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Normal forms of smooth plane quartics and their restrictions

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ABSTRACT: It is well known that smooth plane quartic curves in a two-dimensional complex projective space are curves of genus three and that the dimension of the parameters of the defining equation is less than seven. We show a process for obtaining the normal forms and their restrictions. For a homogeneous 4th-degree polynomial over the complex numbers, the vanishing set $\mathbb{C}$ of the homogeneous polynomial in the complex projective plane $\mathbb{P}^2$ is a curve of genus three, and such curves depend on six-dimensional parameters. By using the Gröbner basis of the elimination ideal, we show the restrictions on smooth plane quartics.

KEYWORDS: Gröbner basis

INTRODUCTION

Let $\mathbb{P}^2$ be a two-dimensional complex projective space with the coordinates $[x, y, z]$, and let $f_4(x, y, z)$ be a homogeneous fourth-degree polynomial in x, y, z in $\mathbb{P}^2$. We consider the set

$$V_4 := \{(x, y, z) \mid f_4(x, y, z) = 0\},$$

which we call complex projective plane quartics (or just plane quartics). Nonhyperelliptic curves of genus three are nonsingular plane quartics. Let M_g be the variety of moduli of curves of genus g. Then the dimension of M_3 is less than or equal to six. This is indeed true and is a classical result[1].

Let $f(x, y, z)$ be a homogeneous fourth-degree polynomial in x, y, z in $\mathbb{C}^3$. Then

$$f(0,0,0) = \frac{\partial f(0,0,0)}{\partial x} = \frac{\partial f(0,0,0)}{\partial y} = \frac{\partial f(0,0,0)}{\partial z} = 0.$$

Hence the analytic set defined by $f(x, y, z) = 0$ has a singular point at the origin in $\mathbb{C}^3$. The analytic set is a nonsingular plane quartic if it has only an isolated singular point at the origin in $\mathbb{C}^3$. For the defining equation of an analytic set that has an isolated singular point at the origin in $\mathbb{C}^3$, we have the following theorem[2].

Theorem 1 *Let $f(z_0, z_1, z_2)$ be a polynomial in $\mathbb{C}^3$, and let V be an analytic set such that*

$$V = \{(z_0, z_1, z_2) \mid f(z_0, z_1, z_2) = 0\},$$

which has an isolated singular point at the origin in $\mathbb{C}^3$. Then, for any i $(i = 0, 1, 2)$, there exists an integer $a_i \geqslant 1$ such that f has a term $z_i^{a_i} z_j$.

We present a procedure to obtain the normal forms of smooth plane quadratics in a two-dimensional complex projective space and their restrictions.

THE NORMAL FORMS

We calculate the restriction on smooth plane quartics. Our purpose is to perform the calculation effectively. Hence we expand the procedure of quadratic form in linear algebra.

In general, a plane quartic with the coordinates $[x, y, z]$ has the following defining equation:

$$a_1x^4 + (a_2y + a_3z)x^3 + (a_4y^2 + a_5yz + a_6z^2)x^2 + (a_7y^3 + a_8y^2z + a_9yz^2 + a_{10}z^3)x + a_{11}y^4 + a_{12}y^3z + a_{13}y^2z^2 + a_{14}yz^3 + a_{15}z^4 = 0.$$

By a suitable transformation of coordinates, we obtain the following defining equation:

$$a_1x^4 + (a_2y + a_3z)x^3 + (a_4y^2 + a_5yz + a_6z^2)x^2 + (a_7y^3 + a_8y^2z + a_9yz^2 + a_{10}z^3)x + a_{11}y^4 + a_{12}y^3z + a_{13}y^2z^2 + a_{14}yz^3 + z^4 = 0.$$

We now show a process for obtaining the normal forms of the smooth plane quartics, as follows.

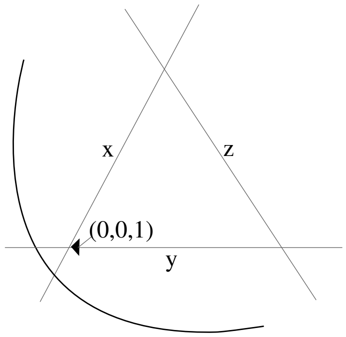

Fig. 1 V_4 does not intercept (0,0,1).

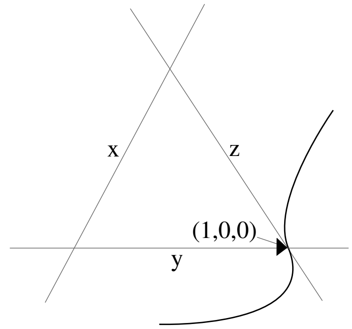

Fig. 2 V_4 intercepts $(1,0,0)$, and $(1,0,0)$ is not a singular point.

