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Mathematical models of nonlinear uniform consensus

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ABSTRACT: We consider a nonlinear protocol for a structured time-invariant and synchronous multi-agent system. In the multi-agent system, we present opinion sharing dynamics as a trajectory of a cubic triple stochastic matrix. We provide a criterion for a uniform consensus of the multi-agent system. We show that the multi-agent system eventually reaches a consensus if either one of the following two conditions is satisfied: (i) every member of the group people has a positive subjective opinion on the given task after some revision steps or (ii) all entries of the given cubic triple stochastic matrix are positive.

KEYWORDS: cubic triple stochastic matrix, multi-agent system, nonlinear protocol, nonlinear consensus

INTRODUCTION

The idea of reaching consensus through repeated averaging was introduced by DeGroot[1] for a structured time-invariant and synchronous environment. Since that time, the consensus, which is the most ubiquitous phenomenon of multi-agent systems, has become popular in various fields such as biology, physics, control engineering, and social science[2,3].

The dynamics of opinion sharing, competing, and the emergence of consensus have become an active topic of the recent research in statistical and nonlinear physics[4]. For example, there are many models which address how consensus can be achieved in the evolution of two competing opinions in a population. These include the voter model, the majority rule model, and the social impact model. Due to the high relevance of complex networks to social and natural systems, opinion dynamics have also been investigated using networks such as regular lattices, random graphs, small-world networks, and scale-free networks. Phase transitions in opinion dynamics are observed[5–7] and the emergence of global consensus, where all agents share the same opinion, has been investigated[8]. It has also been found that both the network structures[9] and the opinion updating strategies[10–12] can affect the time for reaching the final consensus.

Most research papers are concerned with the consensus problem under linear protocols. However, many systems, such as the well-known Kuramoto oscillator, exhibit nonlinear locally passive dynamics[13]. The consensus problem was studied in some nonlinear protocols[14–16].

In this paper, we provide some nonlinear protocols of multi-agent systems and study the consensus problem for the provided nonlinear protocols. It is convenient to first provide a linear protocol for an estimate-modification process of a structured time-invariant and synchronous environment which was presented in Refs. 1, 17.

We consider a group of m individuals each of which can specify their own subjective probability distribution for some given task. Suppose the m individuals have to act together as a team or committee.

For $i = 1, \dots, m$, let $x_i^{(0)}$ denote the subjective distribution that the individual i is assigned to a given task. The subjective distributions,

$$x^{(0)} = (x_1^{(0)}, \dots, x_m^{(0)})^{\mathrm{T}}$$

will be based on the different backgrounds and different levels of expertise of the members of the group. It is assumed that if the individual i is informed of the distributions of each of the other members of the group, they might wish to revise their subjective distribution to accommodate this information. In DeGroot's model[1], it was assumed that when individual i makes this revision, their revised distribution is a linear combination of the distributions $x_1^{(0)}, \dots, x_m^{(0)}$. Let p_{ij} denote the weight that individual i assigns to $x_j^{(0)}$ when they make this revision. It was assumed that the $p_{ij} \geqslant 0$ and

$$\sum_{j=1}^{m} p_{ij} = 1.$$

Hence after being informed of the subjective distributions of the other members of the group, individual i revises their own subjective distribution from $x_i^{(0)}$ to

$$x_i^{(1)} = \sum_{j=1}^{m} p_{ij} x_j^{(0)}.$$

Let $\mathbb{P}$ denote an $m \times m$ matrix whose (i,j)th element is p_{ij}. It is clear that $\mathbb{P}$ is a row-stochastic matrix since the elements are all non-negative and the row-sums are equal to one. Let

$$x^{(n)} = (x_1^{(n)}, \dots, x_m^{(n)})^{\mathrm{T}}.$$

Then the vector of revised subjective distributions can be written as $x^{(1)} = \mathbb{P}x^{(0)}$.

The critical step in this process is that the above revision is iterated, i.e., if $x^{(n)}$ denotes the subjective distribution of the members of the group after n revisions then

$$x^{(n)} = \mathbb{P}x^{(n-1)} = \mathbb{P}^n x^{(0)}.$$

DeGroot states[1] that a consensus is reached if and only if all m components of $x^{(n)}$ converge to the same limit as $n \to \infty$.

