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Bohr inequalities in C*-algebras

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ABSTRACT: The classical Bohr inequality states that for complex numbers a, b and real numbers $p, q > 1$ such that $1/p + 1/q = 1$, we have $|a+b|^2 \leqslant p|a|^2 + q|b|^2$ with equality if and only if $b = (p-1)a$. Various generalizations of the Bohr inequality occur for scalars, vectors, matrices and operators. In this paper, this inequality is generalized from Hilbert space operators to the context of C*-algebras and some extensions and related inequalities are obtained. For each inequality, the necessary and sufficient condition for the equality is also determined. The idea of transforming problems in operator theory to problems in matrix theory, which are easy to handle, plays a key role.

KEYWORDS: operator inequality, positivity

INTRODUCTION

Inequalities are beautiful, useful and powerful tools in mathematics. This paper is focused on inequalities of Bohr type in C*-algebras. The original Bohr inequality[1] asserts that

$$|a+b|^2 \leqslant p|a|^2 + q|b|^2 \tag{1}$$

for complex numbers a, b and real numbers $p, q > 1$ such that $1/p + 1/q = 1$. Such p and q are called *conjugate exponents*. The equality occurs if and only if $ap = bq$, i.e., $b = (p-1)a$ or $a = (q-1)b$.

There are a number of generalizations of (1) in various contexts. Some extensions and variations of this inequality for scalars are obtained in Refs. 2–4. The case of matrices is discussed in Ref. 5. The results in the context of vectors in normed linear spaces and inner product spaces are shown in Refs. 6–9. The results for matrix case are generalized to the operator case in Refs. 10–13. Hirzallah[10] first established this inequality in the context of Hilbert space operators. In Ref. 11, the Bohr inequality for Hilbert space operators is established for all positive conjugate exponents and some other interesting operator inequalities are obtained. The results in Refs. 10–12 are generalized in Ref. 14 by using the idea of transforming problems in operator theory to problems in matrix theory. The idea of using matrix ordering is used again in Ref. 15. Ref. 16 uses superquadracity to extend the Bohr inequality to the cases of $p \geqslant 2$ and $q \leqslant 2$. Some absolute-value operator inequalities related to the Bohr inequality are generalized to the framework of Hilbert C*-modules in Ref. 17.

The algebra of all bounded linear operators on a Hilbert space is the prototypical example of a C*-algebra. See, for example, Ref. 18 for an introduction to C*-algebra. In this paper, the Bohr inequality is generalized from Hilbert space operators to C*-algebras. Extensions of it and some related inequalities are also obtained. For each inequality, a necessary and sufficient condition for the equality is also determined.

KEY LEMMAS AND IMPORTANT REMARK

Throughout, $\mathcal{A}$ shall represent a C*-algebra. The next lemma provides a useful tool for determining the positivity of a quadratic form involving absolute values of elements in C*-algebras.

Lemma 1 *Let $a, b \in \mathcal{A}$ and $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha\gamma \geqslant \beta^2$. Consider $x := \alpha|a|^2 + \beta(a^*b + b^*a) + \gamma|b|^2$.*

(i) If $\alpha, \gamma \geqslant 0$, then x is positive.

(ii) If $\alpha, \gamma \leqslant 0$, then x is negative.

Proof: (i) Suppose that $\alpha, \gamma \geqslant 0$. If $\beta = 0$, we are done. If $\beta \neq 0$, then $\alpha > 0$ and $\gamma > 0$. Set $\lambda =$

$\alpha\gamma-\beta^2$. Since $\alpha=(\lambda+\beta^2)/\gamma$, it follows that

$$\begin{aligned}
&\alpha|a|^2+\beta(a^*b+b^*a)+\gamma|b|^2\\
&\quad= \frac{\lambda+\beta^2}{\gamma}|a|^2+\beta(a^*b+b^*a)+\gamma|b|^2\\
&\quad= \frac{\lambda}{\gamma}|a|^2+\frac{\beta^2}{\gamma}|a|^2+\beta(a^*b+b^*a)+\gamma|b|^2\\
&\quad= \frac{\lambda}{\gamma}|a|^2+\left|\frac{\beta}{\sqrt{\gamma}}a+\sqrt{\gamma}b\right|^2\\
&\quad\geqslant 0.
\end{aligned}$$

