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Some inequalities of Hermite-Hadamard type for *m*-harmonic-arithmetically convex functions

Bo-Yan Xi[a], **Feng Qi**[b,*], **Tian-Yu Zhang**[a]

[a] College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China
[b] Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300160, China

*Corresponding author, e-mail: qifeng618@hotmail.com

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ABSTRACT: We introduce the notion of m-harmonic-arithmetically convex functions and establish some integral inequalities of Hermite-Hadamard type for these functions.

KEYWORDS: m-HA-convex function, Hermite-Hadamard type inequality

MSC2010: 26A51 26D15

INTRODUCTION

A function $f : I \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$. The concept of m-convex functions was introduced as follows[1].

Definition 1 For $f : [0,b] \to \mathbb{R}$ and $m \in (0,1]$, if

$$f(tx + m(1-t)y) \leqslant tf(x) + m(1-t)f(y)$$

is valid for all $x, y \in [0,b]$ and $t \in [0,1]$, then we say that f is an m-convex function on $[0,b]$.

The following inequalities of Hadamard-type were established for the above kinds of convex functions.

Theorem 1 (Ref. 2) *Let $f : \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ be m-convex and $m \in (0,1]$. If $f \in L_1([a,b])$ for $0 \leqslant a < b < \infty$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant \min\left\{\frac{f(a)+mf(b/m)}{2}, \frac{mf(a/m)+f(b)}{2}\right\}.$$

Theorem 2 (Ref. 3) *Let $f : \mathbb{R}_0 \to \mathbb{R}$ be an m-convex function with $m \in (0,1]$. If $0 \leqslant a < b < \infty$ and $f \in L_1([a,b])$, then*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b \frac{f(x)+mf(x/m)}{2}\,\mathrm{d}x \leqslant \frac{m+1}{4}\left[\frac{f(a)+f(b)}{2} + m\frac{f(a/m)+f(b/m)}{2}\right].$$

For more information on notions of various convex functions and their Hermite-Hadamard type inequalities see Refs. 4–14 and references therein.

m-HARMONIC-ARITHMETICALLY CONVEX FUNCTIONS

We first define m-harmonic-arithmetically convex functions.

Definition 2 Let $f : (0,b] \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ and $m \in (0,1]$ be a constant. If

$$f\left(\left(\frac{t}{x} + m\frac{1-t}{y}\right)^{-1}\right) \leqslant tf(x) + m(1-t)f(y) \quad (1)$$

for all $x, y \in (0,b]$ and $t \in [0,1]$, then f is said to be an m-harmonic-arithmetically convex (or m-HA-convex) function. If the inequality (1) is reversed, then f is said to be an m-harmonic-arithmetically concave (or m-HA-concave) function.

Example 1 Let $m \in (0,1]$ and $f(x) = x^{-r}$ for $x \in$

$\mathbb{R}_+$ and $r>0$. If $r \geqslant 1$, we have

$$f\left(\left(\frac{t}{x}+m\frac{1-t}{y}\right)^{-1}\right)=\frac{[ty+m(1-t)x]^r}{(xy)^r} \leqslant \frac{ty^r+(1-t)(mx)^r}{(xy)^r} \leqslant tf(x)+m(1-t)f(y) \quad (2)$$

for all $x,y>0$ and $t\in[0,1]$. If $0<r\leqslant 1$, the inequality (2) is reversed. This implies that
(i) if $0<r\leqslant 1$, the function f is m-HA-convex on $\mathbb{R}_+$;
(ii) if $r\geqslant 1$, the function f is m-HA-concave on $\mathbb{R}_+$.

INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE

We now present several integral inequalities of Hermite-Hadamard type for m-harmonic-arithmetically convex functions.