Step 1

If V_4 does not intercept (0,0,1) (as in Fig. 1), we can transform the coordinates in such a way that V_4 intercepts $(1,0,0)$. If we replace z by $z'+\lambda x$, where λ is a solution of $\lambda^4+a_{10}\lambda^3+a_6\lambda^2+a_3\lambda+a_1=0$, we reduce the defining equation to

$$\begin{aligned}(g_2y+g_3z)x^3&+(g_4y^2+g_5yz+g_6z^2)x^2\\ &+(g_7y^3+g_8y^2z+g_9yz^2+g_{10}z^3)x\\ &+a_{11}y^4+a_{12}y^3z+a_{13}y^2z^2+a_{14}yz^3+z^4=0.\end{aligned}$$

If $g_3\neq 0$, go to Step 2; if $g_3=0$ and $g_2\neq 0$, go to Step 3; if $g_3=g_2=0$, go to Step 4.

Step 2

V_4 intercepts $(1,0,0)$, and $(1,0,0)$ is not a singular point of V_4 (Fig. 2). We can replace the coordinates so that the multiplicity of V_4 and the y-axis at $(1,0,0)$ is greater than or equal to two. By a suitable rescaling of the x, y, z coordinates, we can reduce the defining equation to

$$\begin{aligned}(a_1y+z)x^3&+(a_2y^2+a_3yz+a_4z^2)x^2\\ &+(a_5y^3+a_6y^2z+a_7yz^2+a_8z^3)x+a_9y^4\\ &+a_{10}y^3z+a_{11}y^2z^2+a_{12}yz^3+a_{13}z^4=0.\end{aligned}$$

If we replace z by $z'-a_1y$, we can reduce the defining equation to

$$\begin{aligned}x^3z&+(g_4y^2+g_5yz+a_4z^2)x^2\\ &+(g_7y^3+g_8y^2z+g_9yz^2+a_{10}z^3)x+g_{11}y^4\\ &+g_{12}y^3z+g_{13}y^2z^2+g_{14}yz^3+g_{14}z^4=0.\end{aligned}$$

If $g_4\neq 0$, go to Step 5; if $g_4=0$ and $g_7\neq 0$, go to Step 6; if $g_4=g_7=0$ and $g_{11}\neq 0$, go to Step 7; if $g_4=g_7=g_{11}=0$, go to Step 8.

Step 3

We exchange the y-axis with the z-axis (Fig. 3). By a suitable rescaling of the x, y, z coordinates, we can reduce the defining equation to

$$\begin{aligned}x^3y&+(a_1y^2+a_2yz+a_3z^2)x^2\\ &+(a_4y^3+a_5y^2z+a_6yz^2+a_7z^3)x+a_8y^4\\ &+a_9y^3z+a_{10}y^2z^2+a_{11}yz^3+a_{12}z^4=0.\end{aligned}$$

By transforming the coordinates as

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix},$$

we can reduce the defining equation to

$$\begin{aligned}x^3z&+(a_3y^2+a_2yz+a_1z^2)x^2\\ &+(a_7y^3+a_6y^2z+a_5yz^2+a_4z^3)x+a_{12}y^4\\ &+a_{11}y^3z+a_{10}y^2z^2+a_9yz^3+a_8z^4=0.\end{aligned}$$

If $a_3\neq 0$, go to Step 5; if $a_3=0$ and $a_7\neq 0$, go to Step 6; if $a_3=a_7=0$ and $a_{12}\neq 0$, go to Step 7; if $a_3=a_7=a_{12}=0$, go to Step 8.

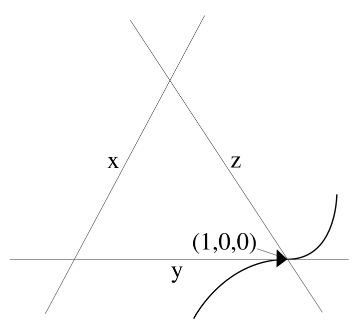

Fig. 3 Exchange the y-axis with the z-axis.

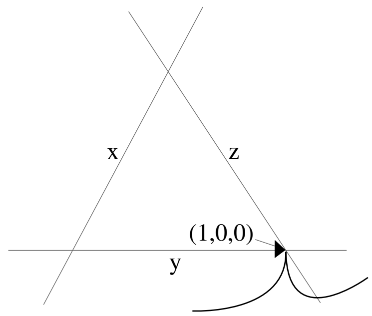

Fig. 4 V_4 has a singular point at $(1,0,0)$.

Step 4

In this case, the defining equation is as follows:

$$f_2(y,z)\,x^2 + f_3(y,z)\,x + f_4(y,z) = 0,$$

where f_i denotes a homogeneous polynomial in y, z of degree i $(2 \leqslant i \leqslant 4)$. By Theorem 1, the set defined by the above equation has a singular point at $(1,0,0)$ in $\mathbb{P}^2$ (the curve defined by the above equation is singular in $\mathbb{P}^2$, Fig. 4).