To prove (ii), apply (i) for α,β,γ to $-\alpha,-\beta,-\gamma$. □

Remark 1 For a hermitian matrix A, the matrix inequality $A\geqslant 0$ in the Löwner partial order means that A is a positive semidefinite matrix. This notion is analogous to the notion of a positive element in a C*-algebra. So the Löwner partial order is the natural ordering on the C*-algebras of complex matrices. The conditions $\alpha,\gamma\geqslant 0$ and $\alpha\gamma\geqslant\beta^2$ in Lemma 1 can be recognized as positive definiteness of the matrix

$$X:=\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}.$$

Similarly, the conditions $\alpha,\gamma\leqslant 0$ and $\alpha\gamma\geqslant\beta^2$ are equivalent to $X\leqslant 0$. This suggests that we should transform the problem of determining the positive definiteness of a quadratic form of an operator to the problem of determining the positive definiteness of a matrix.

In order to determine a necessary and sufficient condition for the equality case in inequalities involving absolute values, the following lemma will be used frequently.

Lemma 2 *Let $a,b\in\mathcal{A}$. For $\alpha,\beta,\gamma\in\mathbb{R}$ such that $\alpha,\gamma\geqslant 0$ and $\alpha\gamma\geqslant\beta^2$, the equality*

$$\alpha|a|^2+\beta(a^*b+b^*a)+\gamma|b|^2 = 0 \tag{2}$$

occurs if and only if one of the following conditions holds:

(i) $\alpha=\gamma=0$,

(ii) $a=b=0$,

(iii) $a=0$ *and* $\gamma=0$,

(iv) $b=0$ *and* $\alpha=0$,

(v) $\alpha a+\beta b=0$ *and* $\alpha\gamma=\beta^2\neq 0$ *(i.e.,* $\beta a+\gamma b=0$ *and* $\alpha\gamma=\beta^2\neq 0$*).*

Proof: The sufficiency is quite obvious. For the necessity, there are 4 choices of α,γ. If $\alpha=\gamma=0$, we are done. If $\alpha>0,\gamma=0$, we get $\beta=0$ and then $a=0$. If $\alpha=0,\gamma>0$, we get $\beta=0$ and then $b=0$. Now for the case $\alpha>0,\gamma>0$, let $\lambda=\alpha\gamma-\beta^2$. We have $\lambda\geqslant 0$ and $\gamma=(\lambda+\beta^2)/\alpha$. Hence

$$\begin{aligned}
&\alpha|a|^2+\beta(a^*b+b^*a)+\gamma|b|^2\\
&\quad= \alpha|a|^2+\beta(a^*b+b^*a)+\frac{\lambda+\beta^2}{\alpha}|b|^2\\
&\quad= \alpha|a|^2+\beta(a^*b+b^*a)+\frac{\beta^2}{\alpha}|b|^2+\frac{\lambda}{\alpha}|b|^2\\
&\quad= \left|\sqrt{\alpha}a+\frac{\beta}{\sqrt{\alpha}}b\right|^2+\frac{\lambda}{\alpha}|b|^2.
\end{aligned}$$

