RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2006.32.241

Robust Discrete-Time Feedback Error Learning

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Received 3 Jun 2005
Accepted 29 Mar 2006

ABSTRACT: In this study, a theoretical foundation and stability analysis on the Discrete-Time Feedback Error Learning (DTFEL) method is discussed. The motivation to propose this method is to develop it as a ready-to-use digital controller. However, it is presently available only for controlling a stable and stably invertible plant. This limitation is an obstacle to apply this method to the real systems where most plants are non-invertible. This study proposes a method to relax this constraint. This extension is based on the addition of a prefilter. Additionally, same as another type of adaptive controlled systems, DTFEL alone is very sensitive to the system disturbance and it may cause the system to be unstable. This robustness problem can be solved by integrating the Anti-Fluctuator(AF) to DTFEL system. The analysis shows how the system fluctuation is removed. Some numerical simulation results are given to illustrate the effectiveness of the method.

KEYWORDS: Discrete-time system, Feedback Error Learning, strictly positive realness, learning control, Adaptive control, feedback and feedforward control, anti-fluctuator.

INTRODUCTION

Neuroscience and systems theory play complementary roles in understanding the mechanisms of adaptive systems. Neuroscientists are faced with complex, high performance adaptive systems and try to understand why they work so nicely. Systems theorists tend to start from simple, idealized systems but try to prove rigorously how they perform under well-defined conditions. Doya and his group[1] introduced some examples of converging efforts from both sides towards understanding and building adaptive autonomous systems, and aim to promote future collaboration between the neuroscience and systems theory communities.

The biological motor system is acceptably considered as an ideal realization of control. It consists of actuators, sensors and controllers, like usual control systems do. Unlike artificial control systems, however, it exhibits much higher performance with great flexibility and versatility in spite of the nonlinearity, uncertainties and large degrees of freedom of animal bodies.

Kawato and his group[2] proposed a novel architecture of the brain motor control called Feedback Error Learning (FEL) method which is a two degree of freedom control system and consists of an adaptive feedforward controller and a fixed feedback controller. For linear time-invariant systems, the stability of this scheme was theoretically discussed by many researchers[1-3].

However, all the presented theoretical results are for continuous-time systems. They must be modified before implementation to the real systems. This is due to the fact that a today controller becomes computer-based. The transformation from continuous-time system to the discrete-time one may create some problems in the stability of the system or lead to poor-tracking performance.

The motivation of this study is to propose the new theoretical control knowledge for the discrete systems that are able to directly applied to the real controller.

This study aims to establish a new theoretical ground and improvement of the Discrete-Time Feedback Error Learning (DTFEL) method. In the first part of this paper, the mathematical knowledge which is useful for analyzing the DTFEL system, is briefly discussed. Then, the stability of the DTFEL system is analyzed. This analysis is based on the strict positive realness, under the assumption that the plant is stable and stably invertible. However, the extension to the noninvertible cases is consequently discussed. In this study, the extension is proposed by integrating the prefilter to the system. After that, the robustness problems due to the system disturbance of DTFEL are discussed. This study suggests the solution to this problem by integrating the Anti-Fluctuator(AF) to DTFEL system. The analysis of the integrated system is also explained. Finally, the simulation results is demonstrated to illustrate the effectiveness of the method from this study.

Notations

Throughout this study, a fairly standard notation is used. The overview is as follow.

$\gamma_{\min}[P]$ the smallest eigenvalue of P.

$\| A \| = \sqrt{tr(A^T A)} = \sqrt{\sum_{i,j} a_{ij}^2}$ the Frobenius norm.

$(A, B, C, D) = D + C(zI - A)^{-1} B$ a minimal realization.
$H^*(z)$ the complex conjugate transpose of $H(z)$.
p.r. positive real.
s.p.r. strictly positive real.
PE persistently exciting.

Mathematical Preliminaries

In this section, the mathematical requirements to analyze the DTFEL method in the next section are discussed. The main and most important area is to study the strictly positive real system.

Definition 1 [4] A square matrix $H(z)$ of real rational functions is a positive real (p.r.) matrix if

(d1) $H(z)$ has elements analytic in $|z| > 1$.

(d2) $H^*(z) + H(z)$ is positive, semidefinite and Hermitian for $|z| > 1$.

Condition *(d2)* can be replaced by

(d3) The poles of the elements of $H(z)$ on $|z| = 1$ are simple and the associated residue matrices of $H(z)$ at these poles are 0.