In this paper, our main assumption is that the subjective distribution $x^{(n)}$ of the members is probabilistic in every step, i.e., $\sum_{k=1}^{m} x_k^{(n)} = 1$ and $x_k^{(n)} \geqslant 0$ for any $k = \overline{1,m}$ and $n \in \mathbb{N}$.

Let S^{m-1} be an $(m-1)$-dimensional simplex, where

$$S^{m-1} = \left\{ x : \sum_{k=1}^{m} x_k = 1, x_k \geqslant 0, \forall k = \overline{1,m} \right\}.$$

In this case, one has that $x^{(n)} \in S^{m-1}$ for any $n \in \mathbb{N}$ in DeGroot's model if and only if $\sum_{i=1}^{m} p_{ij} = \sum_{j=1}^{m} p_{ij} = 1$ and $p_{ij} \geqslant 0$, i.e., $\mathbb{P}$ is a doubly stochastic matrix. Consequently, we may conclude that a trajectory $\{x^{(n)}\}_{n=0}^{\infty}$ of the doubly stochastic matrix $\mathbb{P}$ presents the DeGroot model of a structured time-invariant synchronous environment with the probabilistic subjective distribution for some given task.

In Ref. 18, Chatterjee and Seneta consider a generalization of DeGroot's model in which the individuals can change their weights p_{ij} at each iteration. More precisely, let $\{\mathbb{P}_n\}_{n\in\mathbb{N}}$ be a sequence of doubly stochastic matrices (a non-homogeneous Markov chain) and $x^{(0)} \in S^{m-1}$. A sequence $x^{(n+1)} = \mathbb{P}_{n+1}x^{(n)}$ presents the Chatterjee-Seneta model of a structured time-varying and synchronous environment with the probabilistic subjective distribution for some given task. In this paper, we shall consider a nonlinear model for the estimate modification process of a structured time-invariant and synchronous environment which generalizes both the DeGroot and Chatterjee-Seneta models.

In general, we suppose that doubly stochastic matrices in the Chatterjee-Seneta model depend on subjective distributions $x^{(n)}$ in every step, i.e., entries of doubly stochastic matrices are not constants but functions of $x^{(n)}$,

$$\mathbb{P}_{x^{(n)}} := \Big(p_{ij}(x^{(n)}) \Big)_{i,j=1}^{m}. \tag{1}$$

A general model of a structured synchronous time-varying environment with probabilistic subjective distributions is defined by

$$x^{(n+1)} = \mathbb{P}_{x^{(n)}} x^{(n)} \tag{2}$$

where $\mathbb{P}_{x^{(n)}}$ is a doubly stochastic matrix defined by (1). By choosing $\mathbb{P}_{x^{(n)}}$, we may get different models of multi-agent systems. For instance, if $\mathbb{P}_{x^{(n)}} = \mathbb{P}_0$ (the matrices are free of n and $x^{(n)}$) then we get the DeGroot model. If $\mathbb{P}_{x^{(n)}} = \mathbb{P}_n$ (the matrices are free of $x^{(n)}$ but depended on n) then we get the Chatterjee-Seneta model.

We shall study a consensus problem in a multi-agent system.

Definition 1 We say that a *consensus* is reached with respect to an initial state $x^{(0)}$ in a structured time-varying synchronous multi-agent system given by (2) if a trajectory $x^{(n)}$ starting from the initial point $x^{(0)}$ converges to the centre $C = (1/m, \dots, 1/m)^{\mathrm{T}}$ of the simplex S^{m-1} as $n \to \infty$.

Definition 2 We say that a *uniform consensus* is reached in a structured time-varying synchronous multi-agent system given by (2) if the trajectory $x^{(n)}$ starting from any initial point $x^{(0)}$ converges to the centre $C = (1/m, \dots, 1/m)^{\mathrm{T}}$ of the simplex S^{m-1} as $n \to \infty$.

Remark 1 Uniform consensus is achieved if the system reaches to a consensus regardless of the initial opinion. In this event, the consensus does not depend on an initial opinion.