Theorem 3 *Let $f:\mathbb{R}_+\to\mathbb{R}_0$ and $m\in(0,1]$ be a constant. If f is an m-HA-convex function on $(0,b/m^2]$ and $f\in L_1([a,b/m])$ for $a,b\in\mathbb{R}_+$ with $a<b$, then*

$$\begin{aligned} f(H(a,b)) &\leqslant \frac{1}{b-a}\int_a^b\left[f(x)+m\left(\frac{b-a}{ab}x-1\right)^{-2}f\left(\frac{x}{m}\right)\right]\mathrm{d}x \leqslant \tfrac{1}{2}\min\left\{M(a,b)Q\left(a,\frac{b}{m^2}\right)\right. \\ &+M(b,a)Q\left(\frac{a}{m},\frac{b}{m}\right),M(a,b)Q\left(\frac{a}{m},\frac{b}{m^2}\right) \\ &+M(b,a)Q\left(\frac{a}{m},b\right),M(a,b)Q\left(a,\frac{b}{m}\right) \\ &+M(b,a)Q\left(\frac{a}{m^2},\frac{b}{m}\right),M(a,b)Q\left(\frac{a}{m},\frac{b}{m}\right) \\ &\left.+M(b,a)Q\left(\frac{a}{m^2},b\right)\right\}, \end{aligned} \quad (3)$$

where $H(a,b)=2ab/(a+b)$ is the well-known harmonic mean of two positive numbers a and b,

$$Q\left(\frac{u}{m_1},\frac{v}{m_2}\right)=m_1f\left(\frac{u}{m_1}\right)+m_2f\left(\frac{v}{m_2}\right), \quad (4)$$

and

$$M(u,v)=\frac{v[(v-u)-u(\ln v-\ln u)]}{(v-u)^2}$$

for $v,u>0$ with $v\neq u$ and $m_1,m_2>0$.

Proof: For $0\leqslant t\leqslant 1$, by the m-HA convexity of f on $(0,b/m^2]$, we obtain

$$\begin{aligned} f(H(a,b)) &= f\left(\left[\frac{1/2}{\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}}+\frac{1/2}{\left(\frac{1-t}{a}+\frac{t}{b}\right)^{-1}}\right]^{-1}\right) \leqslant \frac{1}{2}\left[f\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\right)\right. \\ &\left.+mf\left(\left(\frac{1-t}{ma}+\frac{t}{mb}\right)^{-1}\right)\right] \leqslant \tfrac{1}{2}\min\left\{tf(a)\right. \\ &+m(1-t)f\left(\frac{b}{m}\right)+m(1-t)f\left(\frac{a}{m}\right) \\ &+m^2tf\left(\frac{b}{m^2}\right),tf(a)+m(1-t)f\left(\frac{b}{m}\right) \\ &\left.+m^2(1-t)f\left(\frac{a}{m^2}\right)+mtf\left(\frac{b}{m}\right)\right\}. \end{aligned} \quad (5)$$

A straightforward computation gives

$$\int_0^1\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-2}\mathrm{d}t=ab, \quad (6)$$

$$abM(b,a)=\int_0^1 t\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-2}\mathrm{d}t =\frac{a^2b[b(\ln b-\ln a)-(b-a)]}{(b-a)^2}, \quad (7)$$

$$abM(a,b)=\int_0^1 (1-t)\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-2}\mathrm{d}t =\frac{ab^2[(b-a)-a(\ln b-\ln a)]}{(b-a)^2}. \quad (8)$$

Letting $x=(t/a+(1-t)/b)^{-1}$ for $0\leqslant t\leqslant 1$ leads to

$$\frac{ab}{b-a}\int_a^b f(x)\,\mathrm{d}x =\int_0^1\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-2}f\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\right)\mathrm{d}t$$

and using $x=((1-t)/a+t/b)^{-1}$ for $0\leqslant t\leqslant 1$ gives

$$\frac{ab}{b-a}\int_a^b\left(\frac{b-a}{ab}x-1\right)^{-2}f\left(\frac{x}{m}\right)\mathrm{d}x=\int_0^1\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-2}f\left(\left(\frac{1-t}{ma}+\frac{t}{mb}\right)^{-1}\right)\mathrm{d}t. \quad (9)$$

Multiplying both sides of the inequality (5) by $(t/a+(1-t)/b)^{-2}$ for $t\in[0,1]$, integrating with respect to $t\in[0,1]$, and using equations (6)–(9) we obtain the inequalities in (3). □

Corollary 1 *Under the conditions of Theorem 3, if $m=1$ then*

$$f(H(a,b)) \leqslant \frac{1}{b-a}\int_a^b \left[\left(\frac{b-a}{ab}x-1\right)^{-2}+1\right] \times f(x)\,\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}.$$