Step 5

We can transform the coordinates in such a way that the multiplicity of V_4 and the y-axis is equal to two at $(1,0,0)$ (Fig. 5). We reduce the defining equation to

$$\begin{aligned} &x^3z + (y^2 + a_1yz + a_2z^2)x^2 \\ &\quad + (a_3y^3 + a_4y^2z + a_5yz^2 + a_6z^3)x + a_7y^4 \\ &\quad + a_8y^3z + a_9y^2z^2 + a_{10}yz^3 + a_{11}z^4 = 0. \end{aligned} \tag{1}$$

Here, we try to eliminate the monomial x^2y^2. To begin with, we simplify the defining equation. By transforming the coordinates as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix},$$

we can reduce the defining equation to

$$\begin{aligned} &x^3z + (y^2 + g_1yz + g_2z^2)x^2 + g_7y^4 \\ &\quad + (g_3y^3 + g_4y^2z + g_5yz^2 + g_6z^3)x \\ &\quad + g_8y^3z + g_9y^2z^2 + g_{10}yz^3 + g_{11}z^4 = 0. \end{aligned}$$

If we assume

$$\alpha = -\frac{a_3}{2},\ \beta = \frac{4a_1^2 - 16a_2 - 9a_3^2}{48},\ \gamma = \frac{-2a_1 + 3a_3}{4},$$

then we can reduce the defining equation to

$$\begin{aligned} &x^3z + y^2x^2 + (h_1y^2z + h_2yz^2 + h_3z^3)x + h_4y^4 \\ &\quad + h_5y^3z + h_6y^2z^2 + h_7yz^3 + h_8z^4 = 0. \end{aligned}$$

Here, if $h_4 = h_5 = 0$, the curve defined by this equation has a singular point in $\mathbb{P}^2$ (Theorem 1), and we wish to consider the condition for a nonsingular curve in $\mathbb{P}^2$. Hence to simplify, we will consider the following two assumptions.

Assumption A1: $h_4 \neq 0$.
Assumption A2: $h_4 = 0$ and $h_5 \neq 0$.

We first consider the situation under Assumption A1. By a suitable rescaling of the y-coordinate, we can reset the defining equation, as follows:

$$\begin{aligned} &x^3z + y^2x^2 + (a_1y^2z + a_2yz^2 + a_3z^3)x \\ &\quad + y^4 + a_4y^3z + a_5y^2z^2 + a_6yz^3 + a_7z^4 = 0. \end{aligned}$$

By transforming the coordinates as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & \beta & \alpha \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix},$$

we can reduce the defining equation to

$$\begin{aligned} &g_1x^4 + (g_2y + g_3z)x^3 + (g_4y^2 + g_5yz + g_6z^2)x^2 \\ &\quad + (g_7y^3 + g_8y^2z + g_9yz^2 + z^3)x \\ &\quad + g_{11}y^4 + g_{12}y^3z + y^2z^2 = 0, \end{aligned} \tag{2}$$

where

$$\begin{aligned} g_1 &= \gamma^4 + a_4\gamma^3 + a_5\gamma^2 + a_6\gamma + a_7 + \alpha^3 \\ &\quad + \alpha^2\gamma^2 + a_1\alpha\gamma^2 + a_2\alpha\gamma + a_3\alpha, \end{aligned}$$

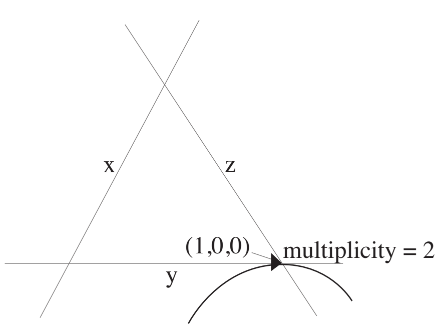

Fig. 5 The multiplicity of V_4 and the y-axis is two.

$$\begin{aligned} g_2 &= a_6 + 2a_1\alpha\gamma + a_1\beta\gamma^2 + a_2\alpha + a_2\beta\gamma \\ &\quad + a_3\beta + 4\gamma^3 + 3a_4\gamma^2 + 2\gamma + 3\alpha^2\beta \\ &\quad + 2\alpha^2\gamma + 2\alpha\beta\gamma, \\ g_4 &= a_1\alpha + 2a_1\beta\gamma + a_2\beta + 6\beta^2 + 3a_4\beta \\ &\quad + a_5 + \alpha^2 + 3\alpha\beta^2 + 4\alpha\beta\gamma + \beta^2\gamma^2. \end{aligned} \tag{3}$$

Note that g_1 is a polynomial in two variables (α, γ), g_2 and g_4 are polynomials in three variables (α, β, γ), and we have

$$\begin{aligned} g_4 &= s_2\beta^2 + s_1\beta + s_0, \\ g_2 &= t_1\beta + t_0, \\ s_2 &= 3\alpha + \gamma^2, \\ s_1 &= 4\alpha\gamma + 2a_1\gamma + a_2, \\ s_0 &= \alpha^2 + a_1\alpha + 6\gamma^2 + 3a_4\gamma + a_5, \\ t_1 &= 3\alpha^2 + 2\alpha\gamma^2 + a_1\gamma^2 + a_2\gamma + a_3, \\ t_0 &= 2\alpha^2\gamma + 2a_1\alpha\gamma + a_2\alpha + 4\gamma^3 \\ &\quad + 3a_4\gamma^2 + 2a_5\gamma + a_6. \end{aligned}$$