The following notation is used in this paper. Let $I = \{1, \dots, m\}$ be an index set, $\mathbb{R}$ be the set of real numbers, and $\mathbb{R}^m$ be the m-dimensional Euclidean space with the standard inner product $(x, y) = \sum_{i=1}^{m} x_i y_i$. Elements of $\mathbb{R}^m$ are column vectors. Let $x = (x_1, \dots, x_m)^{\mathrm{T}} \in \mathbb{R}^m$ and $e_k = (\delta_{k1}, \delta_{k2}, \dots, \delta_{km})^{\mathrm{T}}$, $k = \overline{1,m}$ be the standard basis of the space $\mathbb{R}^m$. Let

$$S^{m-1} = \{x \in \mathbb{R}^m : \sum_{i=1}^m x_i = 1,\ x_k \geqslant 0,\ \forall k = \overline{1,m}\}$$

be the $(m-1)$-dimensional simplex and

$$\operatorname{int} S^{m-1} = \{x \in S^{m-1} : x_k > 0\ \forall k \in \overline{1,m}\}$$

be its interior. Let $C = (1/m, \dots, 1/m)^{\mathrm{T}}$ be the centre of the simplex S^{m-1}. Let $M(x) = \max_{i \in I} x_i$, $m(x) = \min_{i \in I} x_i$, and $d(x) = M(x) - m(x)$ be functions. Let $\mathbb{P} = (p_{ij})_{i,j=1}^m$ be a square matrix and $\mathcal{P} = (P_{ijk})_{i,j,k=1}^m$ be a cubic matrix.

THE MAIN MODEL

In this section, we shall provide some nonlinear protocols of multi-agent systems. We need some preliminary notions and notation.

Definition 3 A cubic matrix $\mathcal{P} = (P_{ijk})_{i,j,k=1}^m$ is called *triple stochastic* if all its entries are non-negative and it is stochastic in three directions, i.e.,

$$\sum_{i=1}^m P_{ijk} = \sum_{j=1}^m P_{ijk} = \sum_{k=1}^m P_{ijk} = 1, \quad P_{ijk} \geqslant 0,$$

for any $i, j, k = \overline{1,m}$.

Let $\mathcal{P} = (P_{ijk})_{i,j,k=1}^m$ be a cubic triple stochastic matrix and $\mathbf{P}_k = (P_{ijk})_{i,j=1}^m$ be its kth-plane (square) matrix for fixed $k = \overline{1,m}$. It is clear that $\mathbf{P}_k = (P_{ijk})_{i,j=1}^m$ is a doubly stochastic matrix for every $k = \overline{1,m}$. We write a cubic matrix $\mathcal{P}$ as $\mathcal{P} = (\mathbf{P}_1 \,|\mathbf{P}_2| \cdots |\mathbf{P}_m)$.

Let $x \in S^{m-1}$ and $\mathcal{P} = (\mathbf{P}_1 \,|\mathbf{P}_2| \cdots |\mathbf{P}_m)$ be a cubic triple stochastic matrix. An *action* $\mathcal{P} \odot x$ of the cubic matrix $\mathcal{P}$ on the vector x is a square matrix $\mathbb{P}_x = \mathcal{P} \odot x$ such that

$$\mathbb{P}_x = \begin{pmatrix} \big(\mathbf{P}_1(x)\big)_1 & \cdots & \big(\mathbf{P}_1(x)\big)_m \\ \big(\mathbf{P}_2(x)\big)_1 & \cdots & \big(\mathbf{P}_2(x)\big)_m \\ \vdots & \ddots & \vdots \\ \big(\mathbf{P}_m(x)\big)_1 & \cdots & \big(\mathbf{P}_m(x)\big)_m \end{pmatrix}. \tag{3}$$

One can see that

$$\mathbb{P}_x = \Big(p_{ki}(x)\Big)_{k,i=1}^m,$$

where $p_{ki}(x) \equiv (\mathbf{P}_k x)_i = \sum_{j=1}^m P_{ijk} x_j$, is a doubly stochastic matrix. Indeed, since $\mathcal{P} = (\mathbf{P}_1 \,|\mathbf{P}_2| \cdots |\mathbf{P}_m)$ is a cubic triple stochastic matrix and $x \in S^{m-1}$, we have that

$$\begin{aligned} \sum_{i=1}^m p_{ki}(x) &= \sum_{i=1}^m (\mathbf{P}_k x)_i = \sum_{i=1}^m \left(\sum_{j=1}^m P_{ijk} x_j\right) \\ &= \sum_{j=1}^m \left(\sum_{i=1}^m P_{ijk}\right) x_j = \sum_{j=1}^m x_j = 1, \\ \sum_{k=1}^m p_{ki}(x) &= \sum_{k=1}^m (\mathbf{P}_k x)_i = \sum_{k=1}^m \left(\sum_{j=1}^m P_{ijk} x_j\right) \\ &= \sum_{j=1}^m \left(\sum_{k=1}^m P_{ijk}\right) x_j = \sum_{j=1}^m x_j = 1. \end{aligned}$$

