

Neighbour isolated scattering number of graphs

Ersin Aslan

Turgutlu Vocational School, Celal Bayar University, Manisa, Turkey

e-mail: ersin.aslan@cbu.edu.tr

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ABSTRACT: The scattering number is a measure of the vulnerability of a graph. In this paper we investigate a refinement that involves the neighbour isolated version of the parameter. The neighbour isolated scattering number of a noncomplete graph G is defined to be $NIS(G) = \max\{i(G/X) - |X| : i(G/X) \geq 1\}$ where the maximum is taken over all X , the cut strategy of G , and $i(G/X)$ is the number of components which are isolated vertices of G/X . Like the scattering number itself, this is a measure of the vulnerability of a graph, but it is more sensitive. The relations between neighbour isolated scattering number and other parameters are determined and the neighbour isolated scattering number of trees and other families are obtained. We also give some results for the neighbour isolated scattering number of the graphs obtained by some graph operations.

KEYWORDS: graph theory, vulnerability, neighbour connectivity, neighbour scattering number

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INTRODUCTION

The problem of quantifying the vulnerability of graphs has received much attention recently, especially in the fields of computers, communication, and spy networks. In a network, the vulnerability parameters measure the resistance of the network to disruption of operation after the failure of certain stations or links. Parameters used to measure the vulnerability include connectivity, integrity¹, scattering number², isolated scattering number³, toughness⁴, tenacity⁵, and rupture degree⁶. However, most of these parameters do not consider the neighbourhoods of the effected vertices. On the other hand, in spy networks, if a spy or a station is captured, then adjacent stations are unreliable. Therefore neighbourhoods should be taken into consideration in spy networks. Nevertheless, there are very few parameters concerning neighbourhoods such as vertex neighbour connectivity⁷, vertex neighbour integrity⁸, and vertex neighbour scattering number⁹.

The scattering number of an incomplete connected graph G is denoted and defined² by

$$s(G) = \max\{\omega(G-S) - |S| : S \subset V(G), \omega(G-S) \geq 2\}$$

where $\omega(G-S)$ denotes the number of components in $G-S$. Replacing $\omega(G-S)$ with $i(G-S)$ in the above definition gives³ the isolated scattering

number, $IS(G)$, as a new parameter to measure the vulnerability of a network:

$$IS(G) = \max\{i(G-S) - |S| : S \in C(G)\}$$

where G is a connected graph, $i(G-S)$ is the number of isolated vertices of $G-S$ and $C(G)$ is the set of vertex cuts of G .

The most common vulnerability parameters concerning with spy networks are as follows. The vertex neighbour connectivity of a graph G is

$$VNC(G) = \min_{S \subseteq V(G)} \{|S|\}$$

where S is a subversion strategy of G ⁷. The vertex neighbour integrity of a graph G is defined to be

$$VNI(G) = \min_{S \subseteq V(G)} \{|S| + m(G/S)\}$$

where S is any vertex subversion strategy of G and $m(G/S)$ is the order of the largest connected component of G/S ⁸. The vertex neighbour scattering number of a graph G is defined as

$$VNS(G) = \max\{\omega(G/X) - |X| : X \text{ is a cut strategy of } G, \omega(G/X) \geq 1\},$$

where $\omega(G/S)$ denotes the number of connected components in G/X ⁹.

Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. Let $\deg(v)$ denote the

degree of the vertex v in G . The set $N(v) = \{v \in V(G) \mid v \neq u, v \text{ and } u \text{ are adjacent}\}$ is the open neighbourhood of v , and $N[v] = \{v\} \cup N(v)$ is the closed neighbourhood of v . A set of vertices $X \subseteq V(G)$ is called a vertex subversion strategy of G if each of the vertices in X is subverted from G . By G/X we denote the survival subgraph that remains after each vertex of X is subverted from G . A vertex set X is called a cut strategy of G if the survival subgraph G/X is disconnected, or is a clique, or is empty.

The known parameters concerning the neighbourhoods do not deal with the number of the removed vertices and the number of the components which are isolated vertices in a disrupted network simultaneously. In order to fill this void in the literature, the current study proposes a definition of neighbour isolated scattering number which is a new parameter concerning these two values. Motivated from the concept of the isolated scattering number and neighbour scattering number, it is natural for us to replace $\omega(G/X)$ with $i(G/X)$ in the above definition.

The neighbour isolated scattering number of a noncomplete graph G is defined to be

$$\text{NIS}(G) = \max\{i(G/X) - |X| : i(G/X) \geq 1\}$$

where the maximum is taken over all X , the cut strategy of G , and $i(G/X)$ is the number of components which are isolated vertices of G/X . A set $X \subset V(G)$ is said to be the NIS-set of G if $\text{NIS}(G) = i(G/X) - |X|$. For the complete graph, subverting any one vertex will betray the entire graph, so we define $\text{NIS}(K_n) = -1$.

The following examples show that the neighbour isolated scattering number is better than the vertex neighbour connectivity and the vertex neighbour integrity in measuring the vulnerability of graphs in some situations.

Example 1 It can be easily seen that the vertex neighbour connectivity of a path P_9 and a comet $C_{6,6}$ are equal: $\text{VNC}(P_9) = \text{VNC}(C_{6,6}) = 1$.

Example 2 It can be easily seen that the vertex neighbour integrity of a cycle C_8 and a comet $C_{6,6}$ are equal: $\text{VNI}(C_8) = \text{VNI}(C_{6,6}) = 3$. On the other hand, the neighbour isolated scattering numbers of a cycle C_8 , a comet $C_{6,6}$, and a path P_9 are different: $\text{NIS}(P_9) = 1$, $\text{NIS}(C_8) = 0$, and $\text{NIS}(C_{6,6}) = 5$.