(d4) $H(e^{j\theta}) + H^T(e^{-j\theta})$ is a positive semidefinite Hermitian matrix for all real θ for which $H(e^{j\theta})$ exists.

Definition 2 [4] A rational transfer matrix $H(z)$ is a strictly positive real (s.p.r.) matrix if $H(\rho z)$ is p.r. for some $0 < \rho < 1$.

Given Definition 2, a necessary and sufficient condition in the frequency domain for s.p.r. transfer matrices in the class $\aleph$ can be defined as following.

Definition 3 [4] An $n \times n$ rational matrix $H(z)$ is said to belong to class $\aleph$ if $H(z) + H^T(z^{-1})$ has rank n almost everywhere in the complex z-plane.

Theorem 1 [4] Consider the $n \times n$ rational matrix $H(z) \in \aleph$ given in Definition 3. Then, $H(z)$ is a s.p.r. matrix if and only if

(a) All elements of $H(z)$ are analytic in $|z| \geq 1$,

(b) $H(e^{j\theta}) + H^T(e^{-j\theta}) > 0, \forall \theta \in [0, 2\pi]$.

Lemma 1 (Discrete-time version of Kalman-Yakubovich-Popov) [4] Assume that the rational transfer matrix $H(z)$ has poles that lie in $|z| < \gamma$, where $0 < \gamma < 1$ and (A, B, C, D) is a minimal realization of $H(z)$. Then, $H(\gamma z)$ is s.p.r., if and only if real matrices $P = P^T > 0$, Q and K exist such that

$$A^T PA - P = -QQ^T - (1-\gamma^2)P,$$
$$APB = C^T - QK,$$
$$K^T K = D + D^T - B^T PB.$$

Remark

If $L(z)$ is a stable transfer function, for a given constant α, there exists sufficiently large K such that $\frac{\alpha}{K}\left(L(z)+K\right)^{-1}$ is s.p.r..

Consider the linear discrete-time varying system given by

$$x(k+1) = A(k)x(k) + B(k)u(k),$$
$$y(k) = C(k)x(k), \qquad (1)$$

with $A(k)$, $B(k)$ and $C(k)$ being appropriately dimensioned matrices.

Lemma 2 [5] Define $\psi(k_1,k_0)$ as the state-transition matrix corresponding to $A(k)$ for the system (1), i.e. $\psi(k_1,k_0) = \prod_{k=k_0}^{k_1-1} A(k)$. Then, if $\|\psi(k_1,k_0)\| \leq 1$, $\forall k_1, k_0 \geq 0$, the system (1) is exponentially stable.

Lemma 3 [5] If $A(k) = I - \beta\phi(k)\phi^T(k)$ in System (1), where $0 < \beta < 2$ and $\phi(k)$ is a regressor vector of past inputs and outputs, then $\|\phi(k_1,k_0)\| < 1$ is guaranteed if there is an $L > 0$ such that $\sum_{k=k_0}^{k_1+L-1} \phi(k)\phi^T(k) > 0$ for all k. Then, Lemma 2 guarantees the exponential stability of the system (1).

Definition 4 [5] An input sequence $x(k)$ is said to be persistently exciting (PE) if there exist $\gamma > 0$ and an integer $k_1 \geq 1$ such that

$$\gamma_{\min}\left[\sum_{k=k_0}^{k_1+L-1} \phi(k)\phi^T(k)\right] > \gamma, \forall k_0 \geq 0. \qquad (2)$$

Note: PE is exactly the stability condition needed in Lemma 3.

Theorem 2 A difference equation

$$z(k+1) = \left(I - \xi(k)L(z)\xi^T(k)\right)z(k) \qquad (3)$$

is asymptotically stable for any time-varying vector $x(k)$ which satisfies the PE condition, if $L(z)$ is s.p.r..

Proof

To prove this theorem, consider the following discrete-time state-space equation of a scalar pulse-transfer function

$$L(z) = \frac{Y(z)}{U(z)} = c^T\left(zI - A\right)^{-1}b + d,$$

$$x(k+1) = Ax(k) + bu(k),$$
$$y(k) = c^T x(k) + du(k).$$

By using this state-space equation form, the difference equation in Equation (3) can then be represented as

$$x(k+1) = Ax(k) + b\xi^T z(k), \quad (4)$$

$$y(k) = c^T x(k) + d\xi^T z(k), \quad (5)$$

$$z(k+1) = z(k) - \xi^T y. \quad (6)$$

The transfer function can be calculated as

$$\frac{y(k)}{z(k)} = d\xi^T + c^T(zI - A)b\xi^T$$
$$= (d + c^T(zI - A)b)\xi^T$$
$$= L\xi^T$$
$$y(k) = L\xi^T z(k)$$
$$z(k) - \xi y(k) = z(k) - \xi L\xi^T z(k)$$
$$= z(k+1)$$
$$z(k+1) = z(k) - \xi y(k)$$