Definition 4 We say that a sequence $\{x^{(n)}\}_{n=0}^\infty \subset S^{m-1}$ is a *trajectory* of a cubic triple stochastic matrix $\mathcal{P} = (\mathbf{P}_1 \,|\mathbf{P}_2| \cdots |\mathbf{P}_m)$ starting from an initial point $x^{(0)}$ if one has that

$$x^{(n+1)} = (\mathcal{P} \odot x^{(n)}) x^{(n)} = \mathbb{P}_{x^{(n)}} x^{(n)}, \tag{4}$$

where $\mathbb{P}_x$ is a doubly stochastic matrix defined by (3).

Let us define a nonlinear stochastic operator $V : S^{m-1} \to S^{m-1}$ by means of a cubic triple stochastic matrix $\mathcal{P} = (\mathbf{P}_1 \,|\mathbf{P}_2| \cdots |\mathbf{P}_m)$ as follows:

$$V : x \to V(x) \equiv (\mathcal{P} \odot x) x = \mathbb{P}_x x. \tag{5}$$

It is clear that $V : S^{m-1} \to S^{m-1}$ has the following form:

$$V(x) = \Big((\mathbf{P}_1 x, x), (\mathbf{P}_2 x, x), \dots, (\mathbf{P}_m x, x)\Big)^{\mathrm{T}}, \tag{6}$$

where $(\cdot, \cdot)$ is the standard inner product in $\mathbb{R}^m$. More precisely, $V : S^{m-1} \to S^{m-1}$ is a quadratic stochastic operator

$$V(x) = \left(\sum_{i,j=1}^m P_{ij1} x_i x_j, \dots, \sum_{i,j=1}^m P_{ijm} x_i x_j\right)^{\mathrm{T}}. \tag{7}$$

Quadratic stochastic operators (QSOs) have applications in population genetics [19]. A QSO describes a distribution of the next generation in the population system if the distribution of the current generation is given. In Ref. 20, a mathematical model of the transmission of human ABO blood groups was described as a QSO on a 7-dimensional simplex and based on some numerical investigations of the QSO, the future ABO blood group distribution of Malaysian people was predicted. See Ref. 21 for a long self-contained exposition of the recent achievements and open problems in the theory of QSOs.

The main problem in nonlinear operator theory is to study the behaviour of nonlinear operators. This problem was not fully finished even in the class of QSOs which are the simplest nonlinear operators[21]. In Refs. 22, 23, a special class of QSO was studied as a generalization of a logistic mapping into the higher dimensions.

The form $V(x) = (\mathcal{P} \odot x)x = \mathbb{P}_x x$ of the QSO gives an advantage during the study of stability (consensus) problems. Moreover, the trajectory $\{x^{(n)}\}_{n=0}^{\infty}$ of V starting from $x^{(0)}$, where $x^{(n+1)} = V(x^{(n)})$, is nothing more than the trajectory of the cubic stochastic matrix defined by (4). In what follows, we shall just examine the cubic stochastic matrix and its trajectory.

Protocol A. In a multi-agent system, an opinion sharing dynamics is given by the following nonlinear rule (or a trajectory of a single cubic triple stochastic matrix):

$$x^{(n+1)} \equiv (\mathcal{P} \odot x^{(n)})x^{(n)} = \mathbb{P}_{x^{(n)}} x^{(n)}, \qquad (8)$$

where $x^{(n)} = (x_1^{(n)}, \dots, x_m^{(n)})^{\mathrm{T}}$ is the subjective distribution of the members of the group after n revisions, $\mathcal{P} = (\mathbf{P}_1\,|\mathbf{P}_2|\cdots|\mathbf{P}_m)$ is a cubic triple stochastic matrix, and $\mathbb{P}_x$ is a doubly stochastic matrix defined by (3). In this case, the opinion sharing dynamics (8) can be written as

$$x^{(n+1)} \equiv \left(\sum_{i,j=1}^{m} P_{ij1} x_i^{(n)} x_j^{(n)}, \dots, \sum_{i,j=1}^{m} P_{ijm} x_i^{(n)} x_j^{(n)} \right)^{\mathrm{T}}.$$