One finds that the neighbour isolated scattering number value of a graph is closely related to its

vertex neighbour scattering number. But graphs with the same vertex neighbour scattering number may have different neighbour isolated scattering numbers.

Example 3 It can be easily seen that the vertex neighbour scattering number of a cycle C_9 and a wheel $W_{1,8}$ are equal: $\text{VNS}(C_9) = \text{VNS}(W_{1,8}) = 0$. On the other hand, the neighbour isolated scattering number of a cycle C_9 and a wheel $W_{1,8}$ differ: $\text{NIS}(C_9) = -1$ and $\text{NIS}(W_{1,8}) = 0$.

The definition of the neighbour isolated scattering number shows that the parameter measures not only the amount of work done to damage the network but also how badly the network is damaged. Graphs with a large neighbour isolated scattering number are more vulnerable.

BOUNDS FOR NEIGHBOUR ISOLATED SCATTERING NUMBER

In this section some lower and upper bounds are given for the neighbour isolated scattering number of a graph using different graph parameters.

Theorem 1 Let G be a connected graph of order n . Then $\text{NIS}(G) \geq 2 - n$.

Proof: Let X be a cut strategy of G . We have $|X| \leq |N[X]|$ and $i(G/X) \geq 1$. Then

$$n - |X| \geq n - |N[X]| \geq i(G/X).$$

If we add $i(G/X)$ to both sides, we have

$$i(G/X) + n - |X| \geq 2i(G/X).$$

Then, by the definition of neighbour isolated scattering number, $\text{NIS}(G) \geq 2 - n$. \square

Theorem 2 Let G be a connected graph of order n . Then $\text{NIS}(G) \leq n - 2\text{VNC}(G)$.

Proof: Let X be an NIS-set of G . For any set X of G we have $\text{VNC}(G) \leq |X| \leq |N[X]|$ and $i(G/X) \leq n - |N[X]|$. Thus

$$i(G/X) - |X| \leq n - |N[X]| - |X|.$$

Hence when we take the maximum of both sides, $\text{NIS}(G) \leq n - 2\text{VNC}(G)$. \square

Theorem 3 Let G be a connected graph of order n and $\delta(G)$ be the minimum vertex degree of G . Then

$$\text{NIS}(G) \leq n - \text{VNC}(G)(\delta(G) + 2).$$

Proof: Let X be a NIS-set of G . For any set X of G we know $VNC(G) \leq |X|$ and we have for any vertex, $v \in V(G)$, $|N[v]| \geq \delta(G) + 1$, and so $i(G/X) \leq n - VNC(G)(\delta(G) + 1)$. Thus

$$i(G/X) - |X| \leq n - VNC(G)(\delta(G) + 1) - VNC(G).$$

Hence we get $NIS(G) \leq n - VNC(G)(\delta(G) + 2)$. \square

Remark 1 The upper bound of NIS in Theorem 3 is tight. This can be shown by considering star graphs.

Theorem 4 Let G be a connected graph and $\alpha(G)$ be the independence number of G . Then

$$NIS(G) \leq \alpha(G) - VNC(G).$$

Proof: Let X be a cut strategy of G . For any set X of G , $VNC(G) \leq |X|$ and $i(G/X) \leq \alpha(G)$. Hence we get $NIS(G) \leq \alpha(G) - VNC(G)$. \square

Theorem 5 Let G be a connected graph. Then $NIS(G) \geq 1 - VNC(G)$.

Proof: If X is a cut strategy of G with connectivity $VNC(G) = |X|$ then $i(G/X) \geq 1$. Thus

$$i(G/X) - |X| \geq 1 - VNC(G).$$

Hence we get $NIS(G) \geq 1 - VNC(G)$. \square

Theorem 6 Let G be a connected graph and $\delta(G)$ be the minimum vertex degree of G . Then $NIS(G) \geq 1 - \delta(G)$.

Proof: By Theorem 5 we know $NIS(G) \geq 1 - VNC(G)$. We have $\delta(G) \geq VNC(G)$ and so $\delta(G) \geq VNC(G) \geq 1 - NIS(G)$. \square

Remark 2 The lower bounds of NIS in Theorem 5 and Theorem 6 are tight. This can be shown by considering P_6 or C_7 .

Theorem 7 Let G be a connected graph. Then $NIS(G) \leq VNS(G)$.

Proof: If X is a cut strategy of G , then $i(G/X) \leq \omega(G/X)$. Hence $i(G/X) - |X| \leq \omega(G/X) - |X|$. Thus when we take the maximum of both sides, the proof is completed. \square

Remark 3 The upper bound of NIS in Theorem 7 is tight. This can be shown by considering P_{12} or $W_{1,8}$.

For any graph G the integrity¹ of G is denoted and defined by $I(G) = \min\{|S| + m(G - S) : S \subseteq V(G)\}$ where $m(G - S)$ denotes the maximum order (vertex-cardinality) of a component of graph $G - S$.

Lemma 1 (Ref. 10) For any graph G , $VNC(G) \leq VNI(G)$.

Lemma 2 (Ref. 10) For any graph G , $VNI(G) \leq I(G) - r$ where r is the maximum degree of the subgraph induced by an I-set of G .

The following results can be easily obtained from Theorem 5 and Lemmas 1 and 2.

Theorem 8 For any graph G , $NIS(G) \geq 1 - VNI(G)$.

Theorem 9 For any graph G , $NIS(G) \geq r + 1 - I(G)$, where r is the maximum degree of the subgraph induced by an I-set of G .