By Sylvester's elimination method, we can eliminate β from g_2 and g_4, and we obtain the resultant, which we denote as $R_1(g_2, g_4)$. Then, $R_1(g_2, g_4)$ is a sixth-degree polynomial in α:

$$R_1(g_2, g_4) = 9\alpha^6 + \sum_{i=0}^{5} k_i\alpha^i,$$

where k_i is a polynomial in $a_1, a_2, \dots, a_7, \gamma$. We eliminate α for $s_2 = 0$ and $t_1 = 0$, and obtain the resultant. Similarly, we eliminate γ for $s_2 = 0$ and $t_1 = 0$, and obtain the resultant. These are as follows:

$$\begin{aligned} \gamma^4 - 3a_1\gamma^2 - 3a_2\gamma - 3a_3 &= 0, \\ 9\alpha^4 + 18a_1\alpha^3 + 9a_1^2\alpha^2 - 6a_3\alpha^2 & \\ + 3a_2^2\alpha - 6a_1a_3 + a_3^2 &= 0. \end{aligned} \tag{4}$$

We eliminate α for g_1 and $R_1(g_2, g_4)$, and obtain the resultant, which we denote as $R_2(g_1, R_1(g_2, g_4))$. Then, $R_2(g_1, R_1(g_2, g_4))$ is a polynomial in the variable γ:

$$R_2(g_1, R_1(g_2, g_4)) = \sum_{i=0}^{24} k_i\gamma^i,$$

where k_i is a polynomial in $a_1, a_2, \dots, a_7, \beta$.

Let γ_j ($j = 1, 2, \dots$) be a solution of $R_2(g_1, R_1(g_2, g_4)) = 0$. If a solution other than the system of algebraic equations (3) exists, then we find a solution α such that $g_1 = R_1(g_2, g_4) = 0$ for the γ_j. We consider the following two assumptions.

Assumption B1: a solution of $R_1(g_2, g_4) = 0$ other than the system of algebraic equations (4) exists.

Assumption B2: there is no solution of $R_1(g_2, g_4) = 0$ other than the system of algebraic equations (4).

We first consider the case of Assumption B1. We calculate the Gröbner basis for the polynomials $k_{24}, k_{23}, \dots, k_1$ in graded reverse lexicographic order. We obtain 100 polynomials as the Gröbner basis. Let $p_1, p_2, \dots, p_{100}$ be the Gröbner basis, and let gs_1 be the list $\{p_1, p_2, \dots, p_{100}\}$.

Next, we calculate the Gröbner basis for the polynomials $k_{24}, k_{23}, \dots, k_0$ in graded reverse lexicographic order. We obtain 100 polynomials as the Gröbner basis. Let $p'_1, p'_2, \dots, p'_{100}$ be the Gröbner basis, and let gs_2 be the list $\{p'_1, p'_2, \dots, p'_{100}\}$. Then $gs_1 = gs_2$. Hence, by $R_2(g_1, R_1(g_2, g_4)) = 0$, we have a solution (β, γ) of the system of algebraic equations (3) for any values of $a_1, a_2, \dots, a_7$, under Assumption A1 and Assumption B1.

We consider the case of Assumption A2. By a suitable rescaling of the y coordinate, we can rewrite the defining equation as follows:

$$\begin{aligned} x^3z + y^2x^2 + (a_1y^2z + a_2yz^2 + a_3z^3)x & \\ + y^3z + a_4y^2z^2 + a_5yz^3 + a_6z^4 &= 0. \end{aligned}$$

Calculating as in the previous case, we obtain $R_2(g_1, R_1(g_2, g_4))$ as follows:

$$R_2(g_1, R_1(g_2, g_4)) = \gamma^{24} + \sum_{i=0}^{23} k_i\gamma^i,$$

where k_i is a polynomial in $a_1, a_2, \dots, a_6, \alpha$. Hence because $R_2(g_1, R_1(g_2, g_4)) = 0$, we can obtain a solution (β, γ) of the system of algebraic equations (3) for any values of $a_1, a_2, \dots, a_6$ under Assumption A2 and Assumption B1. Thus we have shown the existence of a solution (β, γ) of the system of algebraic equations (3) under Assumption B1. Next, we show the existence of a solution (α, β) under Assumption B1. $R_1(g_2, g_4)$ is an eighth-degree polynomial in γ:

$$R_1(g_2, g_4) = 16\gamma^8 + \sum_{i=0}^{7} k_i\gamma^i,$$

where k_i is a polynomial in $a_1, a_2, \dots, a_6, \alpha$. When we eliminate γ for g_1 and $R_1(g_2, g_4)$, we obtain

$$R_2(g_1, R_1(g_2, g_4)) = \sum_{i=0}^{24} k_i\alpha^i,$$

where k_i is a polynomial in $a_1, a_2, \dots, a_7$.