A NONLINEAR UNIFORM CONSENSUS

Let $M(x) = \max_{i \in I} x_i$, $m(x) = \min_{i \in I} x_i$ and $d(x) = M(x) - m(x)$ for any $x \in S^{m-1}$, where $I = \{1, 2, \dots, m\}$. It is clear that all functions $M, m, d : S^{m-1} \to \mathbb{R}$ are continuous and $d(x) = 0$ if and only if $x = C = (1/m, \dots, 1/m)^{\mathrm{T}}$.

Lemma 1 *Let $\{x^{(n)}\}_{n=0}^{\infty} \subset S^{m-1}$ be any sequence. A sequence $\{x^{(n)}\}_{n=0}^{\infty}$ converges to the centre $C = (1/m, \dots, 1/m)^{\mathrm{T}}$ of the simplex S^{m-1} if and only if* $\lim_{n\to\infty} d(x^{(n)}) = 0$.

Proof: Let $\{x^{(n)}\}_{n=0}^{\infty} \subset S^{m-1}$ be any sequence.

The 'only if' part. If $\{x^{(n)}\}_{n=0}^{\infty}$ converges to the centre $C = (1/m, \dots, 1/m)^{\mathrm{T}}$ of the simplex S^{m-1} then $\lim_{n\to\infty} d(x^{(n)}) = \lim_{n\to\infty} M(x^{(n)}) - \lim_{n\to\infty} m(x^{(n)}) = M(C) - m(C) = 0$.

The 'if' part. Suppose that $\lim_{n\to\infty} d(x^{(n)}) = 0$. Let $\omega(\{x^{(n)}\})$ be an omega limiting set of $\{x^{(n)}\}_{n=0}^{\infty}$ and $x^* \in \omega(\{x^{(n)}\})$ be any point. Then there is a subsequence $\{x^{(n_k)}\}_{k=0}^{\infty}$ of $\{x^{(n)}\}_{n=0}^{\infty}$ such that $\lim_{k\to\infty} x^{(n_k)} = x^*$. Since the function $d : S^{m-1} \to \mathbb{R}$ is continuous, we get that $d(x^*) = \lim_{k\to\infty} d(x^{(n_k)}) = 0$. Hence $x^* = C$ and $\omega(\{x^{(n)}\}) = \{C\}$. □

Theorem 1 *Let $\mathcal{P} = (P_{ijk})_{i,j,k=1}^{m}$ be a cubic triple stochastic matrix and $V : S^{m-1} \to S^{m-1}$ be an associated quadratic stochastic operator given by (7). If $x^{(n_0)} \in \operatorname{int} S^{m-1}$ for some n_0 then the trajectory $\{x^{(n)}\}_{n=0}^{\infty}$ of $V : S^{m-1} \to S^{m-1}$ converges to the centre $C = (1/m, \dots, 1/m)^{\mathrm{T}}$ of the simplex S^{m-1}.*

Control system interpretation: Suppose that an opinion sharing dynamics in the multi-agent system is given by nonlinear Protocol A. If every member of the group people has a positive subjective opinion on the given task after some revision steps then the multi-agent system eventually reaches a consensus.

Proof: From Lemma 1 it is enough to show that $\lim_{n\to\infty} d(x^{(n)}) = 0$. Without loss of generality, we may suppose that $x^{(0)} \in \operatorname{int} S^{m-1}$. Let $\mathcal{P} = (\mathbf{P}_1\,|\mathbf{P}_2|\cdots|\mathbf{P}_m)$ be a cubic triple stochastic matrix and $\mathbb{P}_x = \mathcal{P} \odot x$ be an action of the cubic triple stochastic matrix $\mathcal{P}$ on x. Then $\mathbb{P}_x = (p_{ki}(x))_{k,i=1}^{m}$ is the doubly stochastic matrix, where

