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Regularity criteria for weak solution to the 3-d magnetohydrodynamic equations

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ABSTRACT: In this paper, regularity criteria for the 3D magnetohydrodynamic equations are investigated. Some regularity criteria which are related only with $u+B$ or $u-B$ in multiplier spaces $M(\dot{B}^r_{2,1}, L^2)$ are obtained.

KEYWORDS: multiplier space, Besov space

INTRODUCTION

This paper is concerned with the regularity of weak solutions to the viscous incompressible magnetohydrodynamics (MHD) equations in $\mathbb{R}^3 \times (0,T)$

$$\begin{cases} \partial_t u + u\cdot\nabla u - B\cdot\nabla B - \dfrac{1}{\mathrm{Re}}\Delta u \\ +\nabla(p+\dfrac{1}{2}|B|^2) = 0 \\ \partial_t B + u\cdot\nabla B - B\cdot\nabla u - \dfrac{1}{\mathrm{Rm}}\Delta B = 0, \\ \nabla\cdot u = 0, \quad \nabla\cdot B = 0 \end{cases} \tag{1}$$

with the initial condition

$$t=0: \quad u = u_0(x),\ B = B_0(x), \quad x\in\mathbb{R}^3. \tag{2}$$

Here $u=(u_1,u_2,u_3)$, $B=(B_1,B_2,B_3)$ and $P = p+\frac{1}{2}|B|^2$ are non-dimensional quantities corresponding to the flow velocity, the magnetic field, and the total kinetic pressure at the point (x,t), while $u_0(x)$ and $B_0(x)$ are the given initial velocity and initial magnetic fields with $\nabla\cdot u_0 = 0$ and $\nabla\cdot B_0 = 0$, respectively. The non-dimensional number $\mathrm{Re} > 0$ is the Reynolds number and $\mathrm{Rm} > 0$ is the magnetic Reynolds number.

The 3-dimensional magnetohydrodynamic equations govern the dynamics of electrically conducting fluids. Examples of such fluids include plasmas, liquid metals, and salt water. MHD theory has broad applications to the many branches of the sciences, e.g., geophysics, astrophysics, and engineering problems[1].

The local well-posedness of the Cauchy problem (1), (2) in the usual Sobolev spaces $H^s(\mathbb{R}^3)$ is established in Ref. 2 for any given initial data $u_0, B_0 \in H^s(\mathbb{R}^3)$, $s \geqslant 3$. But whether this unique local solution can exist globally is a challenging open problem in mathematical fluid mechanics. There is much important progress on the fundamental issue of the regularity of weak solutions to (1), (2) (see e.g., Refs. 3, 4). Many of the interesting regularity criteria for (1), (2) have been investigated by many authors. Serrin-type regularity criteria in terms of the velocity only were established in Refs. 3, 5. A regularity criterion was proved in Ref. 6 by adding a condition on the velocity in the Besov spaces. Zhou and Gala[7] proved regularity for u and $\nabla\times u$ in the multiplier spaces. Wu[8] considered the velocity field in the homogeneous Besov space. Regularity was obtained by imposing a condition on the pressure and the magnetic field in Ref. 9. As suggested by the results in Ref. 10 and Ref. 11, one may therefore presume that there should be some cancellation between the velocity field and the magnetic field. In particular, some regularity criteria in terms of the combination of u and B were established[12, 13]. Gala[13] proved the regularity criterion in terms of a combination of u and B in multiplier spaces $M(\dot{H}^r, L^2)$. Regularity criteria for the weak solution to the generalized MHD were investigated in Refs. 14, 15. By a multiplier acting from one functional space S_1 into another space S_2, we mean a function which defines a bounded linear mapping of S_1 into S_2 by pointwise multiplication. Thus with any pair of spaces (S_1, S_2), we associate a third, the space of multipliers $M(S_1, S_2)$ with the norm

$$\|f\|_{M(S_1,S_2)} = \sup_{\|g\|_{S_1}\leqslant 1} \|fg\|_{S^2}.$$

The space $M(\dot{H}^r, L^2)$ has been characterized by Mazya[16] in terms of Sobolev capacities. $M(\dot{H}^r, L^2)$ has been used in the study of the uniqueness of weak solutions for the Navier-Stokes equations in Refs. 17, 18, where it is pointed out that

$$L^p \subset L^{p,\infty} \subset \dot{M}^{p,q} \subset M(\dot{H}^r, L^2) \subset M(\dot{B}^r_{2,1}, L^2),$$

$$p = \frac{3}{r} > q > 2.$$

Here $\dot{M}^{p,q}$ and $\dot{B}^s_{p,q}$ stand for the homogeneous Morrey and Besov spaces, respectively.