Consider a Lyapunov function

$$V(k) = x^T(k)Px(k) + ||z(k)||^2. \quad (7)$$

From the assumption and Lemma 1

$$\Delta V(k) = V(k+1) - V(k)$$
$$\leq 0 \text{ if } d^T \; ||z||^2 \geq ||y||^2. \quad (8)$$

From Equations (7) and (8), $x(k)$ and $\xi^T(k)z(k)$ converge to 0. From this result and Equations (4)–(6), for sufficiently large k,

$$z(k+1) = z(k) - d\xi(k)\xi^T(k)z(k). \quad (9)$$

Since $L(z)$ is s.p.r., then, $d > 0$. From the assumption that $\xi(k)$ satisfies the PE condition in Equation (2), then, due to Lemma 1, Equation (9) is asymptotically stable. This implies that $z(k)$ converges to 0. Hence, Theorem 2 has been proved.

Note that a special case of Theorem 2 where $L(z) = 1$ corresponds to Equation (3).

The requirement in Equation (8) can be translated as "the direct input-output transmission gain d is positive and sufficiently large". This clarify the essential differences between continuous-time and discrete-time cases. This is a special feature of discrete-time systems which makes the requirement relatively complicated. The similar requirements are frequently occurred in many discrete-time control systems literatures[6,7].

Analysis of the Discrete-Time Feedback Error Learning

Feedforward adaptive control method without feedback element

The discussion of the feedback error learning method (henceforth, it is simply referred as Kawato scheme), from the viewpoint of adaptive control, is the main objective of this section. Figure 1 illustrates the block diagram of Kawato scheme. In this scheme, the feedforward controller K_2 is chosen to be identical to the inverse P^{-1} of P if P is known. Since P is unknown, some adaptive schemes for K_2 are employed so that K_2 converges to P^{-1}.

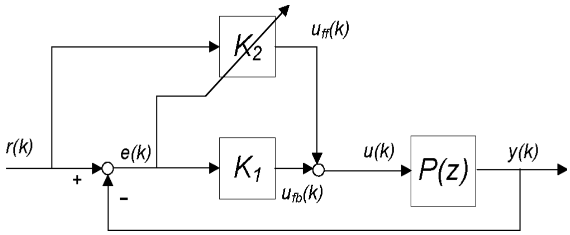

Fig 1. Discrete-time feedback error learning scheme.

Throughout this section, the following assumptions are applied:

Assumptions

(A1) The plant, P, is stable and has stable inverse P^{-1}.

(A2) The upper bound of the order of P is known.

(A3) $l_0 = \lim_{z\to\infty} P(z)$ is assumed to be positive.

(A4) Input signal is bounded and satisfies the PE condition.

The assumption (A1) is rather restrictive in the context of control system design. This may be relaxed without significant difficulty, but in this study, this assumption is kept in order to focus on the intrinsic nature of the Kawato scheme. In the context of motor control, this assumption is not restrictive because the plant is always a neuro-muscular system with low order. This lets the computed torque method, which is essentially equivalent to constructing an inverse model, be applicable.

If l_0 is negative in (A3), the subsequent results are valid by taking $-P(z)$ instead of $P(z)$. Hence, (A3) is relaxed to the assumption that the sign of the high frequency gain is known. For the sake of the simplicity of exposition, however, (A3) is retained. From the assumption (A4), it is obvious that $\xi(k)$ also satisfies PE condition.

Parameterization of unknown systems

To handle adaptation, it is important to decide how to parameterize the adaptive system. Throughout this study, the following parameterization of the unknown system Q is utilized:

$$\xi_1(k+1) = F\xi_1(k) + gr(k) \tag{10}$$

$$\xi_2(k+1) = F\xi_2(k) + gu(k) \tag{11}$$

$$u(k) = c^T(k)\xi_1(k) + d^T(k)\xi_2(k) + l(k)r(k) \tag{12}$$

where F is any stable matrix and g is any vector with $\{F,g\}$ being controllable. In Equations (10)-(12), $c(k)$, $d(k)$ and $l(k)$ are unknown parameters to be estimated. $u(k)$ and $r(k)$ are the output and the desired output of this system, respectively. It is easy to see that appropriate selection of parameters $c(k) = c_0$, $d(k) = d_0$ and $l(k) = l_0$ can yield an arbitrary transfer function from $r(k)$ to $u(k)$.