$$p_{ki}(x) = (\mathbf{P}_k x)_i = \sum_{j=1}^{m} P_{ijk} x_j. \qquad (9)$$

In this case, the opinion sharing dynamics can be written as

$$x^{(n+1)} = \mathbb{P}_{x^{(n)}} x^{(n)}. \qquad (10)$$

We want to show that

$$M(x^{(0)}) \geqslant \cdots \geqslant M(x^{(n)}) \geqslant \cdots, \qquad (11)$$

$$m(x^{(0)}) \leqslant \cdots \leqslant m(x^{(n)}) \leqslant \cdots, \qquad (12)$$

where $M(x) = \max_{i\in I} x_i$ and $m(x) = \min_{i \in I} x_i$. In fact, since $\mathbb{P}_x$ is a doubly stochastic matrix, it follows from (10) that

$$\begin{aligned} M(x^{(n+1)}) = x_{k_0}^{(n+1)} &= \sum_{i=1}^{m} p_{k_0 i}(x^{(n)}) x_i^{(n)} \\ &\leqslant M(x^{(n)}) \sum_{i=1}^{m} p_{k_0 i}(x^{(n)}) \\ &= M(x^{(n)}), \\ m(x^{(n+1)}) = x_{k_0'}^{(n+1)} &= \sum_{i=1}^{m} p_{k_0' i}(x^{(n)}) x_i^{(n)} \end{aligned}$$

$$\geqslant m(x^{(n)})\sum_{i=1}^{m}p_{k_0'i}(x^{(n)})$$
$$= m(x^{(n)}).$$

On the other hand, since $\mathcal{P} = (\mathbf{P}_1\,|\mathbf{P}_2|\cdots|\mathbf{P}_m)$ is a cubic triple stochastic matrix, it follows from (9), (11), and (12) that

$$p_{ki}(x^{(n)}) = \sum_{j=1}^{m} P_{ijk}x_j^{(n)} \leqslant M(x^{(0)})\sum_{j=1}^{m} P_{ijk} = M(x^{(0)}),$$

$$p_{ki}(x^{(n)}) = \sum_{j=1}^{m} P_{ijk}x_j^{(n)} \geqslant m(x^{(0)})\sum_{j=1}^{m} P_{ijk} = m(x^{(0)}),$$

for any $k, i = \overline{1,m}$ and $n \in \mathbb{N}$. This means that for any n, all entries of the matrices $\mathbb{P}_{x^{(n)}}$ lie in the segment $[m(x^{(0)}), M(x^{(0)})]$, i.e., $m(x^{(0)}) \leqslant p_{ki}(x^{(n)}) \leqslant M(x^{(0)})$ for any $k, i = \overline{1,m}$ and $n \in \mathbb{N}$. We have that $m(x^{(0)}) > 0$ since $x^{(0)} \in \operatorname{int} S^{m-1}$. We then obtain from the last argument that

$$\begin{aligned} x_k^{(n+1)} &= \sum_{i=1}^{m} p_{ki}(x^{(n)})\left(x_i^{(n)} - M(x^{(n)})\right) + M(x^{(n)}) \\ &= m(x^{(0)})\left(m(x^{(n)}) - M(x^{(n)})\right) + M(x^{(n)}) \\ &= \left(1 - m(x^{(0)})\right)M(x^{(n)}) + m(x^{(0)})m(x^{(n)}), \end{aligned} \tag{13}$$

$$\begin{aligned} x_k^{(n+1)} &= \sum_{i=1}^{m} p_{ki}(x^{(n)})\left(x_i^{(n)} - m(x^{(n)})\right) + m(x^{(n)}) \\ &\geqslant m(x^{(0)})\left(M(x^{(n)}) - m(x^{(n)})\right) + m(x^{(n)}) \\ &= m(x^{(0)})M(x^{(n)}) + \left(1 - m(x^{(0)})\right)m(x^{(n)}), \end{aligned} \tag{14}$$

for any $k = \overline{1,m}$. Consequently, we obtain from (13) and (14) that

$$\begin{aligned} d(x^{(n+1)}) &= M(x^{(n+1)}) - m(x^{(n+1)}) \\ &= \left(1 - 2m(x^{(0)})\right)d(x^{(n)}), \end{aligned} \tag{15}$$

for any $n \in \mathbb{N}$. Then it follows from (15) that

$$d(x^{(n+1)}) \leqslant \left(1 - 2m(x^{(0)})\right)^{n+1} d(x^{(0)}).$$

Since $m(x^{(0)}) > 0$ and $1 - 2m(x^{(0)}) < 1$, we get that $\lim_{n\to\infty} d(x^{(n)}) = 0$. $\square$

