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Multiple solutions for a p-biharmonic equation with nonlinear boundary conditions

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ABSTRACT: In this paper, we obtain a multiplicity result for the p-biharmonic equation with smooth boundary.

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INTRODUCTION AND PRELIMINARIES

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geqslant 3$ with smooth boundary $\partial\Omega$ and a constant p with $1 < p < N/2$. In this paper, we consider the p-biharmonic equation

$$\begin{cases} \Delta_p^2 u + |u|^{p-2}u = \lambda f(u), & \text{in } \Omega, \\ \frac{\partial(|\Delta u|^{p-2}\Delta u)}{\partial \nu} = \mu g(u), & \text{on } \partial\Omega. \end{cases} \tag{1}$$

The fourth-order equation with nonlinearity furnishes a model to study travelling waves in suspension bridges. Lazer and McKenna[1] give a survey of results in this direction. This fourth-order semilinear elliptic problem can be considered as an analogue of a class of second-order problems which have been studied by many authors (see Refs. 2–4 and references therein). Bonder and Rossi[5] study the existence of nontrivial solutions of the following fourth-order problem with nonlinear boundary conditions:

$$\begin{cases} -\Delta^2 u = u, & \text{in } \Omega, \\ -\frac{\partial \Delta u}{\partial \nu} = f(x,u), & \text{on } \partial\Omega. \end{cases}$$

They also impose one of the following boundary conditions: $\Delta u = 0$ on $\partial\Omega$, or $\partial u/\partial\nu = 0$ on $\partial\Omega$. The authors find infinitely many weak solutions for the above problems under suitable assumptions on the nonlinearity of f. The more general p-biharmonic equation has been considered in Refs. 2, 3. Differential equations with nonlinear boundary conditions have been considered by many authors in the last twenty years[6].

Motivated by Refs. 2–4, 7, we show that problem (1) has at least two nontrivial solutions provided that λ and μ are suitable. More precisely, we are interested in the following case: the functions f, g are $(p-1)$-sublinear at infinity. Our main ingredient is a recent critical point result due to Bonanno[8].

In order to state our main result we introduce some hypotheses. We assume that the functions f and $g : \mathbb{R} \to \mathbb{R}$ satisfy the following conditions.

(H1) There exist constants $C_1, C_2 > 0$ such that for all $t \in \mathbb{R}^N$,

$$|f(t)| \leqslant C_1(1+|t|^{p-1}), \quad |g(t)| \leqslant C_2|t|^{p-1}.$$

(H2) f is superlinear at zero, i.e.,

$$\lim_{t\to 0} \frac{f(t)}{|t|^{p-1}} = 0.$$

(H3) If we set $F(t) = \int_0^t f(s)\,ds$ and $G(t) = \int_0^t g(s)\,ds$, then there exists $t_0 \in \mathbb{R}$ such that

$$F(t_0) = \int_0^{t_0} f(s)\,ds > 0$$

or

$$G(t_0) = \int_0^{t_0} g(s)\,ds > 0.$$

Let $W^{2,p}(\Omega)$ be the usual Sobolev space with respect to the norm

$$\|u\|_{2,p}^p = \int_\Omega (|\Delta u|^p + |u|^p)\,dx$$

which is equivalent to the standard norm and $W_0^{2,p}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ in $W^{2,p}(\Omega)$.

Proposition 1 (see Ref. 7) *For any $1 < p < N/2$ and $1 \leqslant q \leqslant p^\star = Np/(N-2p)$, we denote by $S_{q,\Omega}$ the best constant in the embedding $W^{2,p}(\Omega) \hookrightarrow L^q(\Omega)$ and for all $1 \leqslant q \leqslant p_\star = (N-1)p/(N-2p)$, we also denote by $S_{q,\partial\Omega}$ the best constant in the embedding $W^{2,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$, i.e.,*

$$S_{q,\partial\Omega} = \inf_{u \in W^{2,p}(\Omega)\setminus W_0^{2,p}(\Omega)} \frac{\int_\Omega (|\Delta u|^p + |u|^p)\,\mathrm{d}x}{\left(\int_{\partial\Omega} |u|^q \,\mathrm{d}\sigma\right)^{p/q}}.$$

Moreover, if $1 \leqslant q < p^\star$, then the embedding $W^{2,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact and if $1 \leqslant q < p_\star$, then the embedding $W^{2,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ is compact.