NEIGHBOUR ISOLATED SCATTERING NUMBER OF SEVERAL SPECIFIC CLASSES OF GRAPHS

In this section, we consider the neighbour isolated scattering number of trees, path graphs, complete k -ary trees, comet graphs, cycle graphs, complete k -partite graphs, gear graphs, and star graphs.

Theorem 10 Let T be a tree with order n . Then $0 \leq NIS(T) \leq n - 3$.

Proof: Let X be a cut strategy of T . If $|X| = r$ then we have $i(T/X) \geq r$. Thus $i(T/X) - |X| \geq r - r$. Hence

$$NIS(T) \geq 0. \tag{1}$$

For any vertex, $v \in V(T)$, $|N[v]| \geq 2$, so $i(T/X) \leq n - 2$. If T is connected, then for any NIS-set X , $r \geq 1$ and $i(T/X) \leq n - 2$. Hence $i(T/X) - |X| \leq n - 2 - r$. The function $f(r) = n - 2 - r$ is a decreasing function and it takes its maximum value at $r = 1$. Hence we get

$$NIS(T) \leq n - 3. \tag{2}$$

By (1) and (2), the proof is completed. \square

Theorem 11 Let P_n be a path of order $n \geq 3$. Then

$$NIS(P_n) = \begin{cases} 1, & n \equiv 1 \pmod{4}; \\ 0, & n \equiv 0, 2, 3 \pmod{4}. \end{cases}$$

Proof: Let X be a cut strategy of P_n and $|X| = r$. We distinguish two cases.

Case 1. If $n \equiv 1 \pmod{4}$, then $i(P_n/X) \leq r + 1$. Thus $i(P_n/X) - |X| \leq r + 1 - r = 1$. It can be easily seen that there is a cut strategy X^* of P_n such that $|X^*| = \frac{1}{4}(n - 1)$, $i(P_n/X^*) = \frac{1}{4}(n - 1) + 1$ and so

$$NIS(P_n) = 1 \tag{3}$$

where $n \equiv 1 \pmod{4}$.

Case 2. If $n \equiv 0, 2, 3 \pmod{4}$, then $i(P_n/X) \leq r$. Thus $i(P_n/X) - |X| \leq r - r = 0$. It can be easily seen that there is a cut strategy X^* of P_n such that $|X^*| = \lceil \frac{1}{4}n \rceil$, $i(P_n/X^*) = \lceil \frac{1}{4}n \rceil$ and so

$$\text{NIS}(P_n) = 0 \tag{4}$$

where $n \equiv 0, 2, 3 \pmod{4}$. Hence by (3) and (4),

$$\text{NIS}(P_n) = \begin{cases} 1, & n \equiv 1 \pmod{4}; \\ 0, & n \equiv 0, 2, 3 \pmod{4}. \end{cases}$$

□

Theorem 12 Let $T_{k,d}$ be a complete k -ary tree of depth d where $k \geq 2$. Then

$\text{NIS}(T_{k,d})$

$$= \begin{cases} \frac{k^{d+2} + 1}{k^2 + 1}, & d \equiv 0 \pmod{4}, \\ \frac{k^{d+2} - 2k^2 + k - 2}{k^2 + 1}, & d \equiv 1 \pmod{4}, \\ \frac{k^{d+2} - 1}{k^2 + 1}, & d \equiv 2 \pmod{4}, \\ \frac{k^{d+4} - k^{d+2} - k^3 + k}{k^4 - 1}, & d \equiv 3 \pmod{4}. \end{cases}$$

Proof: Let X be a cut strategy of $T_{k,d}$ and let $|X| = r$ be the number of removed vertices. The proof is similar to that of neighbour rupture degree of $T_{k,d}$ ¹¹. There are four cases according to the depth of $T_{k,d}$.

Case 1: $d \equiv 0 \pmod{4}$. (i) If $0 \leq r \leq k(k^d - 1)/(k^4 - 1)$, then $i(T_{k,d}/X) \leq (k^2 + k - 1)r + 1$. Thus $\text{NIS}(T_{k,d}) \leq \max_r \{(k^2 + k - 1)r + 1 - r\} = \max_r \{(k^2 + k - 2)r + 1\}$. The function $f(r) = (k^2 + k - 2)r + 1$ is an increasing function and it takes its maximum value at $r = k(k^d - 1)/(k^4 - 1)$. Then

$\text{NIS}(T_{k,d})$

$$\leq \frac{k^{d+3} + k^{d+2} - 2k^{d+1} + k^4 - k^3 - k^2 + 2k - 1}{k^4 - 1}. \tag{5}$$

(ii) If $k(k^d - 1)/(k^4 - 1) + 1 \leq r \leq k^2(k^d - 1)/(k^4 - 1)$, then $i(T_{k,d}/X) \leq k^2(k^d - 1)/(k^4 - 1) + r(k^2 - 1) + 1$. Then $\text{NIS}(T_{k,d}) \leq \max_r \{k^2(k^d - 1)/(k^4 - 1) + r(k^2 - 1) + 1 - r\} = \max_r \{k^2(k^d - 1)/(k^4 - 1) + r(k^2 - 2) + 1\}$. The function $f(r) = k^2(k^d - 1)/(k^4 - 1) + r(k^2 - 2) + 1$ is an increasing function and it takes its maximum value at $r = k^2(k^d - 1)/(k^4 - 1)$. Hence

$$\text{NIS}(T_{k,d}) \leq \frac{k^{d+2} + 1}{k^2 + 1}. \tag{6}$$

(iii) If $k^2(k^d - 1)/(k^4 - 1) + 1 \leq r$, then we have $i(T_{k,d}/X) \leq (k^{d+4} - 1)/(k^4 - 1)$. Thus

$$\text{NIS}(T_{k,d}) \leq \max_r \left\{ \frac{k^{d+4} - 1}{k^4 - 1} - r \right\}.$$