We now provide a criterion for a system to reach uniform consensus. Let $e_k^{(0)} = e_k = (\delta_{k1}, \delta_{k2}, \ldots, \delta_{km})^{\mathrm{T}}$ and $e_k^{(n+1)} = \mathbb{P}_{e_k^{(n)}} e_k^{(n)}$, where $k = \overline{1,m}$.

Theorem 2 *Let $\mathcal{P} = (P_{ijk})_{i,j,k=1}^{m}$ be a cubic triple stochastic matrix and $V : S^{m-1} \to S^{m-1}$ be an associated quadratic stochastic operator given by* (7). *Then the trajectory $\{x^{(n)}\}_{n=0}^{\infty}$ starting from any initial point of the simplex S^{m-1} converges to the centre $C = (1/m, \ldots, 1/m)^{\mathrm{T}}$ of the simplex S^{m-1} if and only if there is $N_0 \in \mathbb{N}$ such that $e_k^{(N_0)} \in \operatorname{int} S^{m-1}$ for any $k = \overline{1,m}$.*

Control system interpretation: Suppose that an opinion sharing dynamics in the multi-agent system is given by nonlinear Protocol A. The multi-agent system reaches a uniform consensus if and only if there is $N_0 \in \mathbb{N}$ such that $e_k^{(N_0)} \in \operatorname{int} S^{m-1}$ for any $k = \overline{1,m}$.

Proof: It is evident that if the system reaches a uniform consensus then there is $n_0(k) \in \mathbb{N}$ such that $e_k^{(n_0(k))} \in \operatorname{int} S^{m-1}$ for any $k = \overline{1,m}$. Since $V(\operatorname{int} S^{m-1}) \subset \operatorname{int} S^{m-1}$, we get that $e_k^{(N_0)} \in \operatorname{int} S^{m-1}$ for any $k = \overline{1,m}$, where $N_0 = \max_k n_0(k)$.

Let $x^{(0)} \in S^{m-1}$ be any point. We have that $x = \sum_{i=1}^{m} x_i^{(0)} e_i^{(0)}$. Since $\mathbb{P}_{\lambda x + \mu y} = \lambda\mathbb{P}_x + \mu\mathbb{P}_y$, it follows from (4) that

$$\begin{aligned} x^{(1)} = \mathbb{P}_{x^{(0)}} x^{(0)} &= \sum_{i=1}^{m} x_i^{(0)} \mathbb{P}_{e_i^{(0)}} x^{(0)} \\ &= \sum_{i,j=1}^{m} x_i^{(0)} x_j^{(0)} \mathbb{P}_{e_i^{(0)}} e_j^{(0)} \\ &= \sum_{i} x_i^{(0)} x_i^{(0)} \mathbb{P}_{e_i^{(0)}} e_i^{(0)} + \sum_{i\neq j} x_i^{(0)} x_j^{(0)} \mathbb{P}_{e_i^{(0)}} e_j^{(0)} \\ &= \sum_{i} (x_i^{(0)})^2 e_i^{(1)} + \sum_{i\neq j} x_i^{(0)} x_j^{(0)} \mathbb{P}_{e_i^{(0)}} e_j^{(0)} \\ &= \sum_{i} (x_i^{(0)})^2 e_i^{(1)} + \sum_{i\neq j} x_i^{(0)} x_j^{(0)} e_{ij}^{(00)} \end{aligned}$$

where $e_{ij}^{(00)} = \mathbb{P}_{e_i^{(0)}} e_j^{(0)}$. In a similar manner, by means of $\mathbb{P}_{\lambda x+\mu y} = \lambda\mathbb{P}_x + \mu\mathbb{P}_y$ and (4), we can get that