Definition 1 We say that $u \in W^{2,p}(\Omega)$ is a weak solution of problem (1) if and only if

$$\int_\Omega (|\Delta u|^{p-2}\,\Delta u \Delta\varphi + |u|^{p-2}\,u\varphi)\,\mathrm{d}x - \lambda \int_\Omega f(u)\varphi\,\mathrm{d}x - \mu \int_{\partial\Omega} g(u)\varphi\,\mathrm{d}\sigma = 0$$

for all $\varphi \in W^{2,p}(\Omega)$.

Theorem 1 *Assuming hypotheses* (H1)–(H3) *are fulfilled then there exist an open interval Λ_μ and a constant $\delta_\mu > 0$ such that for all $\lambda \in \Lambda_\mu$, problem* (1) *has at least two weak solutions in $W^{2,p}(\Omega)$ whose $\|\cdot\|_{2,p}$-norms are less than δ_μ.*

We emphasize that the condition (H3) cannot be omitted. Indeed, if $f \equiv 0$ and $g \equiv 0$, then (H1) and (H2) clearly hold, but problem (1) has only the trivial solution. Theorem 1 will be proved by using a result on the existence of at least three critical points by Bonanno[8] which is a refinement of a general principle of Ricceri[9, 10]. For the reader's convenience, we describe it as follows.

Theorem 2 (see Ref. 8) *Let $(X, \|\cdot\|)$ be a separable and reflexive real Banach space, and Φ, $\Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that there exists $x_0 \in X$ such that $\Phi(x_0) = \Psi(x_0) = 0$, $\Phi(x) \geqslant 0$ for all $x \in X$ and there exist $x_1 \in X$, $\rho > 0$ such that*
(i) $\rho < \Phi(x_1)$,
(ii) $\sup_{\{\Phi(x)<\rho\}} \Psi(x) < \rho\Psi(x_1)/\Phi(x_1)$.
Further, put

$$\overline{a} = \frac{\xi\rho}{\rho\dfrac{\Psi(x_1)}{\Phi(x_1)} - \sup\limits_{\{\Phi(x)<\rho\}} \Psi(x)},$$

with $\xi > 1$, and assume that the functional $\Phi - \lambda\Psi$ is sequentially weakly lower semicontinuous, satisfies the Palais-Smale condition, and

(iii) $\lim_{\|x\|\to\infty}[\Phi(x) - \lambda\Psi(x)] = +\infty$ *for every* $\lambda \in [0, \overline{a}]$.

Then, there exist an open interval $\Lambda \subset [0, \overline{a}]$ and a positive real number δ such that for each $\lambda \in \Lambda$, the equation $\Phi'(u) - \lambda\Psi'(u) = 0$ has at least three solutions in X whose norms are less than δ.

PROOF OF THEOREM 1

For λ and $\mu \in \mathbb{R}$, we define the functional $I_{\mu,\lambda} : W^{2,p}(\Omega) \to \mathbb{R}$ by

$$I_{\mu,\lambda}(u) = \Phi(u) - \lambda\Psi(u) \ \text{ for all } u \in W^{2,p}(\Omega),$$

where

$$\Phi(u) = \int_\Omega (|\Delta u|^p + |u|^p)\,\mathrm{d}x - \mu \int_{\partial\Omega} G(u)\,\mathrm{d}\sigma, \quad (2)$$

$$\Psi(u) = \int_\Omega F(u)\,\mathrm{d}x \quad (3)$$

with $F(t) = \int_0^t f(t)\,\mathrm{d}t$ and $G(t) = \int_0^t g(t)\,\mathrm{d}t$. A simple computation implies that the functional $I_{\mu,\lambda}$ is C^1 and hence weak solutions of (1) correspond to the critical points of $I_{\mu,\lambda}$. We now check all assumptions of Theorem 2. For each $\mu \in [0, pS_{p,\partial\Omega}/C_2)$ we have $\Phi(u) \geqslant 0$ for all $u \in W^{2,p}(\Omega)$ and $\Phi(0) = \Psi(0) = 0$ since the assumption (H1) holds. Moreover, by the compact embeddings $W^{2,p}(\Omega) \hookrightarrow L^p(\Omega)$ and $W^{2,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$, a simple computation helps us to obtain the following lemma.