The function $f(r) = (k^{d+4} - 1)/(k^4 - 1) - r$ is a decreasing function and it takes its maximum value at $r = k^2(k^d - 1)/(k^4 - 1) + 1$. Then

$$\text{NIS}(T_{k,d}) \leq \frac{k^{d+4} - k^{d+2} - k^4 + k^2}{k^4 - 1}. \tag{7}$$

It can be easily seen that there is a cut strategy X^* of $T_{k,d}$ such that $|X^*| = k^2(k^d - 1)/(k^4 - 1)$ where X^* contains all the vertices on the 2nd, 6th, 10th, 12th, etc, up to the $(d - 2)$ th levels. Then $i(T_{k,d}/X^*) = (k^{d+4} - 1)/(k^4 - 1)$. Thus

$$\text{NIS}(T_{k,d}) = \frac{k^{d+2} + 1}{k^2 + 1}. \tag{8}$$

The proof is completed by (5), (6), (7) and (8).

Case 2: $d \equiv 1 \pmod{4}$. (i) If $1 \leq r \leq k(k^d - k)/(k^4 - 1)$, then $i(T_{k,d}/X) \leq (k^2 + k - 1)r + 1$. Thus $\text{NIS}(T_{k,d}) \leq \max_r \{(k^2 + k - 1)r + 1 - r\} = \max_r \{(k^2 + k - 2)r + 1\}$. The function $f(r) = (k^2 + k - 2)r + 1$ is an increasing function and it takes its maximum value at $r = k(k^d - k)/(k^4 - 1)$. Then

$\text{NIS}(T_{k,d})$

$$\leq \frac{k^{d+3} + k^{d+2} - 2k^{d+1} - k^3 + 2k^2 - 1}{k^4 - 1}. \tag{9}$$

(ii) If $k(k^d - k)/(k^4 - 1) < r \leq k^2(k^d - k)/(k^4 - 1)$, then $i(T_{k,d}/X) \leq k^2(k^d - k)/(k^4 - 1) + (k^2 - 1)(r - 1)$. Then

$$\begin{aligned} \text{NIS}(T_{k,d}) &\leq \max_r \left\{ \frac{k^2(k^d - k)}{k^4 - 1} + (k^2 - 1)(r - 1) - r \right\} \\ &= \max_r \left\{ \frac{k^2(k^d - k)}{k^4 - 1} + (k^2 - 2)r - k^2 + 1 \right\}. \end{aligned}$$

The function $f(r) = k^2(k^d - k)/(k^4 - 1) + (k^2 - 2)r - k^2 + 1$ is an increasing function and it takes its maximum value at $r = k^2(k^d - k)/(k^4 - 1)$. Hence

$$\text{NIS}(T_{k,d}) \leq \frac{k^{d+2} - k^4 - k^3 + 1}{k^2 + 1}. \tag{10}$$

(iii) If $r = k^2(k^d - k)/(k^4 - 1) + 1$, then $i(T_{k,d}/X) \leq (k^{d+4} - k^4 - k + 1)/(k^4 - 1)$. Hence

$$\text{NIS}(T_{k,d}) \leq \frac{k^{d+2} - 2k^2 + k - 2}{k^2 + 1}. \tag{11}$$

(iv) If $k^2(k^d - k)/(k^4 - 1) + 1 < r$, then $i(T_{k,d}/X) \leq (k^{d+4} - k^4 - k + 1)/(k^4 - 1)$. Then

$$\text{NIS}(T_{k,d}) \leq \max_r \left\{ \frac{k^{d+4} - k^4 - k + 1}{k^4 - 1} - r \right\}.$$

The function $f(r) = (k^{d+4} - k^4 - k + 1)/(k^4 - 1) - r$ is a decreasing function and it takes its maximum value at $r = k^2(k^d - 1)/(k^4 - 1) + 2$. Hence

$$\text{NIS}(T_{k,d}) \leq \frac{k^{d+2} - 3k^2 + k - 3}{k^2 + 1}. \quad (12)$$

It is obvious that there is a cut strategy X^* of $T_{k,d}$ such that $|X^*| = k^2(k^d - k)/(k^4 - 1) + 1$ where X^* contains all the vertices on the 3rd, 7th, 11th, etc., up to the $(d - 2)$ th levels and one of the vertices on the first level. Then $i(T_{k,d}/X^*) = (k^{d+4} - k^4 - k + 1)/(k^4 - 1)$. Hence we get

$$\text{NIS}(T_{k,d}) = \frac{k^{d+2} - 2k^2 + k - 2}{k^2 + 1}. \quad (13)$$

The proof is completed by (9), (10), (11), (12) and (13).