The purpose of this paper is to extend Gala results[13] to the multiplier spaces $M(\dot{B}^r_{2,1}, L^2)$. For this purpose, we reformulate (1), (2) as follows. Formally, if the first equation of (1) is added or subtracted from the second one then (1), (2) can be rewritten as

$$\begin{cases} \partial_t w^+ - \mu\Delta w^+ - \nu\Delta w^- + w^- \cdot \nabla w^+ + \nabla P = 0, \\ \partial_t w^- \mu\Delta w^- \nu\Delta w^+ + w^+ \cdot \nabla w^- + \nabla P = 0, \\ \nabla \cdot w^+ = 0, \nabla \cdot w^- = 0 \end{cases} \tag{3}$$

with the initial condition

$$t = 0: \quad w^+ = w_0^+(x),\ w^- = w_0^-(x). \tag{4}$$

Here

$$w^+ = u + B,\ w^- = u - B,$$

$$w_0^+ = u_0 + B_0,\ w_0^- = u_0 + B_0$$

and

$$\mu = \frac{1}{2\,\mathrm{Re}} + \frac{1}{2\,\mathrm{Rm}}, \quad \nu = \frac{1}{2\,\mathrm{Re}} - \frac{1}{2\,\mathrm{Rm}}.$$

Note that $\mu > |\nu|$.

MAIN RESULTS

Before stating our main results, we introduce some function spaces. Let

$$C^\infty_{0,\sigma}(\mathbb{R}^3) = \{\varphi \in (C^\infty(\mathbb{R}^3))^3 : \nabla \cdot \varphi = 0\} \subset (C^\infty(\mathbb{R}^3))^3.$$

The subspace is obtained as the closure of $C^\infty_{0,\sigma}$ with respect to the L^2-norm $\|\cdot\|_{L^2}$. H^r_σ is the closure of $C^\infty_{0,\sigma}$ with respect to the H^r-norm

$$\|u\|_{H^r} = \|(I - \Delta)^{\frac{r}{2}} u\|_{L^2}, \quad r \geqslant 0.$$

Before stating our main results, we give the definition of a weak solution to (1), (2) (see Ref. 19).

Definition 1 Given u_0, $B_0 \in L^2_\sigma(\mathbb{R}^3)$, a pair (u, B) on $\mathbb{R}^3 \times (0, T)$ is called a weak solution to (1), (2), provided that
(1) u, $B \in L^\infty(0, T; L^2(\mathbb{R}^3)) \bigcap L^2(0, T; H^1(\mathbb{R}^3))$,
(2) $\nabla \cdot u = 0$, $\nabla \cdot B = 0$ in the sense of distribution,
(3) for all $\phi \in C^\infty_0(\mathbb{R}^3 \times (0, T))$ with $\nabla \cdot \phi = 0$,

$$\int_0^T \int_{\mathbb{R}^3} (u\cdot\partial_t\phi - \nabla u\cdot\nabla\phi + \nabla\phi : (u\otimes u - B\otimes B))\,\mathrm{d}x\,\mathrm{d}t = -(u_0, \phi(0))$$

and

$$\int_0^T \int_{\mathbb{R}^3} (B\cdot\partial_t\phi - \nabla B\cdot\nabla\phi + \nabla\phi : (u\otimes B - B\otimes u))\,\mathrm{d}x\,\mathrm{d}t = -(B_0, \phi(0)).$$

The weak solution (w^+, w^-) to (3), (4) can be defined in a similar way as follows.