Lemma 1 *For every $\mu \in [0, pS_{p,\partial\Omega}/C_2)$ and all $\lambda \in \mathbb{R}$, the functional $I_{\mu,\lambda}$ is sequentially weakly lower semicontinuous on $W^{2,p}(\Omega)$.*

Lemma 2 *There exist two positive constants $\overline{\mu}$ and $\overline{\lambda}$ such that for all $\mu \in [0, \overline{\mu})$ and all $\lambda \in [0, \overline{\lambda})$, the functional $I_{\lambda,\mu}$ is coercive and satisfies the Palais-Smale condition in $W^{2,p}(\Omega)$.*

Proof: By (H1), we have

$$\begin{aligned} I_{\mu,\lambda}(u) &= \int_\Omega (|\Delta u|^p + |u|^p)\,\mathrm{d}x - \lambda \int_\Omega F(u)\,\mathrm{d}x \\ &\quad - \mu \int_{\partial\Omega} G(u)\,\mathrm{d}\sigma \\ &\geqslant \|u\|_{2,p}^p - \lambda C_1 \int_\Omega \left(|u| + \frac{|u|^p}{p}\right)\mathrm{d}x \\ &\quad - \mu \frac{C_2}{p} \int_{\partial\Omega} |u|^p \,\mathrm{d}\sigma \end{aligned}$$

$$\geqslant \|u\|_{2,p}^p \left(1-\lambda\frac{C_1}{pS_{p,\Omega}}-\mu\frac{C_2}{pS_{p,\partial\Omega}}\right) - \lambda\frac{C_1}{S_{1,\Omega}}\|u\|_{2,p}. \quad (4)$$

Since relation (4) holds, by choosing

$$\overline{\mu}=\overline{\lambda}=\min\left\{\frac{pS_{p,\Omega}}{2C_1},\frac{pS_{p,\partial\Omega}}{2C_2}\right\},$$

where C_1, C_2 are given in (H1), we conclude that for all $\lambda\in[0,\overline{\lambda})$ and all $\mu\in[0,\overline{\mu})$, the functional $I_{\mu,\lambda}$ is coercive.

Now, let $\{u_n\}$ be a Palais-Smale sequence for the functional $I_{\mu,\lambda}$ in $W^{2,p}(\Omega)$, i.e.,

$$\left|I_{\mu,\lambda}(u_n)\right|\leqslant\overline{c},\quad I'_{\mu,\lambda}(u_n)\to 0 \text{ in } W^{-2,p}(\Omega), \quad (5)$$

where $\overline{c}$ is a constant and $W^{-2,p}(\Omega)$ is the dual space of $W^{2,p}(\Omega)$. Since $I_{\mu,\lambda}$ is coercive, the sequence $\{u_n\}$ is bounded in $W^{2,p}(\Omega)$. Hence there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ such that $\{u_n\}$ converges weakly to some $u\in W^{2,p}(\Omega)$ and hence converges strongly to u in $L^p(\Omega)$ and in $L^p(\partial\Omega)$. We shall prove that $\{u_n\}$ converges strongly to u in $W^{2,p}(\Omega)$. Indeed, we have

$$\begin{aligned}\|u_n-u\|_{2,p}^p &\leqslant \int_\Omega(|\Delta u_n|^{p-2}\Delta u_n-|\Delta u|^{p-2}\Delta u)\times(\Delta u_n-\Delta u)\,\mathrm{d}x\\ &\quad+\int_\Omega(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u)\,\mathrm{d}x\\ &=[I'_{\mu,\lambda}(u_n)-I'_{\mu,\lambda}(u)](u_n-u)\\ &\quad+\lambda\int_\Omega[f(u_n)-f(u)](u_n-u)\,\mathrm{d}x\\ &\quad+\mu\int_{\partial\Omega}[g(u_n)-g(u)](u_n-u)\,\mathrm{d}x.\end{aligned}$$

On the other hand, the compact embeddings and (H1) imply

$$\begin{aligned}&\left|\int_\Omega[f(u_n)-f(u)](u_n-u)\,\mathrm{d}x\right|\\ &\leqslant\int_\Omega|f(u_n)-f(u)|\,|u_n-u|\,\mathrm{d}x\\ &\leqslant C_1\int_\Omega(2+|u_n|^{p-1}+|u|^{p-1})\,|u_n-u|\,\mathrm{d}x\\ &\leqslant C_1\big(2\,\mathrm{meas}(\Omega)^{(p-1)/p}+\|u_n\|_{L^p(\Omega)}^{p-1}+\|u\|_{L^p(\Omega)}^{p-1}\big)\times\|u_n-u\|_{L^p(\Omega)}^p,\end{aligned}$$