Theorem 4 *Let $f:\mathbb{R}_+\to\mathbb{R}_0$ and $m\in(0,1]$ be a constant. If f is an m-HA-convex function on $(0,b/m^2]$ and $f\in L_1([a,b/m])$ for $a,b\in\mathbb{R}_+$ with $a<b$, then*

$$\begin{aligned} L(a,b)f(H(a,b)) &\leqslant \frac{1}{b-a}\int_a^b \frac{1}{x}\left[f(x) + m\left(\frac{b-a}{ab}x-1\right)^{-1} f\left(\frac{x}{m}\right)\right]\mathrm{d}x \\ &\leqslant \tfrac{1}{2}\min\left\{N(a,b)Q\left(a,\frac{b}{m^2}\right) + N(b,a)Q\left(\frac{a}{m},\frac{b}{m}\right), N(a,b)Q\left(\frac{a}{m},\frac{b}{m^2}\right) \right.\\ &\quad + N(b,a)Q\left(\frac{a}{m},b\right), N(a,b)Q\left(a,\frac{b}{m}\right) + N(b,a)Q\left(\frac{a}{m^2},\frac{b}{m}\right), N(a,b)Q\left(\frac{a}{m},\frac{b}{m}\right) \\ &\quad \left. + N(b,a)Q\left(\frac{a}{m^2},b\right)\right\}, \end{aligned} \tag{10}$$

where $Q(u,v)$ is defined as in (4),

$$N(u,v) = \frac{(v-u)-u(\ln v-\ln u)}{(v-u)^2} \tag{11}$$

for $v\neq u$, and $L(u,v)$ is the logarithmic mean defined by

$$L(u,v) = \begin{cases} \dfrac{u-v}{\ln u-\ln v}, & u\neq v, \\ u, & u=v. \end{cases}$$

Proof: For $0\leqslant t\leqslant 1$, from (5), we have

$$\begin{aligned} f(H(a,b))\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1} &\leqslant \frac{1}{2}\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1} \times\left[f\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\right) + mf\left(\left(\frac{1-t}{ma}+\frac{t}{mb}\right)^{-1}\right)\right] \\ &\leqslant \frac{1}{2}\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\min\left\{tf(a)+m(1-t)f\left(\frac{b}{m}\right) + m(1-t)f\left(\frac{a}{m}\right)+m^2tf\left(\frac{b}{m^2}\right),\right. \\ &\quad \left. tf(a)+m(1-t)f\left(\frac{b}{m}\right) + m^2(1-t)f\left(\frac{a}{m^2}\right)+mtf\left(\frac{b}{m}\right)\right\}. \end{aligned} \tag{12}$$

A straightforward computation gives

$$\int_0^1 \left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\mathrm{d}t = abL(a,b), \tag{13}$$

$$\begin{aligned} abN(a,b) &= \int_0^1 t\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\mathrm{d}t \\ &= \frac{ab[(b-a)-a(\ln b-\ln a)]}{(b-a)^2}, \end{aligned} \tag{14}$$

$$\begin{aligned} abN(b,a) &= \int_0^1 (1-t)\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\mathrm{d}t \\ &= \frac{ab[b(\ln b-\ln a)-(b-a)]}{(b-a)^2}. \end{aligned} \tag{15}$$

Taking $x=(t/a+(1-t)/b)^{-1}$ for $0\leqslant t\leqslant 1$ results in

$$\int_0^1 \left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1} f\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\right)\mathrm{d}t = \frac{ab}{b-a}\int_a^b \frac{1}{x}f(x)\,\mathrm{d}x$$

and using $x=((1-t)/a+t/b)^{-1}$ for $0\leqslant t\leqslant 1$ gives

$$\int_0^1 \left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1} f\left(\left(\frac{1-t}{ma}+\frac{t}{mb}\right)^{-1}\right)\mathrm{d}t = \frac{ab}{b-a}\int_a^b \left(\frac{b-a}{ab}x^2-x\right)^{-1} f\left(\frac{x}{m}\right)\mathrm{d}x. \tag{16}$$

Integrating both sides of the inequality (12) with respect to $t\in[0,1]$ and employing (13)–(16) produces the inequalities in (10). □