If $\alpha\gamma=\beta^2$, then $\left|\sqrt{\alpha}a+(\beta/\sqrt{\alpha})b\right|=0$, i.e., $\alpha a+\beta b=0$. So we have $\alpha a+\beta b=0$, $\alpha\gamma=\beta^2$ and $\alpha\neq 0$. But if $\gamma=0$, we have $\beta=0$ and $a=0$ which is included in (iii). The condition $\alpha\neq 0$ and $\gamma\neq 0$ is equivalent to $\alpha\gamma\neq 0$. Therefore, we can simplify the conditions $\alpha a+\beta b=0$, $\alpha\gamma=\beta^2$, and $\alpha\neq 0$ to the conditions $\alpha a+\beta b=0$ and $\alpha\gamma=\beta^2\neq 0$ which are equivalent to the conditions $\beta a+\gamma b=0$ and $\alpha\gamma=\beta^2\neq 0$. If $\alpha\gamma>\beta^2$, then $\left|\sqrt{\alpha}a+(\beta/\sqrt{\alpha})b\right|$ and $|b|$ must be the zero elements. Hence $a=b=0$. □

BOHR INEQUALITIES IN C*-ALGEBRAS

In this section inequalities of Bohr type for operators on Hilbert spaces are generalized to the context of C*-algebras.

Theorem 1 (Bohr Inequality) *Let $a,b\in\mathcal{A}$ and let $p,q>1$ be real numbers such that $1/p+1/q=1$. Then*

$$|a+b|^2 \leqslant p|a|^2+q|b|^2 \tag{3}$$

with the equality if and only if $pa=qb$ (i.e., $b=(p-1)a$ or $a=(q-1)b$).

Proof: It is easy to see the identity

$$|a+b|^2=|a|^2+(a^*b+b^*a)+|b|^2.$$

Hence we get

$$\begin{aligned}
&p|a|^2+q|b|^2-|a+b|^2\\
&\quad=(p-1)|a|^2-(a^*b+b^*a)+(q-1)|b|^2.
\end{aligned}$$

From Lemma 1 and Remark 1 it suffices to show that

$$X:=\begin{pmatrix}p-1 & -1\\ -1 & q-1\end{pmatrix}\geqslant 0.$$

Since $p-1\geqslant 0$, $q-1\geqslant 0$, and $(p-1)(q-1)=1$, the matrix X is positive semidefinite, i.e., (3) holds. If follows from Lemma 2 that the equality in (3) occurs if and only if one of the following holds:

(i) $a = b = 0$,

(ii) $(p-1)a-b = 0$ and $(p-1)(q-1) = (-1)^2 \neq 0$.

Note that the condition $(p-1)(q-1) = (-1)^2 \neq 0$ always holds from the hypothesis. Hence (i) is included in (ii) which is $b = (p-1)a$. The condition $1/p+1/q = 1$ implies that $b = (p-1)a$ is equivalent to $a = (q-1)b$ or $pa = qb$. □

Remark 2 The Bohr inequality (3) can be equivalently stated as

$$|a+b|^2 \leqslant (1+t)|a|^2 + (1+\frac{1}{t})|b|^2 \qquad (4)$$

for any $t > 0$. In this case the equality holds if and only if $b = ta$. In fact, the condition $p, q > 1$ with $1/p + 1/q = 1$ can be extended to the condition $p, q, s > 0$ with $1/p + 1/q \leqslant 1/s$. In this case we have

$$s|a+b|^2 \leqslant p|a|^2 + q|b|^2. \qquad (5)$$

Corollary 1 *Let $a, b \in \mathcal{A}$. If p, q are real numbers such that $p, q < 1$ and $1/p + 1/q = 1$ then*

$$|a+b|^2 \geqslant p|a|^2 + q|b|^2 \qquad (6)$$

with equality if and only if $pa = qb$ (i.e., $b = (p-1)a$ or $a = (q-1)b$).

Proof: Apply the argument in the proof of Theorem 1. In this case we obtain

$$\begin{pmatrix} p-1 & -1 \\ -1 & q-1 \end{pmatrix} \leqslant 0.$$

□

Corollary 2 *Let $a, b \in \mathcal{A}$ and $t > 0$. Then*

(i) $a^*b + b^*a \leqslant t|a|^2 + \frac{1}{t}|b|^2$ with equality if and only if $b = ta$,

(ii) $-(a^*b+b^*a) \leqslant t|a|^2 + \frac{1}{t}|b|^2$ with equality if and only if $b = -ta$.