Definition 2 Given w_0^+, $w_0^- \in L^2_\sigma(\mathbb{R}^3)$, a pair (w^+, w^-) on $\mathbb{R}^3 \times (0, T)$ is called a weak solution to (3), (4), provided that
(1) $w^+, w^- \in L^\infty(0, T; L^2(\mathbb{R}^3)) \bigcap L^2(0, T; H^1(\mathbb{R}^3))$,
(2) $\nabla \cdot w^+ = 0$, $\nabla \cdot w^- = 0$ in the sense of a distribution,
(3) for all $\phi \in C^\infty_0(\mathbb{R}^3 \times (0, T))$ with $\nabla \cdot \phi = 0$,

$$\int_0^T \int_{\mathbb{R}^3} (w^+ \cdot \partial_t\phi - \mu\nabla w^+ \cdot \nabla\phi - \nu\nabla w^- \cdot \nabla\phi)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \nabla\phi : (w^- \otimes w^+))\,\mathrm{d}x\,\mathrm{d}t = -(w_0^+, \phi(0))$$

and

$$\int_0^T \int_{\mathbb{R}^3} (w^- \cdot \partial_t\phi - \mu\nabla w^- \cdot \nabla\phi - \nu\nabla w^+ \cdot \nabla\phi)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \nabla\phi : (w^+ \otimes w^-))\,\mathrm{d}x\,\mathrm{d}t = -(w_0^-, \phi(0)).$$

In this paper, we use the multiplier space $\dot{X}^r(\mathbb{R}^3) = M(\dot{B}^r_{2,1}, L^2)$. We need the following lemma that is basically established in Ref. 20. For completeness, the proof is also sketched here.

Lemma 1 *For $0 < r < 1$, the inequality*

$$\|f\|_{\dot{B}^r_{2,1}} \leqslant C\|f\|^{1-r}_{L^2}\|\nabla f\|^r_{L^2}, \tag{5}$$

holds, where C is a positive constant that depends on r.

Proof: It follows from the definition of Besov spaces that

$$\begin{aligned}\|f\|_{\dot B^r_{2,1}} &= \sum_{i\in\mathbb{Z}} 2^{ir}\|\Delta_i f\|_{L^2}\\ &\leqslant \sum_{i\leqslant j} 2^{ir}\|\Delta_i f\|_{L^2}\\ &\quad+\sum_{i>j} 2^{i(r-1)}2^i\|\Delta_i f\|_{L^2}\\ &\leqslant (\sum_{i\leqslant j} 2^{2ir})^{\frac12}(\sum_{i\leqslant j}\|\Delta_i f\|^2_{L^2})^{\frac12}\\ &\quad+(\sum_{i>j} 2^{2i(r-1)})^{\frac12}(\sum_{i>j}2^{2i}\|\Delta_i f\|^2_{L^2})^{\frac12}\\ &\leqslant C(2^{jr}\|f\|_{L^2}+2^{j(r-1)}\|f\|_{\dot H^1})\\ &= C(2^{jr}A^{-r}+2^{j(r-1)}A^{1-r})\|f\|^{1-r}_{L^2}\|f\|^r_{\dot H^1},\end{aligned} \tag{6}$$

where $A=\|f\|_{\dot H^1}/\|f\|_{L^2}$. Choosing j such that $\frac12\leqslant 2^{jr}A^{-r}\leqslant 1$, from (6) we get

$$\begin{aligned}\|f\|_{\dot B^r_{2,1}} &\leqslant (1+2^{j(r-1)}A^{1-r})\|f\|^{1-r}_{L^2}\|f\|^r_{\dot H^1}\\ &\leqslant C(1+(\frac12)^{-\frac1r})\|f\|^{1-r}_{L^2}\|\nabla f\|^r_{L^2}.\end{aligned}$$

Therefore we have completed the proof of Lemma 1. □

Our main result of this paper is the following theorem.

Theorem 1 *Let $(u_0,B_0)\in H^1_\sigma(\mathbb{R}^3)$. Assume that (u,B) is a weak solution to (1), (2) on some interval $[0,T]$ with $0<T\leqslant\infty$. If w^- satisfies*

$$w^-\in L^{\frac{2}{1-r}}(0,T;\dot X_r(\mathbb{R}^3)),\quad 0<r<1, \tag{7}$$

then

$$w^+,w^-\in L^\infty(0,T;H^1(\mathbb{R}^3))\bigcap L^2(0,T;H^2(\mathbb{R}^3)). \tag{8}$$

Hence (u,B) is smooth in $\mathbb{R}^3\times[0,T]$.

Remark 1 By the arguments given later, it is not difficult to see that the results in Theorem 1 remain valid when w^+ satisfies one of the conditions in (7). As noted in Ref. 5, the loss of regularity in time is balanced by some additional regularities with respect to spatial variables, in just established regularity criteria.