Corollary 2 *Under the conditions of Theorem 4, if $m=1$, then*

$$\begin{aligned} L(a,b)f(H(a,b)) &\leqslant \frac{1}{b-a}\int_a^b \frac{1}{x}\left[\left(\frac{b-a}{ab}x-1\right)^{-1}+1\right]f(x)\,\mathrm{d}x \\ &\leqslant \frac{f(a)+f(b)}{2}L(a,b). \end{aligned}$$

Theorem 5 *Let $f:\mathbb{R}_+\to\mathbb{R}_0$ and $m\in(0,1]$ be a constant. If f is an m-HA-convex function on $(0,b/m^2]$ and $f\in L_1([a,b/m])$ for $a,b\in\mathbb{R}_+$ with $a<b$, then*

$$2f(H(a,b)) \leqslant \frac{1}{b-a}\int_a^b \frac{ab}{x^2}\left[f(x)+mf\left(\frac{x}{m}\right)\right]\mathrm{d}x$$

$$\leqslant \tfrac{1}{2}\min\Big\{Q\Big(a,\frac{b}{m^2}\Big)+Q\Big(\frac{a}{m},\frac{b}{m}\Big),Q\Big(\frac{a}{m},\frac{b}{m^2}\Big)+Q\Big(\frac{a}{m},b\Big),Q\Big(a,\frac{b}{m}\Big)+Q\Big(\frac{a}{m^2},\frac{b}{m}\Big),Q\Big(\frac{a}{m},\frac{b}{m}\Big)+Q\Big(\frac{a}{m^2},b\Big)\Big\}, \quad (17)$$

where $Q(u,v)$ is defined as in (4).

Proof: Letting $x=(t/a+(1-t)/b)^{-1}$ for $0\leqslant t\leqslant 1$ gives

$$\frac{ab}{b-a}\int_a^b \frac{1}{x^2}f(x)\,\mathrm{d}x = \int_0^1 f\Big(\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-1}\Big)\mathrm{d}t \quad (18)$$

and using $x=((1-t)/a+t/b)^{-1}$ for $0\leqslant t\leqslant 1$ gives

$$\frac{ab}{b-a}\int_a^b \frac{1}{x^2}f\Big(\frac{x}{m}\Big)\mathrm{d}x = \int_0^1 f\Big(\Big(\frac{1-t}{ma}+\frac{t}{mb}\Big)^{-1}\Big)\mathrm{d}t. \quad (19)$$

Integrating both sides of the inequality (5) with respect to $t\in[0,1]$ and using (18) and (19) gives (17). □

Corollary 3 *Under the conditions of Theorem 5, if $m=1$, then*

$$\Big(\frac{1}{a}-\frac{1}{b}\Big)f(H(a,b)) \leqslant \int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x \leqslant \Big(\frac{1}{a}-\frac{1}{b}\Big)\frac{f(a)+f(b)}{2}.$$

Theorem 6 *Let $f:\mathbb{R}_+\to\mathbb{R}_0$ and $m\in(0,1]$ be a constant. If f is an m-HA-convex function on $(0,b/m]$ and $f\in L_1([a,b])$ for $a,b\in\mathbb{R}_+$ with $a<b$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant \min\Big\{f(a)M(b,a)+mf\Big(\frac{b}{m}\Big)M(a,b),\, f(b)M(a,b)+mf\Big(\frac{a}{m}\Big)M(b,a)\Big\},$$

$$\frac{1}{b-a}\int_a^b \frac{1}{x}f(x)\,\mathrm{d}x \leqslant \min\Big\{f(a)N(a,b)+mf\Big(\frac{b}{m}\Big)N(b,a),\, mf\Big(\frac{a}{m}\Big)N(a,b)+f(b)N(b,a)\Big\},$$

and

$$\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x \leqslant \Big(\frac{1}{a}-\frac{1}{b}\Big)\times\min\Big\{\frac{f(a)+mf(b/m)}{2},\frac{mf(a/m)+f(b)}{2}\Big\},$$

where $M(u,v)$ and $N(u,v)$ are defined as in (4) and (11).