Case 3: $d \equiv 2 \pmod{4}$. (i) If $1 \leq r \leq k(k^d - k^2)/(k^4 - 1)$, then $i(T_{k,d}/X) \leq (k^2 + k - 1)r + 1$. Thus $\text{NIS}(T_{k,d}) \leq \max_r \{(k^2 + k - 1)r + 1 - r\} = \max_r \{(k^2 + k - 2)r + 1\}$. The function $f(r) = (k^2 + k - 2)r + 1$ is an increasing function and it takes its maximum value at $r = k(k^d - k^2)/(k^4 - 1)$. Then

$$\begin{aligned} \text{NIS}(T_{k,d}) &\leq \frac{k^{d+3} + k^{d+2} - 2k^{d+1} - k^5 + 2k^3 - 1}{k^4 - 1}. \quad (14) \end{aligned}$$

(ii) If $r = k(k^d - k^2)/(k^4 - 1) + 1$ then $i(T_{k,d}/X) \leq (k^2 + k - 1)k(k^d - k^2)/(k^4 - 1) + k^2$. Then

$$\text{NIS}(T_{k,d}) \leq (k^2 + k - 1) \frac{k(k^d - k^2)}{k^4 - 1} + k^2 - r$$

and

$$\begin{aligned} \text{NIS}(T_{k,d}) &\leq \frac{k^d(k^3 + k^2 - 2k) + k^6 - k^5 - 2k^4 + 2k^3 - k^2 + 1}{k^4 - 1}. \quad (15) \end{aligned}$$

(iii) If $k(k^d - k^2)/(k^4 - 1) + 1 < r \leq (k^{d+2} - 1)/(k^4 - 1)$, then $i(T_{k,d}/X) \leq (k^{d+2} - 1)/(k^4 - 1) + (k^2 - 1)r$. Thus

$$\text{NIS}(T_{k,d}) \leq \max_r \left\{ \frac{k^{d+2} - 1}{k^4 - 1} + (k^2 - 1)r - r \right\}.$$

The function $f(r) = (k^{d+2} - 1)/(k^4 - 1) + (k^2 - 2)r$ is an increasing function and it takes its maximum value at $r = (k^{d+2} - 1)/(k^4 - 1)$. Then

$$\text{NIS}(T_{k,d}) \leq \frac{k^{d+2} - 1}{k^2 + 1}. \quad (16)$$

(iv) If $(k^{d+2} - 1)/(k^4 - 1) < r$, then $i(T_{k,d}/X) \leq k^2(k^{d+2} - 1)/(k^4 - 1)$. Then

$$\text{NIS}(T_{k,d}) \leq \max_r \left\{ \frac{k^2(k^{d+2} - 1)}{k^4 - 1} - r \right\}.$$

The function $f(r) = k^2(k^{d+2} - 1)/(k^4 - 1) - r$ is a decreasing function and it takes its maximum value at $r = (k^{d+2} - 1)/(k^4 - 1) + 1$. Hence

$$\text{NIS}(T_{k,d}) \leq \frac{k^{d+2} - k^2 - 2}{k^2 + 1}. \quad (17)$$

It is obvious that there is a cut strategy X^* of $T_{k,d}$ such that $|X^*| = (k^{d+2} - 1)/(k^4 - 1)$ where X^* contains all the vertices on the 0th, 4th, 8th, etc., up to the $(d - 2)$ th levels. Then $i(T_{k,d}/X^*) = k^2(k^{d+2} - 1)/(k^4 - 1)$. Hence we get

$$\text{NIS}(T_{k,d}) = \frac{k^{d+2} - 1}{k^2 + 1}. \quad (18)$$

The proof is completed by (14), (15), (16), (17), and (18).

Case 4: $d \equiv 3 \pmod{4}$. The proof is similar to those of Cases 1-3. \square

Theorem 13 Let $C_{m,n}$ be a comet with $m, n \geq 2$. Then

$$\text{NIS}(C_{m,n}) = \begin{cases} n, & m \equiv 0 \pmod{4}, \\ n - 1, & m \equiv 1, 2, 3 \pmod{4}. \end{cases}$$

Proof: Suppose $V(P_m) = \{v_1, v_2, \dots, v_m\}$ and $\deg(v_1) = n + 1$. Let X be a cut strategy of $C_{m,n}$ and $|X| = r$. We distinguish two cases.

Case 1. If $m \equiv 0 \pmod{4}$ and $X = \{v_2, v_6, \dots, v_{m-2}\}$ then $i(C_{m,n}/X) = n + r$. So we have $i(C_{m,n}/X) - |X| = n + r - r$. Thus

$$\text{NIS}(C_{m,n}) = n. \quad (19)$$

If $m \equiv 0 \pmod{4}$ and $X \neq \{v_2, v_6, \dots, v_{m-2}\}$, then $i(C_{m,n}/X) \leq n + r$ and therefore

$$i(C_{m,n}/X) - |X| \leq n + r - r$$

and

$$\text{NIS}(C_{m,n}) \leq n. \quad (20)$$

Hence by (19) and (20), $NIS(C_{m,n}) = n$ where $m \equiv 0 \pmod{4}$.

Case 2. If $m \equiv 1, 2, 3 \pmod{4}$ and $X = \{v_2\}$ then $C_{m,n}/\{v_2\}$ is a graph composed of a path of order $m - 3$ and n isolated vertices. So we have

$$i(C_{m,n}/X) - |X| = n + r - 1 - r$$

and

$$NIS(C_{m,n}) = n - 1. \tag{21}$$

If $m \equiv 1, 2, 3 \pmod{4}$ and $X \neq \{v_2\}$, then $i(C_{m,n}/X) \leq n + r - 1$. By the definition of neighbour isolated scattering number we have

$$i(C_{m,n}/X) - |X| \leq n + r - 1 - r.$$

Thus

$$NIS(C_{m,n}) \leq n - 1. \tag{22}$$

Hence by (21) and (22), $NIS(C_{m,n}) = n - 1$ where $m \equiv 1, 2, 3 \pmod{4}$. \square

Theorem 14 Let C_n be a cycle of order $n \geq 4$. Then

$$NIS(C_n) = \begin{cases} 0, & n \equiv 0 \pmod{4}, \\ -1, & n \equiv 1, 2, 3 \pmod{4}. \end{cases}$$

Proof: Let X be a cut strategy of C_n and $|X| = r$ be the number of removed vertices of C_n . We have two cases to consider: $n \equiv 0 \pmod{4}$ and $n \equiv 1, 2, 3 \pmod{4}$.