Proof: (i) From Theorem 1 we have that for $p, q > 1$ with $1/p + 1/q = 1$,

$$|a|^2 + (a^*b + b^*a) + |b|^2 \leqslant p|a|^2 + q|b|^2.$$

Hence when $t := p - 1 > 0$ we have

$$\begin{aligned} a^*b + b^*a &\leqslant (p-1)|a|^2 + (q-1)|b|^2 \\ &= (p-1)|a|^2 + \frac{1}{p-1}|b|^2 \\ &= t|a|^2 + \frac{1}{t}|b|^2. \end{aligned}$$

The equality occurs if and only if $b = (p-1)a = ta$. To prove (ii), replace a by $-a$ in (i). □

The Bohr inequality is extended to all possible cases of conjugate exponents in the following theorems. The analogous results for the case of operators on Hilbert spaces are obtained as Theorem 2, Theorem 1 and Corollary 1, respectively, in Ref. 11.

Theorem 2 *Let $a, b \in \mathcal{A}$ and let p, q be real numbers such that $1/p + 1/q = 1$. If $p < 1$, then*

$$|a-b|^2 + |(p-1)a + b|^2 \geqslant p|a|^2 + q|b|^2, \qquad (7)$$
$$|a-b|^2 + |a + (q-1)b|^2 \geqslant p|a|^2 + q|b|^2, \qquad (8)$$

with equality if and only if $b = (1-p)a$ (i.e., $a = (1-q)b$).

Proof: By expanding we have

$$\begin{aligned} &|a-b|^2 + |(p-1)a+b|^2 - p|a|^2 - q|b|^2 \\ &= |a|^2 - (a^*b + b^*a) + |b|^2 + (p-1)^2|a|^2 \\ &\quad + (p-1)(a^*b + b^*a) + |b|^2 - p|a|^2 - q|b|^2 \\ &= (p^2 - 3p + 2)|a|^2 + (p-2)(a^*b + b^*a) \\ &\quad + (2-q)|b|^2. \end{aligned} \qquad (9)$$

Let us define

$$X := \begin{pmatrix} p^2 - 3p + 2 & p-2 \\ p-2 & 2-q \end{pmatrix}.$$

The conditions $p < 1$ and $1/p + 1/q = 1$ imply $p + q \leqslant 4$. It follows that $(p-2)(q-2) = 4 - (p+q) \geqslant 0$ which implies $(2-q)/(p-2) \leqslant 0$. The identity $(p-1)(2-q) = (p-2)$ yields $(p-1)(2-q)/(p-2) \geqslant 1$. Hence we arrive at

$$\frac{1}{p-2}X = \begin{pmatrix} p-1 & 1 \\ 1 & \dfrac{2-q}{p-2} \end{pmatrix} \leqslant 0.$$

From Lemma 1, since $X \geqslant 0$, we conclude that (7) holds. From Lemma 2 the equality in (7) is valid if and only if one of the following holds:

(i) $a = b = 0$,

(ii) $(p^2 - 3p + 2)a + (p-2)b = 0$ and $(p^2 - 3p + 2)(2-q) = (p-2)^2 \neq 0$.

Note that the hypothesis $1/p + 1/q = 1$ implies $(p^2 - 3p + 2)(2-q) = (p-2)^2$ and the condition $p < 1$ yields $(p-2)^2 \neq 0$. Hence the condition (i) is embedded in (ii), i.e., the equality in (7) holds if and only if $(p^2-3p+2)a+(p-2)b=0$ which is $b=(1-p)a$.