Proof: Putting $x=(t/a+(1-t)/b)^{-1}$ for $0\leqslant t\leqslant 1$, using the m-HA-convexity of f on $(0,b/m]$, and using (7), (8), (14), and (15) yields

$$\frac{ab}{b-a}\int_a^b f(x)\,\mathrm{d}x = \int_0^1 f\Big(\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-1}\Big)\times\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-2}\mathrm{d}t \leqslant \int_0^1\Big[tf(a)+m(1-t)f\Big(\frac{b}{m}\Big)\Big]\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-2}\mathrm{d}t = ab\Big[f(a)M(b,a)+mf\Big(\frac{b}{m}\Big)M(a,b)\Big],$$

$$\frac{ab}{b-a}\int_a^b \frac{1}{x}f(x)\,\mathrm{d}x = \int_0^1 f\Big(\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-1}\Big)\times\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-1}\mathrm{d}t \leqslant \int_0^1\Big[tf(a)+m(1-t)f\Big(\frac{b}{m}\Big)\Big]\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-1}\mathrm{d}t = ab\Big[f(a)N(a,b)+mf\Big(\frac{b}{m}\Big)N(b,a)\Big],$$

and

$$\frac{ab}{b-a}\int_a^b \frac{1}{x^2}f(x)\,\mathrm{d}x = \int_0^1 f\Big(\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-1}\Big)\mathrm{d}t \leqslant \int_0^1\Big[tf(a)+m(1-t)f\Big(\frac{b}{m}\Big)\Big]\mathrm{d}t = \frac{f(a)+mf(b/m)}{2}.$$

□

Corollary 4 *Under the conditions of Theorem 6, if $m=1$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant f(a)M(b,a)+f(b)M(a,b),$$

$$\int_a^b \frac{f(x)g(x)}{x}\,\mathrm{d}x \leqslant \left(\frac{1}{a}-\frac{1}{b}\right)[f(a)N(a,b) + f(b)N(b,a)],$$

$$\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x \leqslant \left(\frac{1}{a}-\frac{1}{b}\right)\frac{f(a)+f(b)}{2}.$$

Theorem 7 *Let $f, g : \mathbb{R}_+ \to \mathbb{R}_0$ and $m \in (0,1]$ be a constant. If f and g are m-HA-convex functions on $(0, b/m]$ and $fg \in L_1([a,b])$ for $a, b \in \mathbb{R}_+$ with $a < b$, then*

$$\int_a^b \frac{f(x)g(x)}{x^2}\,\mathrm{d}x \leqslant \frac{1}{6}\left(\frac{1}{a}-\frac{1}{b}\right)\left[2f(a)g(a) + mf(a)g\left(\frac{b}{m}\right) + mf\left(\frac{b}{m}\right)g(a) + 2m^2 f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right].$$

Proof: Putting $x = (t/a + (1-t)/b)^{-1}$ for $0 \leqslant t \leqslant 1$ and using the m-HA-convexity of f and g on $(0, b/m]$ yields

$$\begin{aligned}\frac{ab}{b-a}\int_a^b \frac{1}{x^2} f(x)g(x)\,\mathrm{d}x &= \int_0^1 f\left(\left(\frac{t}{a} + \frac{1-t}{b}\right)^{-1}\right) g\left(\left(\frac{t}{a} + \frac{1-t}{b}\right)^{-1}\right)\mathrm{d}t \\ &\leqslant \int_0^1 \left[tf(a) + m(1-t)f\left(\frac{b}{m}\right)\right]\left[tg(a) + m(1-t)g\left(\frac{b}{m}\right)\right]\mathrm{d}t = \left[f(a)g(a) + m^2 f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]\int_0^1 t^2\,\mathrm{d}t \\ &\quad + m\left[f(a)g\left(\frac{b}{m}\right) + f\left(\frac{b}{m}\right)g(a)\right]\int_0^1 t(1-t)\,\mathrm{d}t \\ &= \frac{1}{6}\left[2f(a)g(a) + mf(a)g\left(\frac{b}{m}\right) + mf\left(\frac{b}{m}\right)g(a) + 2m^2 f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right].\end{aligned}$$

□

Corollary 5 *Under the conditions of Theorem 7, if $m = 1$, then*

$$\int_a^b \frac{f(x)g(x)}{x^2}\,\mathrm{d}x \leqslant \frac{1}{6}\left(\frac{1}{a}-\frac{1}{b}\right)[2f(a)g(a) + f(a)g(b) + f(b)g(a) + 2f(b)g(b)].$$

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