Case 1. If $n \equiv 0 \pmod{4}$, then $i(C_n/X) \leq r$. Hence $i(C_n/X) - |X| \leq r - r = 0$. It can be easily seen that there is a cut strategy X^* of C_n such that $|X^*| = \frac{1}{4}n$ and $i(C_n/X^*) = \frac{1}{4}n$. Thus we get

$$NIS(C_n) = 0 \tag{23}$$

where $n \equiv 0 \pmod{4}$.

Case 2. If $n \equiv 1, 2, 3 \pmod{4}$, then $i(C_n/X) \leq r - 1$. Thus $i(C_n/X) - |X| \leq r - 1 - r = -1$. It can be easily seen that there is a cut strategy X^* of C_n such that $|X^*| = \lceil \frac{1}{4}n \rceil$ and $i(C_n/X^*) = \lceil \frac{1}{4}n \rceil - 1$. Therefore

$$NIS(C_n) = -1 \tag{24}$$

where $n \equiv 1, 2, 3 \pmod{4}$. The proof is completed from (23) and (24). \square

Theorem 15 Let K_{n_1, n_2, \dots, n_k} be a complete k -partite graph. Then

$$NIS(K_{n_1, n_2, \dots, n_k}) = \max\{n_1, n_2, \dots, n_k\} - 2.$$

Proof: We assume $V(K_{n_1, n_2, \dots, n_k}) = V_1 \cup V_2 \cup \dots \cup V_k$ is a partition where $|V_j| = n_j$ for $j = 1, \dots, k$. Let X be a cut strategy of K_{n_1, n_2, \dots, n_k} and let X contain the elements which belongs to only one of the sets V_i . Otherwise $i(K_{n_1, n_2, \dots, n_k}/X) = 0$ and this contradicts the definition. If $X \subseteq V_i$ and $|X| = r$, then $i(K_{n_1, n_2, \dots, n_k}/X) = n_i - r$. By the definition of the neighbour isolated scattering number, we get

$$NIS(K_{n_1, n_2, \dots, n_k}) = \max_r \{n_i - 2r\}.$$

The function $f(r) = n_i - 2r$ is a decreasing function and since $1 \leq r \leq n_i$ we have

$$NIS(K_{n_1, n_2, \dots, n_k}) = n_i - 2.$$

The proof is completed by taking $n_i = \max\{n_1, n_2, \dots, n_k\}$. \square

The following results can be easily obtained from Theorem 15.

Corollary 1 Let $K_{m,n}$ be a complete bipartite graph. Then

$$NIS(K_{m,n}) = \max\{m, n\} - 2.$$

Corollary 2 Let $K_{1, n-1}$ be a star of order $n \geq 4$. Then

$$NIS(K_{1, n-1}) = n - 3.$$

The gear graph¹² is a wheel graph with a vertex added between each pair of adjacent graph vertices of the outer cycle. The gear graph Ge_n has $2n + 1$ vertices and $3n$ edges.

Theorem 16 Let Ge_n be a gear graph. Then

$$NIS(Ge_n) = n - 1.$$

Proof: Let X be a cut strategy of Ge_n , $|X| = r$, and $\deg(u) = n$. If $r \geq 1$ then we have $i(Ge_n/X) \leq n$. Hence $i(Ge_n/X) - |X| \leq n - r$. The function $f(r) = n - r$ is a decreasing function and it takes its maximum value at $r = 1$ and we have $NIS(Ge_n) \leq n - 1$. It is obvious that there is a cut strategy X^* of Ge_n such that $X^* = \{u\}$. Then we have $i(Ge_n/X^*) = n$. Hence $NIS(Ge_n) = n - 1$. \square

GRAPH OPERATIONS AND NEIGHBOUR ISOLATED SCATTERING NUMBER

In this section we consider results on the neighbour isolated scattering number of the join, union, corona, and Cartesian product of two graphs.

Join

The join¹³ $G = G_1 + G_2$ has graph set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1) \text{ and } v \in V(G_2)\}$.

Theorem 17 Let G and H be two connected graphs of order m and n , respectively. Then

$$\text{NIS}(G + H) = \max\{\text{NIS}(G), \text{NIS}(H)\}.$$

Proof: Let X be a cut strategy of $G + H$ and $i((G + H)/X) \geq 1$. Since every vertex of G is adjacent to all vertices of H and conversely, $X \subseteq V(G)$ and $X \cap V(H) = \emptyset$ or $X \subseteq V(H)$ and $X \cap V(G) = \emptyset$. There are two cases according to the elements of X .

Case 1. Let $X = X_1 \subseteq V(G)$ be the NIS-set of G such that $\text{NIS}(G) = i(G/X_1) - |X_1|$. Since every vertex of G is adjacent to all vertices of H , we have

$$i((G + H)/X_1) - |X_1| = i(G/X_1) - |X_1| = \text{NIS}(G). \quad (25)$$

Case 2. Let $X = X_2 \subseteq V(H)$ be the NIS-set of H such that $\text{NIS}(H) = i(H/X_2) - |X_2|$. Since every vertex of H is adjacent to all vertices of G , we have

$$i((G + H)/X_2) - |X_2| = i(H/X_2) - |X_2| = \text{NIS}(H). \quad (26)$$

Hence by (25) and (26),

$$\text{NIS}(G + H) = \max\{\text{NIS}(G), \text{NIS}(H)\}.$$

□

The following result can be easily obtained from Theorem 17.