We calculate the Gröbner basis for the polynomials $k_{24}, k_{23}, \dots, k_1$ in graded reverse lexicographic order, and we obtain 109 polynomials as the Gröbner basis. Let $p_1, p_2, \dots, p_{109}$ be the Gröbner basis, and let gs_1 be the list $\{p_1, p_2, \dots, p_{109}\}$. We calculate the Gröbner basis for the polynomials $k_{24}, k_{23}, \dots, k_0$ in graded reverse lexicographic order, we obtain 109 polynomials as the Gröbner basis. Let $p'_1, p'_2, \dots, p'_{109}$ be the Gröbner basis, and let gs_2 be the list $\{p'_1, p'_2, \dots, p'_{109}\}$. Then $gs_1 = gs_2$. Hence, because $R_2(g_1, R_1(g_2, g_4)) = 0$, we have a solution (α, β) of the system of algebraic equations (3) for any values of $a_1, a_2, \dots, a_7$ under Assumption A1 and Assumption B1.

We now consider the case of Assumption A2. Here, $R_2(g_1, R_1(g_2, g_4))$ is

$$R_2(g_1, R_1(g_2, g_4)) = \alpha^{24} + \sum_{i=0}^{23} k_i \alpha^i,$$

where k_i is a polynomial in $a_1, a_2, \dots, a_6, \gamma$. Hence because $R_2(g_1, R_1(g_2, g_4)) = 0$, we have a solution (α, β) of the system of algebraic equations (3) for any values of $a_1, a_2, \dots, a_6$ under Assumption A2 and Assumption B1. With an extension of the theorem[3], we have a solution (α, β, γ) of the system of algebraic equations (3) for any values of $a_1, a_2, \dots, a_7$ under Assumption B1. We next consider the case under Assumption B2. (Here, we do not require Assumption A1 and Assumption A2.) We take the defining equation to be

$$\begin{aligned} x^3z + x^2y^2 + (a_1y^2z + a_2yz^2 + a_3z^3)x \\ + a_4y^4 + a_5y^3z + a_6y^2z^2 + a_7yz^3 + a_8z^4 = 0. \end{aligned}$$

By transforming the coordinates as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & \beta & \alpha \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix},$$

we obtain (2) and

$$\begin{aligned} g_1 &= a_4\gamma^4 + a_5\gamma^3 + a_6\gamma^2 + a_7\gamma + a_8 + \alpha^3 \\ &\quad + \alpha^2\gamma^2 + a_1\alpha\gamma^2 + a_2\alpha\gamma + a_3\alpha, \\ g_2 &= a_7 + 2a_1\alpha\gamma + a_1\beta\gamma^2 + a_2\alpha + a_2\beta\gamma \\ &\quad + a_3\beta + 4a_4\gamma^3 + 3a_5\gamma^2 + 2a_4\gamma + 3\alpha^2\beta \\ &\quad + 2\alpha^2\gamma + 2\alpha\beta\gamma, \\ g_4 &= a_1\alpha + 2a_1\beta\gamma + a_2\beta + 6a_4\beta^2 + 3a_5\beta \\ &\quad + a_6 + \alpha^2 + 3\alpha\beta^2 + 4\alpha\beta\gamma + \beta^2\gamma^2. \end{aligned}$$

From (4), $\alpha = -\frac{1}{3}\gamma^2$, and $\gamma^4 - 3a_1\gamma^2 - 3a_2\gamma - 3a_3 = 0$. Then $g_1 = \frac{2}{27}\gamma^6 + (a_4 - \frac{1}{3}a_1)\gamma^4 + (a_5 - \frac{1}{3}a_2)\gamma^3 + (a_6 - \frac{1}{3}a_3)\gamma^2 + a_7\gamma + a_8$, and

$$R_1(g_2, g_4) = -\tfrac{7}{81}\gamma^{12} + \sum_{i=0}^{11} k_i \gamma^i,$$

where k_i is a polynomial in $a_1, a_2, \dots, a_7, \alpha$.

Note that $R_1(g_2, g_4)$ does not include a_8. Hence we can let

$$\begin{aligned} a_8 = -\tfrac{2}{27}\gamma^6 - \left(a_4 - \frac{a_1}{3}\right)\gamma^4 \\ - \left(a_5 - \frac{a_2}{3}\right)\gamma^3 - \left(a_6 - \frac{a_3}{3}\right)\gamma^2 - a_7\gamma. \end{aligned} \quad (5)$$

Then,

$$\begin{aligned} R_1(g_2, g_4) &= -\frac{1}{81}\big(Q(\gamma)(\gamma^4 - 3a_1\gamma^2 - 3a_2\gamma - 3a_3) \\ &\quad + r(\gamma)\big)(\gamma^4 - 3a_1\gamma^2 - 3a_2\gamma - 3a_3), \\ Q(\gamma) &= 7\gamma^4 + \sum_{i=0}^{2} h_i\gamma^i, \\ r(\gamma) &= \sum_{i=0}^{3} h'_i\gamma^i, \end{aligned}$$

where $Q(\gamma)$ and $r(\gamma)$ are polynomials in $\gamma, a_1, a_2, \dots, a_7, \alpha$, and h_i and h'_i are polynomials in $a_1, a_2, \dots, a_7, \alpha$.