A proof of (8) is similar to that of (7). In this case, we get

$$\begin{aligned}|a-b|^2&+|a+(q-1)b|^2-p|a|^2-q|b|^2\\&= (2-p)|a|^2+(q-2)(a^*b+b^*a)\\&\quad+(q^2-3q+2)|b|^2.\end{aligned}$$

We can check that

$$\begin{pmatrix}2-p & q-2\\ q-2 & q^2-3q+2\end{pmatrix}\geqslant 0,$$

which means (8) holds. The equality case in (8) is obtained via Lemma 2. Repeating the above procedure yields that the equality holds if and only if

$$(q-2)a+(q^2-3q+2)b \;=\; 0,$$

which is $a=(1-q)b$, i.e., $b=(1-p)a$. □

Theorem 3 *Let $a,b\in\mathcal{A}$ and p,q real numbers such that $1/p+1/q=1$. If $1<p\leqslant 2$, then*

$$|a-b|^2+|(p-1)a+b|^2 \;\leqslant\; p|a|^2+q|b|^2, \quad (10)$$
$$|a-b|^2+|a+(q-1)b|^2 \;\geqslant\; p|a|^2+q|b|^2, \quad (11)$$

with equality if and only if $p=q=2$ or $b=(1-p)a$.

Proof: The proof of (11) is similar to that of (10). As in (9)

$$\begin{aligned}|a-b|^2&+|(p-1)a+b|^2-p|a|^2-q|b|^2\\&=(p^2-3p+2)|a|^2+(p-2)(a^*b+b^*a)\\&\quad+(2-q)|b|^2.\end{aligned}$$

Let us define

$$X:=\begin{pmatrix}p^2-3p+2 & p-2\\ p-2 & 2-q\end{pmatrix}.$$

The condition $1<p\leqslant 2$ is equivalent to $q\geqslant 2$ which implies $p^2-3p+2\leqslant 0$ and $2-q\leqslant 0$. Note that

$$(p^2-3p+2)(2-q)=(p-2)(p-1)(2-q)=(p-2)^2.$$

Since $p>1$ and $1/p+1/q=1$ we have $p+q\geqslant 4$. We can check that $X\geqslant 0$. So (10) holds by Lemma 1. From Lemma 2 the equality in (10) is valid if and only if one of the following holds:

(i) $a=b=0$,

(ii) $p^2-3p+2=2-q=0$,

(iii) $a=0$ and $2-q=0$,

(iv) $b=0$ and $p^2-3p+2=0$,

(v) $(p^2-3p+2)a+(p-2)b=0$ and $(p^2-3p+2)(2-q)=(p-2)^2\neq 0$.

It is easy to see that (i)–(v) can be reduced to $p=q=2$ or $b=(1-p)a$. □

Theorem 4 *Let $a,b\in\mathcal{A}$ and let p,q be real numbers such that $1/p+1/q=1$. If $p>2$, then*

$$|a-b|^2+|(p-1)a+b|^2 \;\geqslant\; p|a|^2+q|b|^2, \quad (12)$$
$$|a-b|^2+|a+(q-1)b|^2 \;\leqslant\; p|a|^2+q|b|^2, \quad (13)$$

with equality if and only if $b=(1-p)a$ (i.e., $a=(1-q)b$).

Proof: Observe that for $p,q\in\mathbb{R}$ such that $1/p+1/q=1$, the condition $1<p<2$ is equivalent to $q>2$. Hence the theorem is proved by swapping a with b and p with q in Theorem 3. □

Remark 3 The Bohr inequality can be obtained from (10) and (13).

SOME RELATED IDENTITIES AND INEQUALITIES IN C*-ALGEBRAS

In this section some identities and inequalities of Bohr type for multiple elements are established. The parallelogram identity for operator algebras appeared in Ref. 11. In the context of C*-algebras we also have this identity:

$$|a+b|^2+|a-b|^2 \;=\; 2|a|^2+2|b|^2. \quad (14)$$

Let us generalize this identity to multiple elements.

Theorem 5 *For $a_i\in\mathcal{A}$ $(i=0,1,2,\dots,n)$, we have*

$$\sum_{\sigma(a_i)}\left|a_0+\sum_{i=1}^{n}\sigma(a_i)\right|^2 \;=\; 2^n\sum_{i=0}^{n}|a_i|^2 \quad (15)$$

where the outer summation is taken over all 2^n permutations $\sigma(a_i)$ of $\{a_i,-a_i\}$.