To see this, let the matrix F and vector g be in a controllable canonical form:

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ -f_1 & -f_2 & -f_3 & \ldots & -f_n \end{bmatrix}, g = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{13}$$

From (10), (11) , and (12), the transfer function from $r(k)$ to $u(k)$ is given by

$$\begin{aligned} \frac{U(z)}{R(z)} &= \left(\frac{U - d^T\xi_2}{R}\right)\left(\frac{U}{U - d^T\xi_2}\right) \\ &= \left(\frac{U - d^T\xi_2}{R}\right)\left(\frac{1}{1 - d^T\dfrac{\xi_2}{U}}\right) \\ &= \frac{l_0 + c_0^T(zI - F)^{-1}g}{1 - d_0^T(zI - F)^{-1}g} \\ &= \frac{l_o z^n + (f_n l_o + c_n)z^{n-1} + \ldots + (f_1 l_o + c_1)}{z^n + (f_n - d_n)z^{n-1} + \ldots + (f_1 - d_1)}, \end{aligned} \tag{14}$$

Therefore, any transfer function of degree less than or equal to n can be constructed by selecting parameters c_0, d_0 and l_0 appropriately. The advantage of the parameterization (10)-(12) is that the unknown parameters enter linearly in the system description. The continuous version of this parameterization was firstly used in adaptive observer[8].

Adaptation law

The same parameterization of the adaptive feedforward controller K_2 as in Equations (10)-(12) is taken.

$$\xi_1(k+1) = F\xi_1(k) + gr(k) \tag{15}$$

$$\xi_2(k+1) = F\xi_2(k) + gu(k) \tag{16}$$

$$u_{ff}(k) = c^T(k)\xi_1(k) + d^T(k)\xi_2(k) + l(k)r(k) \tag{17}$$

$$u(k) = u_{ff}(k) + K_1 e(k), \tag{18}$$

where F is stable and $\{F,g\}$ is controllable and $e(k)$ is the error signal defined as

$$e(k) = r(k) - u(k).$$

In the ideal situation, K_2 is identical to P^{-1}. In that case, $e(k) = 0$, $u(k) = u_{ff}(k) = P^{-1}(z)r(k)$. The true values c_0, d_0 and l_0 of $c(k)$, $d(k)$ and $l(k)$, respectively, satisfy

$$\frac{l_0 + c_0^T(zI - F)^{-1}g}{1 - d_0^T(zI - F)^{-1}g} = P^{-1}(z) \tag{19}$$

as given in Equation (14).

The cost function for adaptation is defined as

$$J(k) = \frac{1}{2}\sum_{i=0}^{k} e^2(i). \tag{20}$$

The unknown parameters $c(k)$, $d(k)$ and $l(k)$ must be updated so that the error signal $e(k)$ decreases.

The vector $\xi(k)$ is defined as,

$$\xi(k) := [\xi_1(k)^T \quad \xi_2(k)^T \quad r(k)]^T. \tag{21}$$

The usual gradient method gives rise to the updating rule. Then, the adaptation law of parameters is obtained as

$$\theta(k) := [c(k)^T \quad d(k)^T \quad l(k)]^T, \tag{22}$$

$$\theta(k+1) = \theta(k) + \frac{\alpha}{K_1} e(k)\xi(k), \tag{23}$$

where α is a adaptive gain. Note: This is adapted from the continuous-time adaptation algorithm by using gradient method presented by Miyamura[9].

Define the desired control input $u_d(k)$ as

$$u_d(k) = P^{-1}(z)r(k).$$

The adaptation of DTFEL is finally be writen as

$$\xi_1(k+1) = F\xi_1(k) + gr(k) \tag{24}$$

$$\xi_2(k+1) = F\xi_2(k) + g\left(u_d(k) - P^{-1}(z)e(k)\right) \tag{25}$$

$$u_{ff}(k) = c^T(k)\xi_1(k) + d^T(k)\xi_2(k) + l(k)r(k) \tag{26}$$

$$\left.\begin{aligned} c(k+1) &= c(k)+\frac{\alpha}{K_1}e(k)\xi_1(k), \\ d(k+1) &= d(k)+\frac{\alpha}{K_1}e(k)\xi_2(k), \\ l(k+1) &= l(k)+\frac{\alpha}{K_1}e(k)r(k). \end{aligned}\right\} \quad (27)$$

By using such the above parameterization algorithm and adaptation law, together with some control theorems proved previously, the convergence of DTFEL system can be proved. The following fundamental result was established[10].

