_{i,j+1},2\},\\ \beta^x_{i,j}&\geqslant \max\{4D^x_{i+1,j}/\Delta^x_{i,j},2\},\\ \beta^x_{i,j+1}&\geqslant \max\{4D^x_{i+1,j+1}/\Delta^x_{i,j+1},2\},\\ \alpha^y_{i,j}&\geqslant \max\{2h^y_jD^y_{i,j}/h^x_i\Delta^x_{i,j},2\},\\ \alpha^y_{i+1,j}&\geqslant \max\{D^y_{i+1,j}\alpha^y_{i,j}/D^y_{i,j},\alpha^y_{i,j}\},\\ \beta^y_{i+1,j}&\geqslant \max\{2h^y_jD^y_{i+1,j+1}/h^x_i\Delta^x_{i,j+1},2\},\\ \beta^y_{i,j}&\geqslant \max\{D^y_{i,j+1}\beta^y_{i+1,j}/D^y_{i+1,j+1},\beta^y_{i+1,j}\}.\end{aligned}\right\}\tag{15}$$

In the same fashion, we can conclude that the following conditions are sufficient for $\partial S(x,y)/\partial y>0$, $\forall(x,y)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]$:

$$\left.\begin{aligned}\alpha^y_{i,j}&\geqslant \max\{4D^y_{i,j}/\Delta^y_{i,j},2\},\\ \alpha^y_{i+1,j}&\geqslant \max\{4D^y_{i+1,j}/\Delta^y_{i+1,j},2\},\\ \beta^y_{i,j}&\geqslant \max\{4D^y_{i,j+1}/\Delta^y_{i,j},2\},\\ \beta^y_{i+1,j}&\geqslant \max\{4D^y_{i+1,j+1}/\Delta^y_{i+1,j},2\},\\ \alpha^x_{i,j}&\geqslant \max\{2h^x_iD^x_{i,j}/h^y_j\Delta^y_{i,j},2\},\\ \alpha^x_{i,j+1}&\geqslant \max\{D^x_{i,j+1}\alpha^x_{i,j}/D^x_{i,j},\alpha^x_{i,j}\},\\ \beta^x_{i,j+1}&\geqslant \max\{2h^x_iD^x_{i+1,j+1}/h^y_j\Delta^y_{i+1,j},2\},\\ \beta^x_{i,j}&\geqslant \max\{D^x_{i+1,j}\beta^x_{i,j+1}/D^x_{i+1,j+1},\beta^x_{i,j+1}\}.\end{aligned}\right\}\tag{16}$$

In summary, we conclude that for a monotonic increasing data set, the following conditions are sufficient to ensure $\partial S(x,y)/\partial x>0$, $\partial S(x,y)/\partial x>0$,