where meas(Ω) denotes the Lebesgue measure of Ω, which approaches 0 as $n\to\infty$. Similarly, we obtain

$$\begin{aligned}&\left|\int_{\partial\Omega}[g(u_n)-g(u)](u_n-u)\,\mathrm{d}x\right|\\ &\leqslant\int_{\partial\Omega}|g(u_n)-g(u)|\,|u_n-u|\,\mathrm{d}x\\ &\leqslant C_2\int_{\partial\Omega}(|u_n|^{p-1}+|u|^{p-1})\,|u_n-u|\,\mathrm{d}x\\ &\leqslant C_2(\|u_n\|_{L^p(\partial\Omega)}^{p-1}+\left\|u\|_{L^p(\partial\Omega)}^{p-1}\right)\left\|u_n-u\right\|_{L^p(\partial\Omega)}^p\end{aligned}$$

which approaches 0 as $n\to\infty$. Hence by (5) we have $\|u_n-u\|_{2,p}\to 0$ as $n\to\infty$. □

Lemma 3 *For every $\mu\in[0,\overline{\mu})$ with $\overline{\mu}$ as in Lemma 2, we have*

$$\lim_{\rho\to 0^+}\frac{\sup\{\Psi(u):\Phi(u)<\rho\}}{\rho}=0.$$

Proof: Let $\lambda\in[0,\overline{\lambda})$ and $\mu\in[0,\overline{\mu})$ be fixed. By (H2), for any $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ such that

$$|f(s)|<\varepsilon pS_{p,\Omega}\left(1-\mu\frac{C_2}{pS_{p,\partial\Omega}}\right)|s|^{p-1}\ \ \forall\,|s|<\delta.$$

We first fix $q\in(p,p^\star)$. Combining the above inequalities with (H1) we deduce that

$$|F(s)|\leqslant\varepsilon S_{p,\Omega}\left(1-\mu\frac{C_2}{pS_{p,\partial\Omega}}\right)|s|^p+C_\delta\,|s|^q, \quad (6)$$

for all $s\in\mathbb{R}$, where C_δ is a constant depending on δ. Now, for every $\rho>0$, we define the sets

$$\mathscr{B}_\rho^1=\{u\in W^{2,p}(\Omega):\Phi(u)<\rho\}$$

and

$$\mathscr{B}_\rho^2=\left\{u\in W^{2,p}(\Omega):\left(1-\mu\frac{C_2}{pS_{p,\partial\Omega}}\right)\|u\|_{2,p}^p<\rho\right\}.$$

Then $\mathscr{B}_\rho^1\subset\mathscr{B}_\rho^2$. From (6) we get

$$|\Psi(u)|\leqslant\varepsilon\left(1-\mu\frac{C_2}{pS_{p,\partial\Omega}}\right)\left\|u\right\|_{2,p}^p+\frac{C_\delta}{S_{q,\Omega}^{q/p}}\left\|u\right\|_{2,p}^q. \quad (7)$$

It is clear that $0\in\mathscr{B}_\rho^1$ and $\Psi(0)=0$. Hence, $0\leqslant\sup_{u\in\mathscr{B}_\rho^1}\Psi(u)$. Using (7) we get

$$\begin{aligned}0\leqslant\frac{\sup_{u\in\mathscr{B}_\rho^1}\Psi(u)}{\rho}&\leqslant\frac{\sup_{u\in\mathscr{B}_\rho^2}\Psi(u)}{\rho} \quad (8)\\ &\leqslant\varepsilon+\frac{C_\delta}{S_{q,\Omega}^{q/p}}\left(1-\mu\frac{C_2}{pS_{p,\partial\Omega}}\right)^{-q/p}\rho^{q/p-1}.\end{aligned}$$

We complete the proof of the lemma by letting $\rho \to 0^+$, since $\varepsilon > 0$ is arbitrary. □