Theorem 3. Under the assumptions (A1)-(A4), the feedback error learning method (24)-(27) is converging, i.e. the controller K_2 converges to $P^{-1}(z)$

DTFEL for general plants

For the traditional DTFEL, the controlling plant is assumed to be stable and stably invertible. In practice, almost all plants are not in that case. However, there are many method to relax this assumption.

Feedback adaptive control method for the non-invertible plant case

In the previous sections, it is assumed that the plant $P(z)$ has a stable inverse $P^{-1}(z)$. But most plants do not have stable inverses. This section considers the case where $P(z)$ is strictly proper, i.e. $P(z)$ has positive relative degree. This section also proposes a method to deal with this problem by introducing a prefilter $W(z)$.

When the plant $P(z)$ does not have a stable inverse, an approximated inverse P_a^{-1} is introduced as,

$$P(z)P_a^{-1}(z) = W(z) \quad (28)$$

$$P_a^{-1}(z) = P^{-1}(z)W(z) \quad (29)$$

Using this approximation, the relative degree of $P(z)$, which is the cause of non-invertibility, is compensated by the relative degree of $W(z)$.

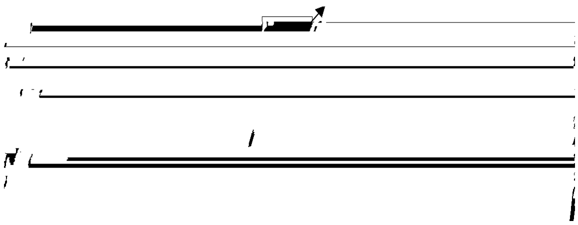

Fig 2. The system block diagram described by DTFEL with prefilter.

Then, consider the system illustrated in Figure 2.

This section aims to construct $P_a^{-1}(z) = W(z)P^{-1}(z)$ as a feedforward controller by the scheme of the feedback error learning method. In the other word, an adaptive scheme, of the case when a part of the adapted controller is known, is proposed.

Throughout this section, the following assumptions are made:

Assumptions

1. The plant P is stable.
2. The upper bound of the order of P is known.
3. $l_w = \lim_{z\to\infty} P(z)$ is assumed to be positive.
4. Prefilter $W(z)$ is given and known.
5. The upper bound of relative degree of P is known.

The parameterization of the unknown system for feedforward controller K_2 is the same as the stable case. With that parameterization, any transfer function of degree less than or equal to n,

$$K_2(z) = \frac{U(z)}{R(z)} = \frac{l_o z^n + (f_n l_o + c_n)z^{n-1} + \ldots + (f_1 l_o + c_1)}{z^n + (f_n - d_n)z^{n-1} + \ldots + (f_1 - d_1)}, \quad (30)$$

can be constructed by selecting appropriate parameters.

Since $W(z)$ is known dynamics, parameter in K_2 are subjected to some constraints. In other word, because of the information from $W(z)$, the dimension of unknown parameters is reduced.

To show the constraints in the case of relative degree 1 which is corresponded to the concerned servo plant system, $P(z)$ would be written generally as,

$$P(z) = \frac{b_1 z^{n-1} + b_2 z^{n-2} + \ldots + b_n}{z^n + a_1 z^{n-1} + \ldots + a_n}. \quad (31)$$

Select a prefilter with relative degree 1 as

$$W(z) = \frac{v_o}{z + w_1} \quad (32)$$

where w_1 and v_0 are known.

Since $P^{-1}(z)W(z)$ is represented by Equation (14), then

$$\begin{aligned} K(z) &= \frac{U(z)}{R(z)} \\ &= \frac{U(z)}{R(z)} = \frac{l_o z^n + (f_n l_o + c_n)z^{n-1} + \ldots + (f_1 l_o + c_1)}{z^n + (f_n - d_n)z^{n-1} + \ldots + (f_1 - d_1)} \\ &= \frac{v_o}{z + w_1} \cdot \frac{z^n + a_1 z^{n-1} + \ldots + a_n}{b_1 z^{n-1} + b_2 z^{n-2} + \ldots + b_n} \end{aligned} \quad (33)$$

Comparing the coefficients of both sides, the following equations are obtained

$$f_1 - d_1 = \frac{b_n}{b_1} w_1$$
$$f_2 - d_2 = \frac{b_{n-1}}{b_1} w_1 + \frac{b_n}{b_1}$$
$$\vdots$$
$$f_{n-1} - d_{n-1} = \frac{b_2}{b_1} w_1 + \frac{b_3}{b_1} \quad (34)$$
$$f_n - d_n = w_1 + \frac{b_2}{b_1}$$