Proof: We use an induction on n. The case $n=1$ is just the parallelogram law (14). Suppose that (15) is valid for $n=k$ for some integer $k\geqslant 1$. Replacing a_k

with $a_k + a_{k+1}$ and $a_k - a_{k+1}$ yields

$$\begin{aligned}\sum_{\sigma(a_i)} \quad & \left|a_0 + \sum_{i=1}^{k-1} \sigma(a_i) + \sigma(a_k + a_{k+1})\right|^2 \\ &= 2^k \left[\sum_{i=0}^{k-1} |a_i|^2 + |a_k + a_{k+1}|^2\right], \\ \sum_{\sigma(a_i)} \quad & \left|a_0 + \sum_{i=1}^{k-1} \sigma(a_i) + \sigma(a_k - a_{k+1})\right|^2 \\ &= 2^k \left[\sum_{i=0}^{k-1} |a_i|^2 + |a_k - a_{k+1}|^2\right],\end{aligned}$$

respectively. Since $\sigma(a_k) + \sigma(a_{k+1})$ is the combination between $\sigma(a_k + a_{k+1})$ and $\sigma(a_k - a_{k+1})$, it follows from summing these two inequalities that

$$\begin{aligned}\sum_{\sigma(a_i)} & \left|a_0 + \sum_{i=1}^{k-1} \sigma(a_i) + \sigma(a_k) + \sigma(a_{k+1})\right|^2 \\ &= 2 \cdot 2^k \sum_{i=0}^{k-1} |a_i|^2 + 2^k |a_k + a_{k+1}|^2 \\ &\quad + 2^k |a_k - a_{k+1}|^2 \\ &= 2 \cdot 2^k \sum_{i=0}^{k-1} |a_i|^2 + 2 \cdot 2^k |a_k|^2 \\ &\quad + 2 \cdot 2^k |a_{k+1}|^2 \qquad \text{(by (14))} \\ &= 2^{k+1} \sum_{i=0}^{k+1} |a_i|^2.\end{aligned}$$

Hence (15) holds for $n = k + 1$ and the theorem is proved. □

Lemma 3 *For $a_i \in \mathcal{A}$ $(i = 1, 2, \ldots, n)$ we have the following identities:*

$$\left|\sum_{i=1}^{n} a_i\right|^2 = \sum_{i=1}^{n} |a_i|^2 + \sum_{1 \leqslant i < j \leqslant n} a_i^* a_j + a_j^* a_i \quad (16)$$

$$\begin{aligned}\sum_{1 \leqslant i < j \leqslant n} |a_i - a_j|^2 &= (n-1) \sum_{i=1}^{n} |a_i|^2 \\ &\quad - \sum_{1 \leqslant i < j \leqslant n} a_i^* a_j + a_j^* a_i. \quad (17)\end{aligned}$$

Proof: The proof is done by expanding. □

Corollary 3 *For $a_i \in \mathcal{A}$ $(i = 1, 2, \ldots, n)$ the inequality*

$$\left|\sum_{i=1}^{n} a_i\right|^2 \leqslant n \sum_{i=1}^{n} |a_i|^2 \quad (18)$$

holds and the equality holds if and only if all the a_i are equal.