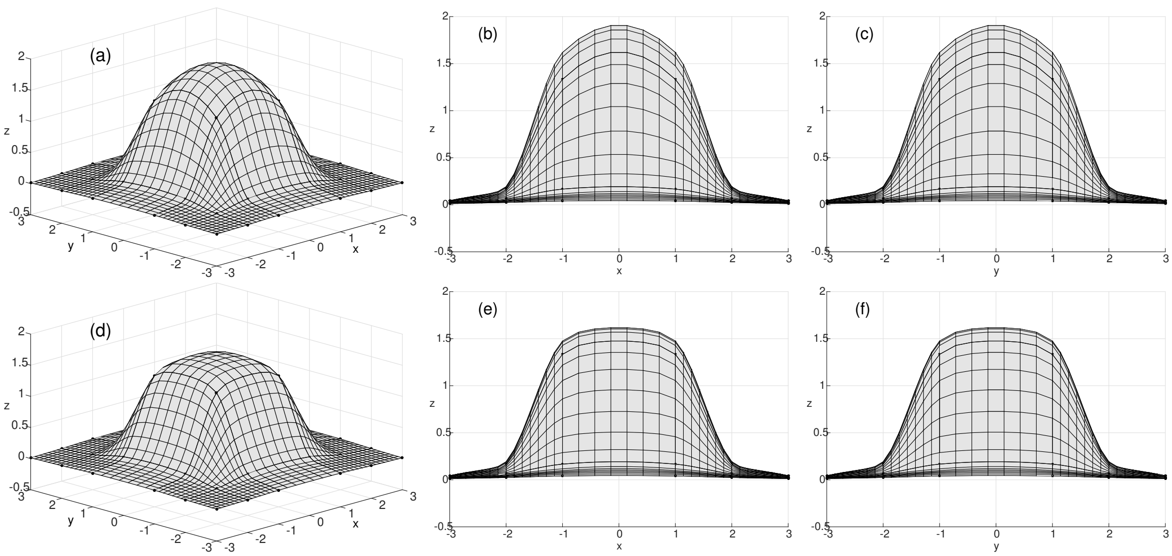

Fig. 1 C^1 positivity-preserving interpolation surfaces for the 3D positive data set given in Example 1: (a) $S_1(x,y)$; (b) xz-view of $S_1(x,y)$; (c) yz-view of $S_1(x,y)$; (d) $S_2(x,y)$; (e) xz-view of $S_2(x,y)$; (f) yz-view of $S_2(x,y)$. In this and the other figures, the given data points are shown by black dots.

$\forall(x,y)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]$:

$$\left.\begin{aligned}
\alpha^x_{i,j} &= \max\left\{\frac{4D^x_{i,j}}{\Delta^x_{i,j}}, \frac{2h^x_i D^x_{i,j}}{h^y_j \Delta^y_{i,j}}, 2\right\} + c^x_{i,j},\\
\alpha^x_{i,j+1} &= \max\left\{\frac{4D^x_{i,j+1}}{\Delta^x_{i,j+1}}, \frac{D^x_{i,j+1}\alpha^x_{i,j}}{D^x_{i,j}}, \alpha^x_{i,j}\right\} + c^x_{i,j+1},\\
\beta^x_{i,j+1} &= \max\left\{\frac{4D^x_{i+1,j+1}}{\Delta^x_{i,j+1}}, \frac{2h^x_i D^x_{i+1,j+1}}{h^y_j \Delta^y_{i+1,j}}, 2\right\} + d^x_{i,j+1},\\
\beta^x_{i,j} &= \max\left\{\frac{4D^x_{i+1,j}}{\Delta^x_{i,j}}, \frac{D^x_{i+1,j}\beta^x_{i,j+1}}{D^x_{i+1,j+1}}, \beta^x_{i,j+1}\right\} + d^x_{i,j},\\
\alpha^y_{i,j} &= \max\left\{\frac{4D^y_{i,j}}{\Delta^y_{i,j}}, \frac{2h^y_j D^y_{i,j}}{h^x_i \Delta^x_{i,j}}, 2\right\} + c^y_{i,j},\\
\alpha^y_{i+1,j} &= \max\left\{\frac{4D^y_{i+1,j}}{\Delta^y_{i+1,j}}, \frac{D^y_{i+1,j}\alpha^y_{i,j}}{D^y_{i,j}}, \alpha^y_{i,j}\right\} + c^y_{i+1,j},\\
\beta^y_{i+1,j} &= \max\left\{\frac{4D^y_{i+1,j+1}}{\Delta^y_{i+1,j}}, \frac{2h^y_j D^y_{i+1,j+1}}{h^x_i \Delta^x_{i,j+1}}, 2\right\} + d^y_{i+1,j},\\
\beta^y_{i,j} &= \max\left\{\frac{4D^y_{i,j+1}}{\Delta^y_{i,j}}, \frac{D^y_{i,j+1}\beta^y_{i+1,j}}{D^y_{i+1,j+1}}, \beta^y_{i+1,j}\right\} + d^y_{i,j},
\end{aligned}\right\} \tag{17}$$

where $i=1,2,\dots,n-1$, $j=1,2,\dots,m-1$, and $c^x_{i,j}$, $c^y_{i,j}$, $d^x_{i,j}$, and $d^y_{i,j}$ are arbitrary nonnegative real numbers and serve as free shape parameters.