Under Assumption B2, $h_2 = h_1 = h_0 = -21$, and $h'_3 = h'_2 = h'_1 = h'_0 = 0$. Then, $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 0$, and from (5), $a_8 = 0$. A solution to (2) exists for $g_1 = g_2 = g_4 = 0$ ($\alpha = \beta = \gamma = 0$). As a result, in this case, the curve defined by (2) is singular in $\mathbb{P}^2$. Hence we can eliminate the monomial x^2y^2 in the defining equation (1). If we return to (2) and take $g_1 = g_2 = g_4 = 0$, then we can reduce the defining equation to

$$\begin{aligned} h_3x^3z + (h_5y + h_6z)x^2z \\ + (h_7y^3 + h_8y^2z + h_9yz^2 + z^3)x \\ + (h_{11}y^2 + h_{12}yz + z^2)y^2 = 0. \end{aligned}$$

If $h_3 = 0$, go to Step 4; if $h_3 \neq 0$ and $h_7 \neq 0$, go to Step 6; if $h_3 \neq 0$, $h_7 = 0$, and $h_{11} \neq 0$, go to Step 7; if $h_3 \neq 0$ and $h_7 = h_{11} = 0$, go to Step 8.

Step 6

We can transform the coordinates in such a way that the multiplicity of V_4 and the y'-axis is equal to three

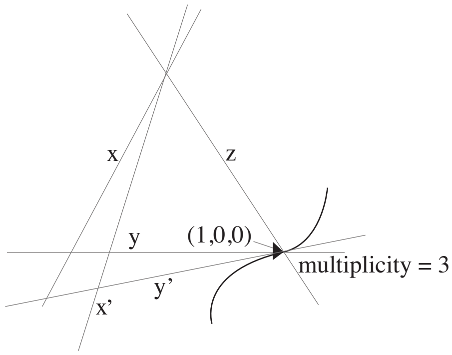

Fig. 6 The multiplicity of V_4 and the y'-axis is three.

at $(1,0,0)$ (Fig. 6). We can then reduce the defining equation to

$$x^3z+(a_1y+a_2z)x^2z+(y^3+a_3y^2z+a_4yz^2+a_5z^3)x+a_6y^4+a_7y^3z+a_8y^2z^2+a_9yz^3+a_{10}z^4=0.$$

If we transform the coordinates as

$$\begin{pmatrix} x\\ y\\ z\end{pmatrix}=\begin{pmatrix} 1 & \alpha & \beta\\ 0 & 1 & \gamma\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} x'\\ y'\\ z'\end{pmatrix},$$

and we let

$$\alpha=-\frac{a_1}{3},\ \beta=\frac{-a_1^3+3a_1a_3-9a_2}{27},\ \gamma=\frac{a_1^2-3a_3}{9},$$

then the defining equation is

$$x^3z+(y^3+h_1yz^2+h_2z^3)x+h_3y^4+h_4y^3z+h_5y^2z^2+h_6yz^3+h_7z^4=0.$$

From Theorem 1, if $(h_2,h_6,h_7)=(0,0,0)$, the curve defined by this equation is singular in $\mathbb{P}^2$. Furthermore, if $(h_3,h_4,h_5,h_6,h_7)=(0,0,0,0,0)$, then the curve defined by this equation is reducible (and singular) in $\mathbb{P}^2$. If $(h_2,h_6,h_7)\neq(0,0,0)$ and $(h_3,h_4,h_5,h_6,h_7)\neq(0,0,0,0,0)$, go to Step 9.

Step 7

We can transform the coordinates in such a way that the multiplicity of V_4 and the y'-axis is equal to four at $(1,0,0)$ (Fig. 7). We can then reduce the defining equation to

$$x^3z+(a_1y+a_2z)x^2z+(a_3y^2+a_4yz+a_5z^2)xz+y^4+a_6y^3z+a_7y^2z^2+a_8yz^3+a_9z^4=0.$$

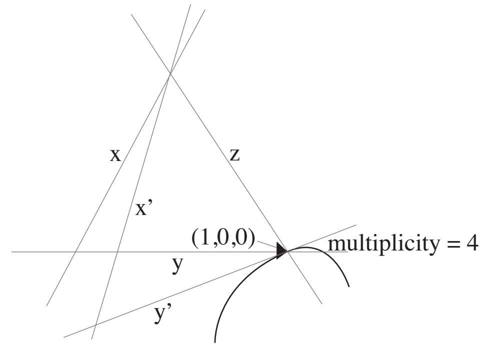

Fig. 7 The multiplicity of V_4 and the y'-axis is four.