From these relations, it is easy to derive the relation

$$\sum_{j=0}^{n-1} (-w_1)^j (d_{j+1} - f_{j+1}) = (-w_1)^n \quad (35)$$

This relation is written as

$$[h_o \quad h_1 \quad h_2 \quad \ldots \quad h_{n-1}] \cdot [d(k) - f] = h_n \quad (36)$$

where h_j are defined recursively as

$$h_o = 1$$
$$h_{i+1} = -w_1 h_i$$
$$h_m = 0, \, m < 0$$

where f and d are defined as

$$f := \begin{bmatrix} f_1 & f_2 & f_3 & \ldots & f_n \end{bmatrix}^T$$
$$d := \begin{bmatrix} d_1 & d_2 & d_3 & \ldots & d_n \end{bmatrix}^T$$

The relationship written as Equation (36) tells that one element of $d(k)$ is determined by other elements of $d(k)$, i.e. there exists a function $\xi_1(d_2, d_3, \ldots, d_n)$ such as

$$d_1(k) = \xi_1(d_2, d_3, \ldots, d_n). \quad (37)$$

So in this case of $rd[P(z)] = 1$, the number of free parameters decreases with one, i.e. the dimensions in which parameters can move decreases with one.

For the general case that the plant $P(z)$ has relative degree k ($k \le n$), $P(z)$ would be written generally as,

$$P(z) = \frac{b_k z^{n-k} + b_{k+1} z^{n-k-1} + \ldots + b_n}{z^n + a_1 z^{n-1} + \ldots + a_n}. \quad (38)$$

Select a prefilter with relative degree k as

$$W(z) = \frac{w_{k+1}}{z^k + w_1 z^{k-1} + \ldots + w_k} \quad (w_i, i = 1, 2, \ldots, k+1 : \text{known}). \quad (39)$$

Assume that $P^{-1}(z)W(z)$ is represented as

$$\xi_1(k+1) = F\xi_1(k) + gr(k) \quad (40)$$
$$\xi_2(k+1) = F\xi_2(k) + gu(k) \quad (41)$$
$$u(k) = c_w^T \xi_1(k) + d_w^T \xi_2(k) + l_w r(k), \quad (42)$$

where F is given as before. Then, $P^{-1}(z)W(z)$ is written as

$$P^{-1}W(z) = \frac{l_w + c_w^T (zI - F)^{-1} g}{1 - d_w^T (zI - F)^{-1} g}$$
$$= \frac{l_w z^n + (f_n l_w + c_{w,n}) z^{n-1} + \ldots + (f_1 l_w + c_{w,1})}{z^n + (f_n - d_{w,n}) z^{n-1} + \ldots + (f_1 - d_{w,1})}, \quad (43)$$

where

$$c_w = \begin{bmatrix} c_{w,1} & c_{w,2} & \cdots & c_{w,n} \end{bmatrix}^T, \quad d_w = \begin{bmatrix} d_{w,1} & d_{w,2} & \cdots & d_{w,n} \end{bmatrix}^T.$$

Hence, c_w, d_w and l_w must satisfy the identity

$$\frac{l_w z^n + (f_n l_w + c_{w,n}) z^{n-1} + \ldots + (f_1 l_w + c_{w,1})}{z^n + (f_n - d_{w,n}) z^{n-1} + \ldots + (f_1 - d_{w,1})}$$
$$= \frac{b_k z^{n-k} + b_{k+1} z^{n-k-1} + \ldots + b_n}{z^n + a_1 z^{n-1} + \ldots + a_n} \cdot \frac{w_{k+1}}{z^k + w_1 z^{k-1} + \ldots + w_k} \quad (44)$$

Let $f := [f_1 \; f_2 \; f_3 \; \ldots \; f_n]^T$. The above identity yields the relation

$$f - d_w = \begin{bmatrix} w_k & 0 & \cdots & 0 \\ w_{k-1} & w_k & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ w_1 & w_2 & \cdots & 0 \\ 1 & w_1 & \cdots & w_k \\ 0 & 1 & \cdots & w_{k-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_{n-k+1}/b_1 \\ b_{n-k}/b_1 \\ \vdots \\ b_2/b_1 \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ w_k \\ w_{k-1} \\ \vdots \\ w_2 \\ w_1 \end{bmatrix}. \quad (45)$$