Theorem 2 *Let $(u_0,B_0)\in H^1_\sigma(\mathbb{R}^3)$. Assume that (u,B) is a weak solution to (1), (2) on some interval $[0,T]$ with $0<T\leqslant\infty$. If ∇w^- satisfies*

$$\nabla w^-\in L^{\frac{2}{2-r}}(0,T;\dot X_r(\mathbb{R}^3)),\quad 0<r<1, \tag{9}$$

then

$$w^+,w^-\in L^\infty(0,T;H^1(\mathbb{R}^3))\bigcap L^2(0,T;H^2(\mathbb{R}^3)). \tag{10}$$

Hence (u,B) is smooth in $\mathbb{R}^3\times[0,T]$.

Remark 2 Of course, Theorem 2 remains valid if ∇w^+ satisfy assumption (9).

PROOF OF THEOREM 1

Multiplying the first equation of (3) by $-\Delta w^+$ and using integration by parts, we obtain

$$\begin{aligned}&\frac12\frac{\mathrm d}{\mathrm dt}\|\nabla w^+(t)\|^2_{L^2}+\mu\|\Delta w^+(t)\|^2_{L^2}\\ &=-\nu\int_{\mathbb{R}^3}\Delta w^-\cdot\Delta w^+\,\mathrm dx+\int_{\mathbb{R}^3}(w^-\cdot\nabla)w^+\cdot\Delta w^+\,\mathrm dx.\end{aligned} \tag{11}$$

Similarly, we get

$$\begin{aligned}\frac12\frac{\mathrm d}{\mathrm dt}\|\nabla w^-(t)\|^2_{L^2}&+\mu\|\Delta w^-(t)\|^2_{L^2}\\ &=-\nu\int_{\mathbb{R}^3}\Delta w^-\cdot\Delta w^+\,\mathrm dx\\ &\quad+\sum_{i=1}^3\int_{\mathbb{R}^3}\partial_i w^+\cdot\nabla\partial_i w^-\cdot w^-\,\mathrm dx.\end{aligned} \tag{12}$$

Adding (11) and (12), we deduce that

$$\begin{aligned}&\frac{\mathrm d}{\mathrm dt}(\|\nabla w^+(t)\|^2_{L^2}+\|\nabla w^-(t)\|^2_{L^2})\\ &\quad+2\mu(\|\Delta w^+(t)\|^2_{L^2}+\|\Delta w^-(t)\|^2_{L^2})\\ &=-4\nu\int_{\mathbb{R}^3}\Delta w^-\cdot\Delta w^+\,\mathrm dx\\ &\quad+2\int_{\mathbb{R}^3}(w^-\cdot\nabla)w^+\cdot\Delta w^+\,\mathrm dx\\ &\quad+2\sum_{i=1}^3\int_{\mathbb{R}^3}\partial_i w^+\cdot\nabla\partial_i w^-\cdot w^-\,\mathrm dx.\end{aligned} \tag{13}$$

With the help of the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&-4\nu\int_{\mathbb{R}^3}\Delta w^-\cdot\Delta w^+\,\mathrm dx\\ &\qquad\leqslant 2|\nu|(\|\Delta w^+(t)\|^2_{L^2}+\|\Delta w^-(t)\|^2_{L^2})\end{aligned} \tag{14}$$

Using the Hölder inequality, (5), and the Young inequality, we have

$$\begin{aligned} 2\int_{\mathbb{R}^3}&(w^-\cdot\nabla)w^+\cdot\Delta w^+\,\mathrm{d}x\\ &\leqslant C\|w^-\cdot\nabla w^+\|_{L^2}\|\Delta w^+\|_{L^2}\\ &\leqslant C\|w^-\|_{\dot X_r}\|\nabla w^+\|_{\dot B^r_{2,1}}\|\Delta w^+\|_{L^2}\\ &\leqslant C\|w^-\|_{\dot X_r}\|\nabla w^+\|^{1-r}_{L^2}\|\Delta w^+\|^{1+r}_{L^2}\\ &\leqslant \varepsilon\|\Delta w^+\|^2_{L^2}+C\|w^-\|^{\frac{2}{1-r}}_{\dot X_r}\|\nabla w^+\|^2_{L^2}, \end{aligned}\tag{15}$$