Proof: The identities (16) and (17) imply

$$\begin{aligned} & n \sum_{i=1}^{n} |a_i|^2 - \left|\sum_{i=1}^{n} a_i\right|^2 \\ &= n \sum_{i=1}^{n} |a_i|^2 - \sum_{i=1}^{n} |a_i|^2 - \sum_{1 \leqslant i < j \leqslant n} a_i^* a_j + a_j^* a_i \\ &= (n-1) \sum_{i=1}^{n} |a_i|^2 - \sum_{1 \leqslant i < j \leqslant n} a_i^* a_j + a_j^* a_i \\ &= \sum_{1 \leqslant i < j \leqslant n} |a_i - a_j|^2 \\ &\geqslant 0.\end{aligned}$$

Hence we arrive at (18). The equality in (18) occurs if and only if $a_i - a_j = 0$ for $i \neq j$ which is $a_i = a_j$ for $i \neq j$. □

Theorem 6 *Let $n > 1$ be an integer and $a_i \in \mathcal{A}$ for each $i = 1, 2, \ldots, n$. The following identity holds:*

$$\begin{aligned} & \left|\sum_{i=1}^{n} a_i\right|^2 - \left(\sum_{i=1}^{n} |a_i|\right)^2 \\ &= \sum_{1 \leqslant i < j \leqslant n} \left(|a_i + a_j|^2 - (|a_i| + |a_j|)^2\right). \quad (19)\end{aligned}$$

Proof: We shall use an induction on n. The case $n = 2$ is trivially true. Suppose that (19) holds for $n = k$ for some integer $k > 1$. For convenience write

$$A_k = \sum_{i=1}^{k} a_i, B_k = \sum_{i=1}^{k} |a_i|, X = |A_{k+1}|^2 - B_{k+1}^2.$$

Then

$$\begin{aligned} X &= |A_k|^2 + {A_k}^* a_{k+1} + a_{k+1}^* A_k - {B_k}^2 \\ &\quad - {B_k}^* |a_{k+1}| - |a_{k+1}|^* B_k \\ &= {A_k}^* a_{k+1} + a_{k+1}^* A_k - B_k |a_{k+1}| - |a_{k+1}| B_k \\ &\quad + \sum_{1 \leqslant i < j \leqslant k} \left(|a_i + a_j|^2 - (|a_i| + |a_j|)^2\right) \\ &= \sum_{i=1}^{k} (a_i^* a_{k+1} + a_{k+1}^* a_i - |a_i||a_{k+1}| - |a_{k+1}||a_i|) \\ &\quad + \sum_{1 \leqslant i < j \leqslant k} \left(|a_i + a_j|^2 - (|a_i| + |a_j|)^2\right).\end{aligned}$$

It follows that

$$X = \sum_{1\leqslant i<j\leqslant k} \left(|a_i + a_j|^2 - (|a_i| + |a_j|)^2\right) + \sum_{i=1}^{k} \left[|a_i + a_{k+1}|^2 - (|a_i| + |a_{k+1}|)^2\right] = \sum_{1\leqslant i<j\leqslant k+1} \left(|a_i + a_j|^2 - (|a_i| + |a_j|)^2\right).$$

Hence (19) holds for all $n > 1$. □

Remark 4 For each $i = 1, 2, \ldots, n$, let $a_i, b_i \in \mathcal{A}$ and let $p_i, q_i \in \mathbb{R}$ be such that $1/p_i + 1/q_i = 1$. It is easy to see that if $p_i, q_i > 1$ for all i then

$$\sum_{i=1}^{n} |a_i + b_i|^2 \leqslant \sum_{i=1}^{n} \left(p_i|a_i|^2 + q_i|b_i|^2\right). \tag{20}$$

and the inequality is reversed if $p_i, q_i < 1$ for all i. We can verify that all equalities hold if and only if $b_i = (p_i - 1)a_i$ for all i.

The next theorem is an extension of the Bohr inequality for multiple elements. This result generalizes Theorem 4 in Ref. 11.