Notice that any monotonicity-preserving interpolation surfaces to 3D positive and monotonic data on rectangular grids must then also be positive. Thus we can see that for 3D positive and monotonic data on rectangular grids, the conditions (17) are sufficient for the interpolation surface $S(x,y)$ to preserve both positivity and monotonicity.

NUMERICAL EXAMPLES

We now give several numerical examples to show that the proposed C^1 interpolation surface $S(x,y)$ given in (2) can be used to nicely visualize the shape of 3D positive or monotonic data on rectangular grids.

Example 1 We use the 3D positive data set[6] $P_{x,y}$ for $x,y=-3,-2,-1,1,2,3$ where $P_{-x,y}=Q_{x,y}$, $P_{x,-y}=Q_{x,y}$, $P_{-x,-y}=Q_{x,y}$, and $Q_{x,y}$ for $x,y=1,2,3$ is given by

$$Q=\begin{pmatrix} 1.3333 & 0.1667 & 0.0404\\ 0.1667 & 0.0635 & 0.0238\\ 0.0404 & 0.0238 & 0.0124 \end{pmatrix}.$$

Fig. 1a shows the interpolation surface $S_1(x,y)$ generated by using the sufficient conditions given in (7) with all the free shape parameters $a^x_{i,j}=a^y_{i,j}=b^x_{i,j}=b^y_{i,j}=0$. Fig. 1d shows the interpolation surface $S_2(x,y)$ generated by changing the free shape parameters $a^x_{i,j}$, $b^x_{i,j}$, $i=3$, $j=3,4$, and $a^y_{i,j}$, $b^y_{i,j}$, $i=3,4$, $j=3$ from 0 to 5. It can be seen that both the interpolation surfaces preserve the shape of the data set well and the

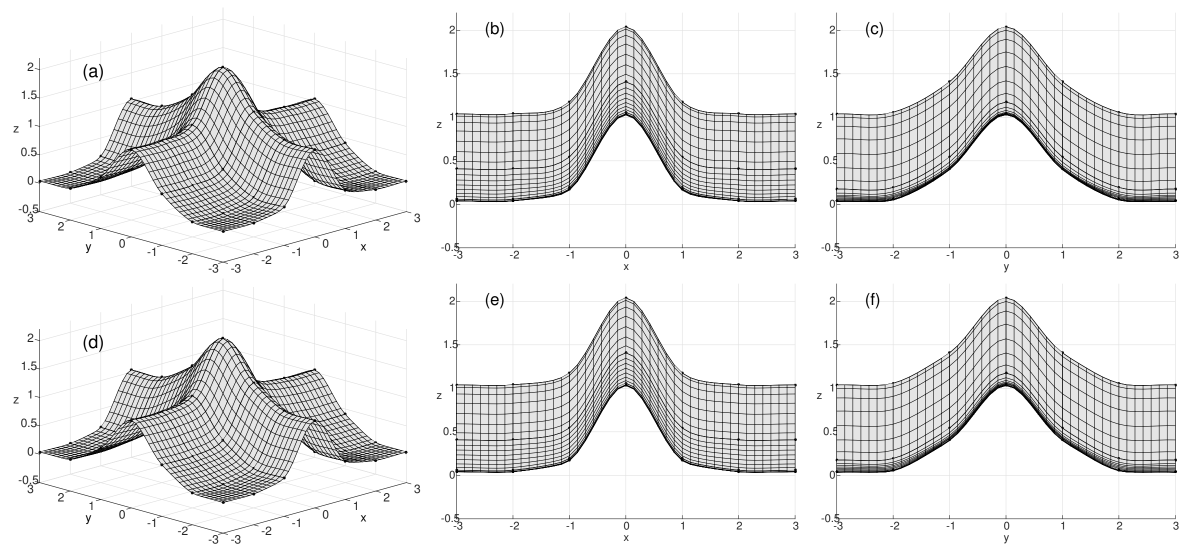

Fig. 2 C^1 positivity-preserving interpolation surfaces for the 3D positive data set given in Example 2: (a) $S_3(x,y)$; (b) xz-view of $S_3(x,y)$; (c) yz-view of $S_3(x,y)$; (d) $S_4(x,y)$; (e) xz-view of $S_4(x,y)$; (f) yz-view of $S_4(x,y)$.

shape of the positivity-preserving interpolation surfaces can be adjusted locally by using the shape parameters.