Let h_i, $i = 0, \ldots, n-1$ be a sequence of solutions of a difference equation

$$h_i + h_{i-1} w_1 + h_{i-2} w_2 + \cdots + h_{i-k} w_k = 0. \quad (46)$$

Using (46), we have

$$\begin{bmatrix} h_0 & h_1 & h_2 & \cdots & h_{n-1} \end{bmatrix} . \begin{bmatrix} f - d_w \end{bmatrix} = w_1 h_{n-1} + w_2 h_{n-2} + \cdots + w_k h_{n-k}. \quad (47)$$

The difference equation (46) has independent solutions $h_i^{(j)}, j = 1, \ldots, k; i = 0, \ldots, n-k$ as

$$h_i^{(j)} = -w_1 h_{i-1}^{(j)} - w_2 h_{i-2}^{(j)} - \cdots - w_k h_{i-k}^{(j)}, \quad i \ge k,$$
$$h_i^{(j)} = \begin{cases} 0, & i \ne j-1, \quad i \le k-1, \\ 1, & i = j-1. \end{cases}$$

Thus, the following equation is obtained

$$\begin{bmatrix} h_0^{(1)} & h_1^{(1)} & \cdots & h_{n-1}^{(1)} \\ h_0^{(2)} & h_1^{(2)} & \cdots & h_{n-1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ h_0^{(k)} & h_1^{(k)} & \cdots & h_{n-1}^{(k)} \end{bmatrix} [f - d_w]$$

$$= \begin{bmatrix} h_{n-k+1}^{(1)} & h_{n-k+2}^{(1)} & \cdots & h_{n-1}^{(1)} \\ h_{n-k+1}^{(2)} & h_{n-k+2}^{(2)} & \cdots & h_{n-1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ h_{n-k+1}^{(k)} & h_{n-k+2}^{(k)} & \cdots & h_{n-1}^{(k)} \end{bmatrix} \begin{bmatrix} w_k \\ w_{k-1} \\ \vdots \\ 1 \end{bmatrix}. \tag{48}$$

Actually, from the selection of $h_i^{(j)}$, the relation (48) implies

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & h_k^{(1)} & h_{k+1}^{(1)} & \cdots & h_{n-1}^{(1)} \\ 0 & 1 & \ddots & 0 & h_k^{(2)} & h_{k+1}^{(2)} & \cdots & h_{n-1}^{(2)} \\ \cdots & \ddots & \ddots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & 1 & h_k^{(k)} & h_{k+1}^{(k)} & \cdots & h_{n-1}^{(k)} \end{bmatrix} [f - d_w]$$

$$= \begin{bmatrix} h_{n-k+1}^{(1)} & h_{n-k+2}^{(1)} & \cdots & h_{n-1}^{(1)} \\ h_{n-k+1}^{(2)} & h_{n-k+2}^{(2)} & \cdots & h_{n-1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ h_{n-k+1}^{(k)} & h_{n-k+2}^{(k)} & \cdots & h_{n-1}^{(k)} \end{bmatrix} \begin{bmatrix} w_k \\ w_{k-1} \\ \vdots \\ 1 \end{bmatrix}. \tag{49}$$

Using this relation, we can represent $d_1, d_2, \ldots, d_k$ as affine functions of the rest $(n - k)$ parameters $d_{k+1}, \ldots, d_n$. More precisely, we have affine relations

$$\bar{d}(k) = M\hat{d}(k) + m,$$

where M is a known matrix, m is a known vector and

$$\bar{d} = [d_1 \quad d_2 \quad \cdots \quad d_n]^T, \quad \hat{d} = [d_{k+1} \quad d_{k+2} \quad \cdots \quad d_n]^T$$

The parameters $d_1, d_2, \ldots, d_k$ are determined once $d_{k+1}, \ldots, d_n$ are given. Hence, it is sufficient to estimate $(n - k)$ unknowns $\hat{d}$ for estimating d.

Adaptation law

Using the result of the previous section, we construct an adaptation law. From Figure 2, the error signal $e(k)$ is defined as

$$e(k) = W(z)r(k) - y(k).$$

The unknown parameters $c(k)$, $d(k)$ and $l(k)$ must be updated so that the error signal $e(k)$ decreases. Let

$$\bar{d}(k) = [d_1 \quad d_2 \quad \cdots \quad d_n]^T = M\hat{d}(k) + m, \tag{50}$$

$$\hat{d}_w(k) = [d_{w,k+1} \quad d_{w,k+2} \quad \cdots \quad d_{w,n}]^T,$$

$$\xi_2(k) = [\xi_{21} \quad \xi_{22} \quad \cdots \quad \xi_{2n}]^T, \quad \bar{\xi}_2(k) = [\xi_{21} \quad \xi_{22} \quad \cdots \quad \xi_{2k}]^T,$$

$$\hat{\xi}_2(k) = [\xi_{2(k+1)} \quad \xi_{2(k+2)} \quad \cdots \quad \xi_{2n}]^T,$$

$$\hat{\theta}(k) = [c^T(k) \quad \hat{d}^T(k) \quad l(k)]^T, \quad \hat{\theta}_w(k) = [c_w^T \quad \hat{d}_w^{\,T} \quad l_w]^T,$$