If we transform the coordinates as

$$\begin{pmatrix} x\\ y\\ z\end{pmatrix}=\begin{pmatrix} 1 & \alpha & \beta\\ 0 & 1 & \gamma\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} x'\\ y'\\ z'\end{pmatrix},$$

and we let

$$\alpha=-\frac{a_1}{3},\ \beta=\frac{2a_1^4-9a_1^2a_3+27a_1a_6-108a_2}{324},\ \gamma=\frac{-2a_1^3+9a_1a_3-27a_6}{108},$$

then the defining equation is

$$x^3z+(h_1y^2+h_2yz+h_3z^2)xz+y^4+h_4y^2z^2+h_5yz^3+h_6z^4=0.$$

Because of Theorem 1, if $(h_3,h_5,h_6)=(0,0,0)$, the curve defined by this equation is singular in $\mathbb{P}^2$. If $(h_3,h_5,h_6)\neq(0,0,0)$, go to Step 10.

Step 8

We can reduce the defining equation to

$$z\big(x^3+(a_1y+a_2z)x^2+(a_3y^2+a_4yz+a_5z^2)x+a_6y^3+a_7y^2z+a_8yz^2+a_9z^3\big)=0.$$

The curve defined by this equation is reducible in $\mathbb{P}^2$ (Fig. 8).

Step 9

We can transform the coordinates in such a way that the multiplicity of V_4 and the y-axis is equal to three at $(1,0,0)$ (Fig. 9). We can then reduce the defining equation to

$$x^3z+(y^3+a_1yz^2+a_2z^3)x+a_3y^4+a_4y^3z+a_5y^2z^2+a_6yz^3+a_7z^4=0.$$

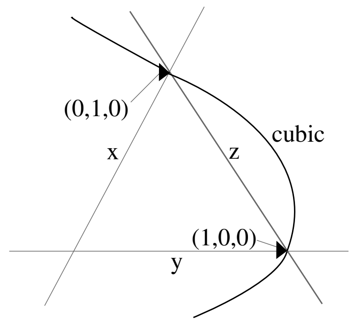

Fig. 8 V_4 is a reducible curve.

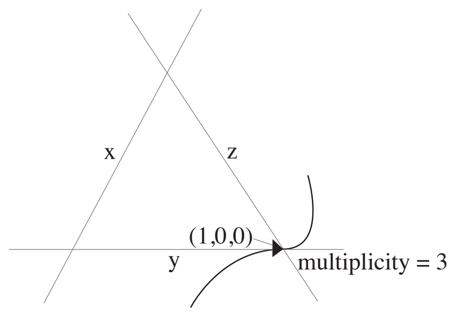

Fig. 9 The multiplicity of the intersection of V_4 and the y-axis is equal to three.

Step 10

We can transform the coordinates in such a way that the multiplicity of V_4 and the y-axis is equal to four at $(1,0,0)$ (Fig. 10). We can then reduce the defining equation to

$$x^3z+(a_1y^2+a_2yz+a_3z^2)xz \\ +y^4+a_4y^2z^2+a_5yz^3+a_6z^4=0.$$

Using the process defined above, we can rewrite the defining equation of any smooth plane quartic as a normal form. As a result, the defining equation of a smooth plane quartic becomes either the one given in Step 9 or the one given in Step 10.

Note that the above process is a weaker version of Propositions 1 and 2 by Shioda[4], and the results of Step 9 and Step 10 are in accord with those results.

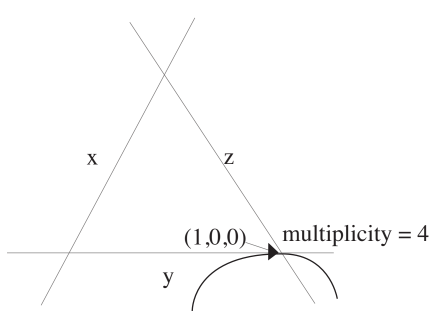

Fig. 10 The multiplicity of the intersection of V_4 and the y-axis is four.

RESTRICTIONS

We now consider the restrictions on a defining equation that results in a nonsingular curve. For example, for this nonsingular elliptic curve in the Weierstrass normal form, $yz^2=x^3+pxz^2+qz^3$, the restriction on the defining equation is $4p^3+27q^2\neq 0$.

In the case of cubics, the structure of the various moduli of smooth plane quartics is greatly complicated.

The calculation of the elimination ideal is a key to a better understanding of the various phenomena, and it is also helps us to understand the variety of modes of the defining equation.