Example 2 We use the 3D positive data set[7] $P_{x,y}$ for $x,y=-3,-2,-1,0,1,2,3$ where $P_{-x,y}=Q_{x,y}$, $P_{x,-y}=Q_{x,y}$, $P_{-x,-y}=Q_{x,y}$, and $Q_{x,y}$ for $x,y=0,1,2,3$ is given by

$$Q=\begin{pmatrix} 2.04 & 1.4079 & 1.0583 & 1.0401\\ 1.1753 & 0.5432 & 0.1936 & 0.1755\\ 1.0403 & 0.4082 & 0.0586 & 0.0404\\ 1.04 & 0.4078 & 0.0583 & 0.0401 \end{pmatrix}.$$

Fig. 2a shows the interpolation surface $S_3(x,y)$ generated by using the sufficient conditions given in (7) with all the free shape parameters $a^x_{i,j}=a^y_{i,j}=b^x_{i,j}=b^y_{i,j}=0$. Fig. 2d shows the interpolation surface $S_4(x,y)$ generated by changing all the free shape parameters $a^x_{i,j}$, $a^y_{i,j}$, $b^x_{i,j}$, $b^y_{i,j}$ from 0 to 2. It is seen in Fig. 2 that the positive shape of the data is preserved nicely.

Example 3 We use the 3D monotonic data set[9] $M_{x,y}$ where $x,y=1,100,200,300$ and

$$M=\begin{pmatrix} 0.6931 & 9.2104 & 10.5967 & 11.4076\\ 9.2104 & 9.9035 & 10.8198 & 11.5129\\ 10.5967 & 10.8198 & 11.2898 & 11.7753\\ 11.4076 & 11.5129 & 11.7753 & 12.1007 \end{pmatrix}.$$

Fig. 3a shows the interpolation surface $S_5(x,y)$ generated by using the sufficient conditions given in (17) with all the free shape parameters $a^x_{i,j}=a^y_{i,j}=b^x_{i,j}=b^y_{i,j}=0$. Fig. 3d shows the interpolation surface $S_6(x,y)$ generated by changing two free shape parameters $d^x_{1,1}$, $d^y_{1,1}$ from 0 to 200. As can be seen from Fig. 3, both surfaces visualize the shape of the data well.

Example 4 We use the 3D monotonic data set[2] $M_{x,y}$ where $x,y=1,2,3,4$ and

$$M=\begin{pmatrix} 0 & 2 & 19.998 & 19.999\\ 2.999 & 3 & 19.999 & 20\\ 3 & 9 & 20 & 20.001\\ 8 & 10 & 20.001 & 20.002 \end{pmatrix}.$$

Fig. 4a shows the interpolation surface $S_7(x,y)$ generated by using the sufficient conditions given in (17) with $a^x_{i,j}=a^y_{i,j}=b^x_{i,j}=b^y_{i,j}=0$. Fig. 4d shows the interpolation surface $S_8(x,y)$ generated by changing two free shape parameters $d^x_{2,2}$, $d^y_{3,1}$ from 0 to 10. As can be seen from Fig. 4, both surfaces nicely visualize the shape of the data.