By the Hölder inequality, (5), and the Young inequality, we get

$$\begin{aligned} 2\sum_{i=1}^3&\int_{\mathbb{R}^3}\partial_i w^+\cdot\nabla\partial_i w^-\cdot w^-\,\mathrm{d}x\\ &\leqslant C\|w^-\cdot\nabla w^+\|_{L^2}\|\Delta w^-\|_{L^2}\\ &\leqslant C\|w^-\|_{\dot X_r}\|\nabla w^+\|_{\dot B^r_{2,1}}\|\Delta w^-\|_{L^2}\\ &\leqslant C\|w^-\|_{\dot X_r}\|\nabla w^+\|^{1-r}_{L^2}\|\Delta w^+\|^{r}_{L^2}\|\Delta w^-\|_{L^2}\\ &\leqslant \varepsilon(\|\Delta w^+\|^2_{L^2}+\|\Delta w^-\|^2_{L^2})+C\|w^-\|^{\frac{2}{1-r}}_{\dot X_r}\|\nabla w^+\|^2_{L^2}. \end{aligned}\tag{16}$$

Choosing $0<\varepsilon\leqslant(\mu-|\nu|)/2$ and then combining (13)-(16) yields

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla w^+(t)\|^2_{L^2}+\|\nabla w^-(t)\|^2_{L^2})\\ &\quad+(\mu-|\nu|)(\|\Delta w^+(t)\|^2_{L^2}+\|\Delta w^-(t)\|^2_{L^2})\\ &\qquad\leqslant C\|w^-\|^{\frac{2}{1-r}}_{\dot X_r}\|\nabla w^+\|^2_{L^2}. \end{aligned}$$

Thus by the Gronwall inequality, we obtain, for every $t\in[0,T]$

$$\begin{aligned} &\|\nabla w^+(t)\|^2_{L^2}+\|\nabla w^-(t)\|^2_{L^2}\\ &+(\mu-|\nu|)\int_0^t(\|\Delta w^+(\tau)\|^2_{L^2}+\|\Delta w^-(\tau)\|^2_{L^2})\,\mathrm{d}\tau\\ &\leqslant C(\|\nabla w_0^+\|^2_{L^2}+\|\nabla w_0^-\|^2_{L^2})\exp\Big(\int_0^t\|w^-(\tau)\|^{\frac{2}{1-r}}_{\dot X_r}\,\mathrm{d}\tau\Big), \end{aligned}$$

which implies that $(u,B)\in L^\infty(0,T;H^1(\mathbb{R})^3)$. Thus according to the regularity results in Ref. 2, (u,B) is smooth on $[0,T]$. We have completed the proof of Theorem 1.

PROOF OF THEOREM 2

Multiplying the first equation of (3) by $-\Delta w^+$ and using integration by parts, we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\|\nabla w^+(t)\|^2_{L^2}+\mu\|\Delta w^+(t)\|^2_{L^2}\\ &=4-\nu\int_{\mathbb{R}^3}\Delta w^-\cdot\Delta w^+\,\mathrm{d}x\\ &\quad-\sum_{i=1}^3\int_{\mathbb{R}^3}(\partial_i w^-\cdot\nabla)w^+\cdot\partial_i w^+\,\mathrm{d}x. \end{aligned}\tag{17}$$

Similarly, we get

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\|\nabla w^-(t)\|^2_{L^2}+\mu\|\Delta w^-(t)\|^2_{L^2}\\ &=-\nu\int_{\mathbb{R}^3}\Delta w^-\cdot\Delta w^+\,\mathrm{d}x\\ &\quad-\sum_{i=1}^3\int_{\mathbb{R}^3}\partial_i w^+\cdot\nabla w^-\cdot\partial_i w^-\,\mathrm{d}x. \end{aligned}\tag{18}$$

Adding (17) and (18), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}&(\|\nabla w^+(t)\|^2_{L^2}+\|\nabla w^-(t)\|^2_{L^2})\\ &+2\mu(\|\Delta w^+(t)\|^2_{L^2}+\|\Delta w^-(t)\|^2_{L^2})\\ &=-4\nu\int_{\mathbb{R}^3}\Delta w^-\cdot\Delta w^+\,\mathrm{d}x\\ &\quad-\sum_{i=1}^3\int_{\mathbb{R}^3}(\partial_i w^-\cdot\nabla)w^+\cdot\partial_i w^+\,\mathrm{d}x\\ &\quad-\sum_{i=1}^3\int_{\mathbb{R}^3}\partial_i w^+\cdot\nabla w^-\cdot\partial_i w^-\,\mathrm{d}x. \end{aligned}\tag{19}$$