For the restriction on the Type I smooth plane quartic, we consider the following defining equation:

$$x^3z+(y^3+a_1yz^2+a_2z^3)x+a_3y^4+a_4y^3z \\ +a_5y^2z^2+a_6yz^3+a_7z^4=0.$$

If we transform the coordinates as

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&0&\alpha\\0&1&\beta\\0&0&1\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix},$$

then the defining equation is as follows (note that we replaced x', y', z' by x, y, z, respectively, following the transformation):

$$x^3z+g_1x^2z^2+(y^3+g_2y^2z+g_3yz^2+g_4z^3)x \\ +g_5y^4+g_6y^3z+g_7y^2z^2+g_8yz^3+g_9z^4=0,$$

where

$$\begin{aligned} g_4&=a_1\alpha+a_2+3\alpha^2+\beta^3,\\ g_8&=a_1\alpha+4a_3\beta^3+2a_5\beta+a_6+3\alpha\beta^2,\\ g_9&=a_1\alpha\beta+a_2\alpha+a_3\beta^4+a_4\beta^3+a_5\beta^2\\ &\quad+a_6\beta+a_7+\alpha^3+\alpha\beta^3. \end{aligned} \tag{6}$$

If $(g_4,g_8,g_9)=(0,0,0)$, then the curve has a singular point at (0,0,1) in $\mathbb{P}^2$. We consider the existence of a solution for the system of algebraic equations (6), and we use the elimination theorem.

Let I be the polynomial ideal $\langle g_4, g_8, g_9\rangle$, and let G be a Gröbner basis of I with respect to a lexicographic order such that $\alpha > \beta > a_1 > a_2 > a_3 > a_4 > a_5 > a_6 > a_7$. Then the curve has a singular point at (0,0,1) if and only if the Gröbner basis of $G \cap \mathbb{C}[a_1, a_2, a_3, a_4, a_5, a_6, a_7]$ is equal to 0. We take their second elimination ideal, as follows:

$$G_2 = G \cap \mathbb{C}[a_1, a_2, a_3, a_4, a_5, a_6, a_7].$$

G_2 is a Gröbner basis of I_2, and it consists of one polynomial in $a_1, a_2, \dots, a_7$.

We now show the existence of the solution (α, β) in the system of algebraic equations (6) for any value of the parametric coefficients. We take their first elimination ideals as follows:

$$G_1(\alpha) = G \cap \mathbb{C}[\alpha, a_1, a_2, a_3, a_4, a_5, a_6, a_7],$$
$$G_1(\beta) = G \cap \mathbb{C}[\beta, a_1, a_2, a_3, a_4, a_5, a_6, a_7].$$

$G_1(\alpha)$ consists of 78 polynomials in α, a_1, a_2, a_3, a_4, a_5, a_6, a_7. Here,

$$3\alpha^7 + \sum_{i=0}^{6} k_i \alpha^i \in G_1(\alpha),$$

where k_i is a polynomial in $a_1, a_2, \dots, a_7$. $G_1(\beta)$ consists of 248 polynomials in β, a_1, a_2, a_3, a_4, a_5, a_6, a_7. Here,

$$9\beta^7 + \sum_{i=0}^{5} k'_i \beta^i \in G_1(\beta),$$

where k'_i is a polynomial in $a_1, a_2, \dots, a_7$. Hence, the solution (α, β) in the system of algebraic equations (6) exists for any value of the parametric coefficients. The above Gröbner basis (one polynomial) is the restriction of Type I smooth plane quartic[5]. The dimension of this variety is equal to six.

Next, we consider the following defining equation:

$$x^3 z + (a_1 y^2 + a_2 yz + a_3 z^2)xz + y^4 + a_4 y^2 z^2 + a_5 yz^3 + a_6 z^4 = 0.$$

If we transform the coordinates as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix},$$

then the defining equation is as follows (note that we replaced x', y', z' by x, y, z, respectively, following the transformation):

$$x^3 z + g'_1 x^2 z^2 + (g'_2 y^2 + g'_3 yz + g'_4 z^2)xz + y^4 + g'_5 y^3 z + g'_6 y^2 z^2 + g'_7 yz^3 + g'_8 z^4 = 0,$$

where

$$\begin{aligned} g'_4 &= a_1\beta^2 + a_2\beta + a_3 + 3\alpha^2, \\ g'_7 &= 2a_1\alpha\beta + a_2\alpha + 2a_4\beta + a_5 + 4\beta^3, \\ g'_8 &= a_1\alpha\beta^2 + a_2\alpha\beta + a_3\alpha + a_4\beta^2 + a_5\beta \\ &\quad + a_6 + \alpha^3 + \beta^4. \end{aligned} \tag{7}$$

If $(g'_4, g'_7, g'_8) = (0, 0, 0)$, then the curve defined by the equation of Type II has a singular point at (0,0,1) in $\mathbb{P}^2$. Finally, we consider the existence of a solution for the system of algebraic equations (7), and we use the elimination theorem.

In a similar way to what we did above, we can obtain the restriction on Type II smooth plane quartic[6]. The dimension of this is equal to five.

OBSERVATION

For a smooth plane quartic, it is well known that the curve has at least one point of inflexion. If we transform this point to $(1, 0, 0)$, the tangent to the inflection point at $z = 0$ leads to the Type I normal form, or, if the curve is hyperflexed, it leads to the Type II normal form. Thus we have shown the configurations and their restrictions for flexes and hyperflexes in smooth plane quartics.

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