$$\begin{aligned}x^{(2)} &= \mathbb{P}_{x^{(1)}}x^{(1)}\\ &= \sum_i (x_i^{(0)})^2 \mathbb{P}_{e_i^{(1)}} x^{(1)} + \sum_{i\neq j} x_i^{(0)} x_j^{(0)} \mathbb{P}_{e_{ij}^{00}} x^{(1)}\\ &= \sum_{i,l} (x_i^{(0)})^2 (x_l^{(0)})^2 \mathbb{P}_{e_i^{(1)}} e_l^{(1)}\\ &\quad + \sum_{i,l\neq r} (x_i^{(0)})^2 x_l^{(0)} x_{\mathrm{r}}^{(0)} \mathbb{P}_{e_i^{(1)}} e_{\mathrm{lr}}^{(00)}\\ &\quad + \sum_{i\neq j,l} x_i^{(0)} x_j^{(0)} (x_l^{(0)})^2 \mathbb{P}_{e_{ij}^{00}} e_l^{(1)}\\ &\quad + \sum_{i\neq j,l\neq r} x_i^{(0)} x_j^{(0)} x_l^{(0)} x_{\mathrm{r}}^{(0)} \mathbb{P}_{e_{ij}^{00}} e_{\mathrm{lr}}^{00}\\ &= \sum_i (x_i^{(0)})^4 e_i^{(2)} + \cdots .\end{aligned}$$

Analogously, in general, one can show that

$$x^{(n)} = \sum_i (x_i^{(0)})^{2^n} e_i^{(n)} + \cdots .$$

Consequently, if $e_i^{(N_0)} \in \operatorname{int} S^{m-1}$ for some $N_0 \in \mathbb{N}$, where $i = \overline{1,m}$, then $x^{(N_0)} \in \operatorname{int} S^{m-1}$ for any $x^{(0)} \in S^{m-1}$. Then due to Theorem 1, the system reaches to a consensus for any $x^{(0)} \in S^{m-1}$. □

Corollary 1 *Suppose that an opinion sharing dynamics in the multi-agent system is given by nonlinear Protocol A. If all entries of a cubic triple stochastic matrix* $\mathcal{P} = (\mathbf{P}_1 | \mathbf{P}_2 | \cdots | \mathbf{P}_m)$ *are positive, i.e.,* $P_{ijk} > 0$ *for any* $i, j, k = \overline{1,m}$*, then the multi-agent system eventually reaches a (uniform) consensus.*

Proof: It is easy to check that if all entries of the cubic triple stochastic matrix $\mathcal{P} = (\mathbf{P}_1 | \mathbf{P}_2 | \cdots | \mathbf{P}_m)$ are positive, i.e., $P_{ijk} > 0$ for any $i, j, k = \overline{1,m}$, then $x^{(1)} = (\mathcal{P} \odot x^{(0)}) x^{(0)} \in \operatorname{int} S^{m-1}$. From Theorem 1, we reach to a consensus in the multi-agent system. □

Remark 2 We know from the theory of Markov chains that if all entries of a doubly stochastic matrix $\mathbb{P}$ are positive then its trajectory $\{x^{(n)}\}_{n=0}^{\infty}$, where $x^{(n)} = \mathbb{P}^n x^{(0)}$, starting from any initial point $x^{(0)} \in S^{m-1}$ converges to the centre $C = (1/m, \ldots, 1/m)^{\mathrm{T}}$ of the simplex S^{m-1} (i.e., it is regular). The similar result was open for cubic triple stochastic matrices. From Corollary 1, this result is generalized for cubic triple stochastic matrices. To the best of our knowledge, Corollary 1 is a new result for higher-dimensional stochastic matrices.

CONCLUSIONS

In this paper, we have studied a nonlinear protocol for a structured time-invariant and synchronous multi-agent system which generalizes both the DeGroot and Chatterjee-Seneta classical models. In the multi-agent system, we present an opinion sharing dynamics as a trajectory of a cubic triple stochastic matrix (Protocol A). We provide a criterion for a uniform consensus of the multi-agent system. We showed that the multi-agent system eventually reaches to a consensus if either one of the following two conditions is satisfied: (i) every member of the group people has a positive subjective opinion on the given task after some revision steps or (ii) all entries of the given cubic triple stochastic matrix are positive. The consensus problem for a nonlinear protocol given by polynomial stochastic operators will be studied in a forthcoming paper.

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