By the Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned} -4\nu&\int_{\mathbb{R}^3}\Delta w^-\cdot\Delta w^+\,\mathrm{d}x\\ &\leqslant 2|\nu|(\|\Delta w^+(t)\|^2_{L^2}+\|\Delta w^-(t)\|^2_{L^2}) \end{aligned}\tag{20}$$

Using the Hölder inequality, (5), and the Young inequality, we have

$$\begin{aligned} -\sum_{i=1}^3&\int_{\mathbb{R}^3}(\partial_i w^-\cdot\nabla)w^+\cdot\partial_i w^+\,\mathrm{d}x\\ &\leqslant C\|\partial_i w^-\cdot\nabla w^+\|_{L^2}\|\nabla w^+\|_{L^2}\\ &\leqslant C\|\nabla w^-\|_{\dot X_r}\|\nabla w^+\|_{\dot B^r_{2,1}}\|\nabla w^+\|_{L^2}\\ &\leqslant C\|\nabla w^-\|_{\dot X_r}\|\nabla w^+\|^{2-r}_{L^2}\|\Delta w^+\|^{r}_{L^2}\\ &\leqslant \varepsilon\|\Delta w^+\|^2_{L^2}+C\|\nabla w^-\|^{\frac{2}{2-r}}_{\dot X_r}\|\nabla w^+\|^2_{L^2} \end{aligned}\tag{21}$$

and

$$
\begin{aligned}
&-\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i w^+\cdot\nabla w^-\cdot\partial_i w^-\,\mathrm{d}x\\
&\leqslant C\|\nabla w^-\cdot\partial_i w^-\|_{L^2}\|\nabla w^+\|_{L^2}\\
&\leqslant C\|\nabla w^-\|_{\dot{X}_r}\|\nabla w^-\|_{\dot{B}^r_{2,1}}\|\nabla w^+\|_{L^2}\\
&\leqslant C\|\nabla w^-\|_{\dot{X}_r}\|\nabla w^-\|_{L^2}^{1-r}\|\Delta w^-\|_{L^2}^{r}\|\nabla w^+\|_{L^2}\\
&\leqslant \varepsilon\|\Delta w^-\|_{L^2}^2\\
&\quad + C\|\nabla w^-\|_{\dot{X}_r}^{\frac{2}{2-r}}(\|\nabla w^+\|_{L^2}^2+\|\nabla w^-\|_{L^2}^2).
\end{aligned}
\tag{22}
$$

It follows from (19)-(22) and choosing $0<\varepsilon\leqslant\mu-|\nu|$ that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla w^+(t)\|_{L^2}^2+\|\nabla w^-(t)\|_{L^2}^2)\\
&\quad+(\mu-|\nu|)(\|\Delta w^+(t)\|_{L^2}^2+\|\Delta w^-(t)\|_{L^2}^2)\\
&\qquad\leqslant C\|\nabla w^-\|_{\dot{X}_r}^{\frac{2}{2-r}}(\|\nabla w^+\|_{L^2}^2+\|\nabla w^-\|_{L^2}^2).
\end{aligned}
$$

Thus by the Gronwall inequality, we obtain, for every $t\in[0,T]$

$$
\begin{aligned}
&\|\nabla w^+(t)\|_{L^2}^2+\|\nabla w^-(t)\|_{L^2}^2\\
&\quad+(\mu-|\nu|)\int_0^t(\|\Delta w^+(\tau)\|_{L^2}^2+\|\Delta w^-(\tau)\|_{L^2}^2)\,\mathrm{d}\tau\\
&\qquad\leqslant C(\|\nabla w_0^+\|_{L^2}^2+\|\nabla w_0^-\|_{L^2}^2)\\
&\qquad\qquad\times\exp\Big(\int_0^t\|\nabla w^-(\tau)\|_{\dot{X}_r}^{\frac{2}{2-r}}\,\mathrm{d}\tau\Big),
\end{aligned}
$$

which implies that $(u,B)\in L^\infty(0,T;H^1(\mathbb{R})^3)$. Thus according to the regularity results in[2], (u,B) is smooth on $[0,T]$. Thus Theorem 2 is proved.

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