CONCLUSIONS

The constructed bi-cubic partially blended rational quartic/linear interpolation spline with four families of local control parameters can be C^1 continuous without making use of the first mixed partial derivatives at the data points. For 3D positive and/or monotonic data on rectangular grids, by developing new constraint conditions on the boundary curves of each local interpolation surface patch, differing

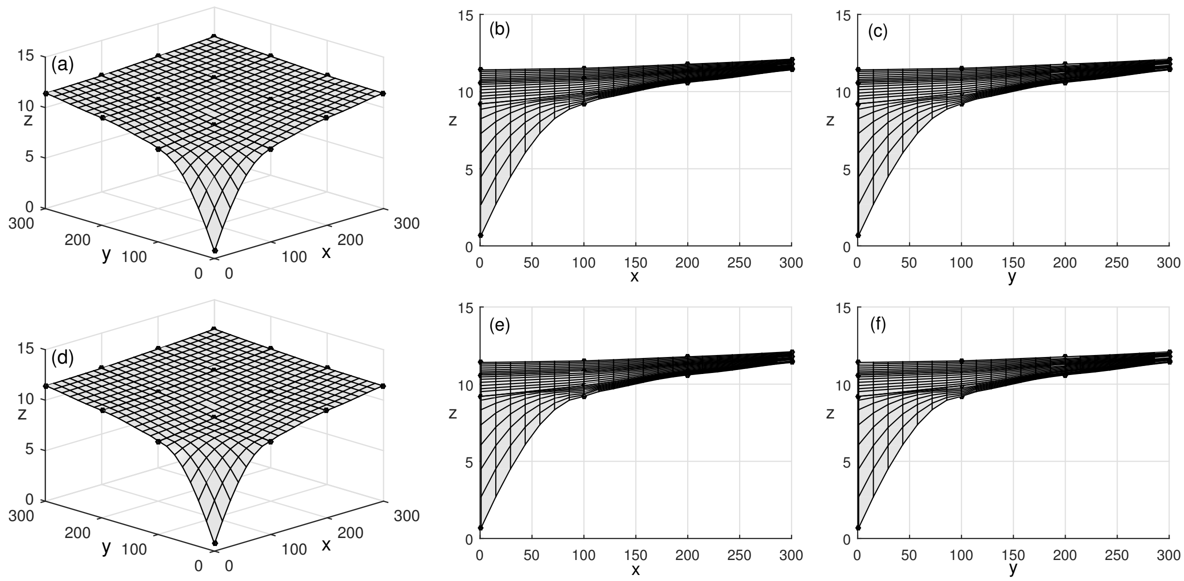

Fig. 3 C^1 monotonicity-preserving interpolation surfaces for the 3D monotonic data set given in Example 3: (a) $S_5(x,y)$; (b) xz-view of $S_5(x,y)$; (c) yz-view of $S_5(x,y)$; (d) $S_6(x,y)$; (e) xz-view of $S_6(x,y)$; (f) The yz-view of $S_6(x,y)$.