Proof of Theorem 1: Let s_0 be as in (H3). We choose a constant $r_0 > 0$ such that $r_0 < \text{dist}(0, \partial\Omega)$. For each $\sigma \in (0,1)$ we define the function $u_\sigma(x) = 0$, if $x \in \mathbb{R}^N \backslash B_{r_0}(0)$, $u_\sigma(x) = s_0$, if $x \in B_{\sigma r_0}(0)$, $u_\sigma(x) = \frac{1}{2}s_0 \sin[(\pi/(1-\sigma)r_0)(\frac{1}{2}(1+\sigma)r_0 - |x - x_0|)] + \frac{1}{2}s_0$, if $x \in B_{r_0}(0) \backslash B_{\sigma r_0}(0)$, where $B_{r_0}(0)$ denotes the open ball with centre 0 and radius $r_0 > 0$. Then it is clear that $u_\sigma \in W^{2,p}(\Omega)$. We have that $u_\sigma(x) \in W^{2,p}$ and $|u_\sigma(x)| \leqslant s_0$ for all $x \in \mathbb{R}^N$. Moreover, we have

$$\Psi(u_\sigma) \geqslant [F(s_0)\sigma^N - \max_{|t|\leqslant|s_0|} |F(t)|(1-\sigma^N)]\omega_N r_0^N, \tag{9}$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$. From (9), there is $\sigma_0 > 0$ such that $\left\|u_{\sigma_0}\right\|_{2,p} > 0$ and $\Psi(u_{\sigma_0}) > 0$. Now, by Lemma 3, we can choose $\rho_0 \in (0,1)$ such that

$$\rho_0 < \left(1 - \mu \frac{C_2}{pS_{p,\partial\Omega}}\right) \left\|u_{\sigma_0}\right\|_{2,p}^p \leqslant \Phi(u_{\sigma_0})$$

and satisfies

$$\frac{\sup_{\Phi(u) < \rho_0} \Psi(u)}{\rho_0} < \frac{\Psi(u_{\sigma_0})}{2\Phi(u_{\sigma_0})}. \tag{10}$$

To apply Theorem 2, we choose $x_1 = u_{\sigma_0}$ and $x_0 = 0$. Then the assumptions (i) and (ii) of Theorem 2 are satisfied. Next, we define

$$a_\mu = \frac{1 + \rho_0}{\dfrac{\Psi(u_{\sigma_0})}{\Phi(u_{\sigma_0})} - \dfrac{\sup\{\Psi(u) : \Phi(u) < \rho_0\}}{\rho_0}} > 0, \tag{11}$$

$$\overline{a}_\mu = \min\{a_\mu, \overline{\lambda}\}. \tag{12}$$

and a simple computation implies that (iii) is verified. Hence, there exist an open interval $\Lambda_\mu \subset [0, \overline{a}_\mu]$ and a real positive number δ_μ such that for each $\lambda \in \Lambda_\mu$, the equation $I'_{\mu,\lambda}(u) = \Phi'_\mu(u) - \lambda\Psi'(u) = 0$ has at least three solutions in $W^{2,p}(\Omega)$ whose norms are less than δ_μ. By (H1) and (H2), one of them may be the trivial one. Thus (1) has at least two weak solutions in $W^{2,p}(\Omega)$. The proof is complete. □

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REFERENCES

1. Lazer AC, McKenna PJ (1990) Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis. *SIAM Rev* **32**, 537–78.
2. Candito P, Li L, Livrea R (2012) Infinitely many solutions for a perturbed nonlinear Navier boundary value problem involving the p-biharmonic. *Nonlin Anal* **75**, 6360–9.
3. Li L, Heidarkhani S (2012) Existence of three solutions to a double eigenvalue problem for the p-biharmonic equation. *Ann Polon Math* **104**, 71–80.
4. Li L, Tang CL (2010) Existence of three solutions for (p,q)-biharmonic systems. *Nonlin Anal* **73**, 796–805.
5. Bonder JF, Rossi JD (2002) A fourth order elliptic equation with nonlinear boundary conditions. *Nonlin Anal* **49**, 1037–47.
6. Rossi JD (2005) Elliptic problems with nonlinear boundary conditions and the Sobolev trace theorem. In: Chipot M, Quittner P (eds) *Stationary Partial Differential Equations* Vol II, Elsevier, Amsterdam, pp 311–406.
7. El Khalil A, Kellati S, Touzani A (2005) A nonlinear boundary problem involving the p-bilaplacian operator. *Int J Math Math Sci* **2005**, 1525–37.
8. Bonanno G (2003) Some remarks on a three critical points theorem. *Nonlin Anal* **54**, 651–65.
9. Ricceri B (2000) On a three critical points theorem. *Arch Math* **75**, 220–6.
10. Ricceri B (2000) Existence of three solutions for a class of elliptic eigenvalue problems. *Math Comput Model* **32**, 1485–94.