Corollary 3 Let $W_{1,n}$ be a wheel, where $n \geq 4$. Then

$$\text{NIS}(W_{1,n}) = \begin{cases} 0, & n \equiv 0 \pmod{4}, \\ -1, & n \equiv 1, 2, 3 \pmod{4}. \end{cases}$$

Union

The union¹³ of G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph $G = G_1 \cup G_2$ with vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$.

Theorem 18 Let G_1, G_2, \dots, G_n be connected graphs. Then

$$\begin{aligned} \text{NIS}(G_1 \cup G_2 \cup \dots \cup G_n) \\ \geq \text{NIS}(G_1) + \text{NIS}(G_2) + \dots + \text{NIS}(G_n). \end{aligned}$$

Proof: Let X_1, X_2, \dots, X_n be NIS-sets of G_1, G_2, \dots, G_n , respectively, and let $X = X_1 \cup X_2 \cup \dots \cup X_n$ be a cut strategy of $G_1 \cup G_2 \cup \dots \cup G_n$. By the definition of neighbour isolated scattering number we have

$$\begin{aligned} \text{NIS}(G_1 \cup G_2 \cup \dots \cup G_n) \\ \geq i((G_1 \cup G_2 \cup \dots \cup G_n)/(X_1 \cup X_2 \cup \dots \cup X_n)) \\ - |X_1 \cup X_2 \cup \dots \cup X_n| \\ = i(G_1/X_1) + i(G_2/X_2) + \dots + i(G_n/X_n) \\ - |X_1| - |X_2| - \dots - |X_n| \\ = i(G_1/X_1) - |X_1| + i(G_2/X_2) \\ - |X_2| + \dots + i(G_n/X_n) - |X_n| \\ = \text{NIS}(G_1) + \text{NIS}(G_2) + \dots + \text{NIS}(G_n). \end{aligned}$$

Thus $\text{NIS}(G_1 \cup G_2 \cup \dots \cup G_n) \geq \text{NIS}(G_1) + \text{NIS}(G_2) + \dots + \text{NIS}(G_n)$. □

Theorem 19 Let G and H be two disjoint connected graphs. Then

$$\text{NIS}(G \cup H) = \max\{\text{NIS}(G) + \text{NIS}(H), \text{NIS}(G), \text{NIS}(H)\}.$$

Proof: The proof is similar to that of Theorem 18. □

Corona

The corona¹³ $G_1 \circ G_2$ is defined as the graph G obtained by taking one copy of G_1 of order n and n copies of G_2 , and then joining the i th vertex of G_1 to every vertex in the i th copy of G_2 .

Theorem 20 Let G and H be two connected graphs of order m and n , respectively. Then

$$\text{NIS}(G \circ H) = \begin{cases} m \text{NIS}(H), & \text{NIS}(H) \geq 0, \\ \text{NIS}(H), & \text{NIS}(H) < 0. \end{cases}$$

Proof: Let X be a cut strategy of $G \circ H$ and X_1 be the NIS-set of H such that $\text{NIS}(H) = i(H/X_1) - |X_1|$.

If $\text{NIS}(H) \geq 0$, then $|X| = m|X_1|$ and $i((G \circ H)/X) = m(i(H/X_1))$. Thus $i((G \circ H)/X) - |X| = m(i(H/X_1)) - m|X_1| = m(i(H/X_1) - |X_1|) = m\text{NIS}(H)$.

If $\text{NIS}(H) < 0$, then $|X| = |X_1|$ and $i((G \circ H)/X) = i(H/X_1)$. Thus $i((G \circ H)/X) - |X| = i(H/X_1) - |X_1| = \text{NIS}(H)$. □

The following results can be easily obtained from Theorem 20.

Corollary 4 Let G be a connected graph of order m and P_n be a path of order $n \geq 3$. Then

$$\text{NIS}(G \circ P_n) = \begin{cases} m, & n \equiv 1 \pmod{4}, \\ 0, & n \equiv 0, 2, 3 \pmod{4}. \end{cases}$$

Corollary 5 Let G be a connected graph of order m and C_n be a cycle of order $n \geq 4$. Then

$$\text{NIS}(G \circ C_n) = \begin{cases} 0, & n \equiv 0 \pmod{4}, \\ -1, & n \equiv 1, 2, 3 \pmod{4}. \end{cases}$$

Cartesian product

The Cartesian product¹³ $G_1 \times G_2$ of graphs G_1 and G_2 has $V(G_1) \times V(G_2)$ as its vertex set and (u_1, u_2) is adjacent to (v_1, v_2) if either $u_1 = v_1$ and u_2 is adjacent to v_2 or $u_2 = v_2$ and u_1 is adjacent to v_1 .

Theorem 21 Let G and H be two disjoint connected graphs. Then

$$\text{NIS}(G \times H) \geq \max\{\text{NIS}(G), \text{NIS}(H)\}.$$

Proof: The proof is similar to that of Theorem 17 and 18. □

Corollary 6 Let G be a connected graph of order m and P_n be a path of order $n \geq 3$. Then

$$\text{NIS}(K_2 \times P_n) = \begin{cases} \frac{1}{3}n, & n \equiv 0 \pmod{3}, \\ \frac{1}{3}(n-4), & n \equiv 1 \pmod{3}, \\ \frac{1}{3}(n-2), & n \equiv 2 \pmod{3}. \end{cases}$$

Proof: Let X be a cut strategy of $K_2 \times P_n$ and $|X| = r$. We distinguish three cases.