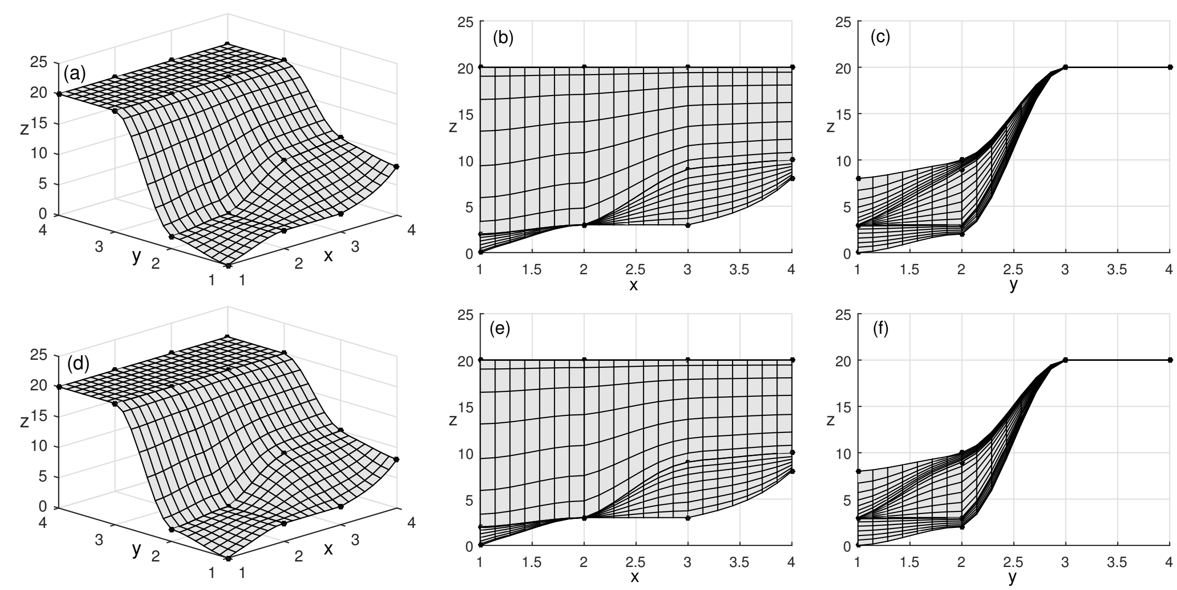

Fig. 4 C^1 monotonicity-preserving interpolation surfaces for the 3D monotonic data set given in Example 4: (a) $S_7(x,y)$; (b) xz-view of $S_7(x,y)$; (c) yz-view of $S_7(x,y)$; (d) $S_8(x,y)$; (e) xz-view of $S_8(x,y)$; (f) yz-view of $S_8(x,y)$.

from those given in Refs. 9, 10, simple sufficient data dependent conditions are given on the local control parameters to generate positivity and/or monotonicity preserving interpolation surfaces. The given method also allows extensions to generate C^1 shape-preserving interpolation surfaces for 3D non-gridded data. There remain some problems we intend to study, such as the construction of convexity preserving interpolation surfaces with local shape parameters.

Acknowledgements: The authors thank the anonymous referees for their inspiring comments which helped to improve this manuscript significantly.

REFERENCES

1. Carlson RE, Fritsch FN (1985) Monotone piecewise bicubic interpolation. *SIAM J Numer Anal* **22**, 386–400.
2. Carlson RE, Fritsch FN (1989) An algorithm for monotone piecewise bicubic interpolatino. *SIAM J Numer Anal* **26**, 230–8.
3. Costantini P, Fontanella F (1990) Shape-preserving bivariate interpolation. *SIAM J Numer Anal* **27**, 488–506.
4. Coons SA (1967) Surfaces in computer-aided design of space forms. Technical Report MAC-TR-41, MIT.
5. Hussain MZ, Hussain M (2006) Monotonic surfaces for computer graphics. *J Prime Res Math* **2**, 170–86.
6. Hussain MZ, Sarfraz M (2008) Positivity-preserving interpolation of positive data by rational cubics. *J Comput Appl Math* **218**, 446–58.
7. Abbas M, Majid AA, Ali JM (2014) Positivity-preserving rational bi-cubic spline interpolation for 3D positive data. *Appl Math Comput* **234**, 460–76.
8. Costantini P (1996) A bicubic shape-preserving blending scheme. *Comput Aided Geom Des* **13**, 307–31.
9. Hussain MZ, Hussain M (2007) Visualization of data preserving monotonicity. *Appl Math Comput* **190**, 1353–64.
10. Hussain MZ, Hussain M, Aqeel B (2014) Shape-preserving surfaces with constraints on tension parameters. *Appl Math Comput* **247**, 442–64.
11. Casciola G, Romani L (2002) Rational interpolants with tension parameters. In: Lyche T, Mazure M, Schumaker LL (eds) *Proceedings of Curve and Surface Design*, Saint-Malo, Nashboro Press, Brentwood, TN, pp 41–50.
12. Zhu YP, Han XL, Liu SJ (2014) Quartic rational Said-Ball-like basis with tension shape parameters and its application. *J Appl Math* **2014**, 857840.