Theorem 7 *For any integer $n > 2$ let $a_i \in \mathcal{A}$ $(i = 1, 2, \ldots, n)$ and $p_{ij}, q_{ij} \in \mathbb{R}$ such that $1/p_{ij} + 1/q_{ij} = 1$ for $1 \leqslant i < j \leqslant n$.*

(i) If $p_{ij} > 1$ for all $1 \leqslant i < j \leqslant n$ then

$$\left|\sum_{i=1}^{n} a_i\right|^2 \leqslant \left(\sum_{j=2}^{n} p_{1j} + 2 - n\right)|a_1|^2 + \sum_{k=2}^{n-1}\left(\sum_{j=k+1}^{n} p_{kj} + \sum_{j=1}^{k-1} q_{jk} + 2 - n\right)|a_k|^2 + \left(\sum_{j=1}^{n-1} q_{jn} + 2 - n\right)|a_n|^2. \tag{21}$$

(ii) If $p_{ij} < 1$ for all $1 \leqslant i < j \leqslant n$ then the reverse inequality of (21) is obtained.

All equalities hold if and only if $a_j = (p_{ij} - 1)a_i$ for all $1 \leqslant i < j \leqslant n$.

Proof: We shall prove only (i) since the proof of (ii) is similar to that of (i). From (16) in Lemma 3 we have

$$\left|\sum_{i=1}^{n} a_i\right|^2 - \sum_{i=1}^{n} |a_i|^2 = \sum_{1\leqslant i<j\leqslant n} (a_i^* a_j + a_j^* a_i) = \sum_{1\leqslant i<j\leqslant n} \left[|a_i + a_j|^2 - (|a_i|^2 + |a_j|^2)\right].$$

The Bohr inequality and Remark 4 yield

$$\sum_{1\leqslant i<j\leqslant n} \left[|a_i + a_j|^2 - (|a_i|^2 + |a_j|^2)\right] \leqslant \sum_{1\leqslant i<j\leqslant n} \left[(p_{ij} - 1)|a_i|^2 + (q_{ij} - 1)|a_j|^2\right]$$

with equality if and only if $(p_{ij} - 1)a_i = a_j$ for all $1 \leqslant i < j \leqslant n$. Let $\tilde{p}_{ij} = p_{ij} - 1$ and $\tilde{q}_{ij} = q_{ij} - 1$ for each i, j. Then

$$\begin{aligned}
\left|\sum_{i=1}^{n} a_i\right|^2 &\leqslant \sum_{i=1}^{n} |a_i|^2 + \sum_{1\leqslant i<j\leqslant n} [\tilde{p}_{ij}|a_i|^2 + \tilde{q}_{ij}|a_j|^2] \\
&= |a_1|^2 + \sum_{j=2}^{n} \tilde{p}_{1j}|a_1|^2 + \sum_{j=1}^{n-1} \tilde{q}_{jn}|a_n|^2 + |a_n|^2 \\
&\quad + \sum_{k=2}^{n-1}\left(1 + \sum_{j=k+1}^{n} \tilde{p}_{kj} + \sum_{j=1}^{k-1} \tilde{q}_{jk}\right)|a_k|^2 \\
&= \left(1 + \sum_{j=2}^{n} \tilde{p}_{1j}\right)|a_1|^2 + \left(1 + \sum_{j=1}^{n-1}(q_{jn} - 1)\right)|a_n|^2 \\
&\quad + \sum_{k=2}^{n-1}\left(\sum_{j=k+1}^{n} p_{kj} + \sum_{j=1}^{k-1} q_{jk} + 2 - n\right)|a_k|^2 \\
&= \left(\sum_{j=2}^{n} p_{1j} + 2 - n\right)|a_1|^2 + \left(\sum_{j=1}^{n-1} q_{jn} + 2 - n\right)|a_n|^2 \\
&\quad + \sum_{k=2}^{n-1}\left(\sum_{j=k+1}^{n} p_{kj} + \sum_{j=1}^{k-1} q_{jk} + 2 - n\right)|a_k|^2,
\end{aligned}$$

with equality if and only if $(p_{ij} - 1)a_i = a_j$ for all $1 \leqslant i < j \leqslant n$. □

Remark 5 Corollary 3 can be obtained from Theorem 7 by setting $p_{ij} = q_{ij} = 2$ for all $1 \leqslant i < j \leqslant n$.

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