Case 1: $n \equiv 0 \pmod{3}$. If $1 \leq r \leq \frac{1}{3}n$, then $i((K_2 \times P_n)/X) \leq 2r$. Thus

$$i((K_2 \times P_n)/X) - |X| \leq 2r - r = r.$$

The function $f(r) = r$ is an increasing function and it takes its maximum value at $r = \frac{1}{3}n$.

$$\text{NIS}(K_2 \times P_n) \leq \frac{n}{3}. \tag{27}$$

If $\frac{1}{3}n < r \leq 2n$, then $i((K_2 \times P_n)/X) \leq n-r$. Thus

$$i((K_2 \times P_n)/X) - |X| \leq n - r - r = n - 2r.$$

The function $f(r) = n - 2r$ is a decreasing function and it takes its maximum value at $r = \frac{1}{3}n + 1$.

$$\text{NIS}(K_2 \times P_n) \leq \frac{n}{3} - 2. \tag{28}$$

It can be easily seen that there is a cut strategy X^* of $(K_2 \times P_n)$ such that $|X^*| = \frac{1}{3}n$, $i((K_2 \times P_n)/X^*) = \frac{1}{3}2n$ and so

$$\text{NIS}(K_2 \times P_n) = \frac{n}{3}. \tag{29}$$

By (27), (28), and (29) we have

$$\text{NIS}(K_2 \times P_n) = \frac{n}{3}, \tag{30}$$

where $n \equiv 0 \pmod{3}$.

Case 2: $n \equiv 1 \pmod{3}$. If $1 \leq r \leq \frac{1}{3}(n-1) + 1$ then $i((K_2 \times P_n)/X) \leq 2r - 2$. Thus

$$i((K_2 \times P_n)/X) - |X| \leq 2r - 2 - r = r - 2.$$

The function $f(r) = r - 2$ is an increasing function and it takes its maximum value at $r = \frac{1}{3}(n-1) + 1$.

$$\text{NIS}(K_2 \times P_n) \leq \frac{n-4}{3}. \tag{31}$$

If $\frac{1}{3}(n-1) + 1 < r \leq 2n$, then $i((K_2 \times P_n)/X) \leq n-r$. Thus

$$i((K_2 \times P_n)/X) - |X| \leq n - r + 1 - r = n - 2r + 1.$$

The function $f(r) = n - 2r + 1$ is a decreasing function and it takes its maximum value at $r = \frac{1}{3}(n+5)$.

$$\text{NIS}(K_2 \times P_n) \leq \frac{n-7}{3}. \tag{32}$$

It is obvious that there exists X^* of $(K_2 \times P_n)$ such that $|X^*| = \frac{1}{3}(n-1) + 1$, $i((K_2 \times P_n)/X^*) = \frac{1}{3}(2n-2)$ and so

$$\text{NIS}(K_2 \times P_n) = \frac{n-4}{3}. \tag{33}$$

By (31), (32), and (33) we have

$$\text{NIS}(K_2 \times P_n) = \frac{n-4}{3}, \tag{34}$$

where $n \equiv 1 \pmod{3}$.

Case 3: $n \equiv 2 \pmod{3}$. If $1 \leq r \leq \frac{1}{3}(n+1)$ then $i((K_2 \times P_n)/X) \leq 2r - 1$. Thus

$$i((K_2 \times P_n)/X) - |X| \leq 2r - 1 - r = r - 1.$$

The function $f(r) = r$ is an increasing function and it takes its maximum value at $r = \frac{1}{3}(n+1)$.

$$\text{NIS}(K_2 \times P_n) \leq \frac{n-2}{3}. \tag{35}$$

If $\frac{1}{3}(n+1) < r \leq 2n$ then $i((K_2 \times P_n)/X) \leq n-r + 1$. Thus

$$i((K_2 \times P_n)/X) - |X| \leq n - r + 1 - r = n - 2r + 1.$$

The function $f(r) = n - 2r + 1$ is a decreasing function and it takes its maximum value at $r = \frac{1}{3}(n + 4)$.

$$\text{NIS}(K_2 \times P_n) \leq \frac{n-5}{3}. \tag{36}$$

It can be easily seen that there is a cut strategy X^* of $(K_2 \times P_n)$ such that $|X^*| = \frac{1}{3}n$, $i((K_2 \times P_n)/X^*) = \frac{1}{3}(2n - 1)$ and so

$$\text{NIS}(K_2 \times P_n) = \frac{n-2}{3}. \tag{37}$$

By (35), (36), and (37) we have

$$\text{NIS}(K_2 \times P_n) = \frac{n-2}{3}, \tag{38}$$

where $n \equiv 0 \pmod{3}$. By (30), (34), and (38) we have

$$\text{NIS}(K_2 \times P_n) = \begin{cases} \frac{1}{3}n, & n \equiv 0 \pmod{3}, \\ \frac{1}{3}(n-4), & n \equiv 1 \pmod{3}, \\ \frac{1}{3}(n-2), & n \equiv 2 \pmod{3}. \end{cases}$$

□

Corollary 7 Let P_n be a path of order $n \geq 3$. Then

$$\text{NIS}(P_3 \times P_n) = \begin{cases} \frac{1}{2}n, & n \equiv 0 \pmod{2}, \\ 3 + \frac{1}{2}(n-3), & n \equiv 1 \pmod{2}. \end{cases}$$

CONCLUSIONS

We investigated a new measure for reliability of a graph called the neighbour isolated scattering number. The vertex neighbour connectivity is sensitive to the number of edges present in a graph, and the vertex neighbour connectivity or vertex neighbour integrity or vertex neighbour scattering number cannot distinguish the vulnerability of different networks very well in some situations. If we want to choose the more stable graph among the graphs which have the same order and the same size, one way is to choose the graph whose neighbour isolated scattering number is less than those of the others.

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