Note that the dimension of the unknown vector $\theta(k)$ is now $2n - k$ instead of $2n$ as in the previous section. The output of $Q(\hat{\theta})$ given by (12) is written as

$$u(k) = c^T(k)\xi_1(k) + \bar{d}^T(k)\bar{\xi}_2(k) + \hat{d}^T(k)\hat{\xi}_2(k) + l(k)r(k)$$

$$= c^T(k)\xi_1(k) + \hat{d}^T(k)\left(M^T\bar{\xi}_2(k) + \hat{\xi}_2(k)\right) + m^T\bar{\xi}_2(k) + l(k)r(k).$$

As in the invertible case, we use the same adaptation law (23), which can be written as

$$\theta(k+1) = \theta(k) + \frac{\alpha}{K_1}\begin{bmatrix} \xi_1(k) \\ M^T\bar{\xi}_2(k) + \hat{\xi}_2(k) \\ r(k) \end{bmatrix} e(k). \tag{51}$$

Convergence Proof

The error signal can be rewritten as

$$e(k) = W(z)r(k) - P(z)u(k).$$

Hence,

$$u(k) = u_d(k) - P^{-1}(z)e(k),$$

$$u_d(k) = P^{-1}(z)W(z)r(k).$$

Then, the adaptive controller is written as

$$\xi_1(k+1) = F\xi_1(k) + gr(k) \tag{52}$$

$$\xi_2(k+1) = F\xi_2(k) + g\left(u_d(k) - P^{-1}(z)e(k)\right) \tag{53}$$

$$u_{ff}(k) = c^T(k)\xi_1(k) + d^T(k)\xi_2(k) + l(k)r(k) \tag{54}$$

$$\left.\begin{aligned} c(k+1) &= c(k) + \frac{\alpha}{K_1}e(k)\xi_1(k), \\ d(k+1) &= d(k) + \frac{\alpha}{K_1}e(k)\xi_2(k), \\ l(k+1) &= l(k) + \frac{\alpha}{K_1}e(k)r(k). \end{aligned}\right\} \tag{55}$$

Assume that the true system $P^{-1}(z)W(z)$ is written as

$$z_1(k+1) = Fz_1(k) + gr(k), \tag{56}$$

$$z_2(k+1) = Fz_2(k) + gu_d(k), \tag{57}$$

$$u_d(k) = c_w^T z_1(k) + d_w^T z_2(k) + l_w r(k). \tag{58}$$

Then,

$$u_{ff}(k) - u_d(k) = \left(c(k) - c_w\right)^T\xi_1(k) + \left(d(k) - d_w\right)^T\xi_2(k) + \left(l(k) - l_w\right)r(k) - d_w^T\left(zI - F\right)^{-1}gP^{-1}(z)e(k). \tag{59}$$

Here, the following asymptotic relations are used

$$\xi_1(k) \to z_1(k)$$
$$\xi_2(k) \to z_2(k) - d_w^T \left(zI - F\right)^{-1} g P^{-1}(z) e(k).$$

The relation (59) is written as

$$u_{ff}(k) - u_d(k) = \hat{\psi}(k)^T \xi(k) - d_w^T \left(zI - F\right)^{-1} g P^{-1}(z) e(k),$$

where

$$\hat{\psi}(k) := \hat{\theta}(k) - \hat{\theta}_w. \tag{60}$$

From the relations

$$\begin{aligned} u(k) &= u_{ff}(k) + K_1 e(k), \\ &-\left[P^{-1}(z)e(k) + K_1 e(k)\right] \\ &= \psi(k)^T \xi(k) - d_w^T \left(zI - F\right)^{-1} g P^{-1}(z) e(k), \end{aligned} \tag{61}$$

which results in

$$\left(G(z) + K_1(z)\right) e(k) = \hat{\psi}(k)^T \hat{\xi}(k), \tag{62}$$

where

$$G(z) := \left(1 - c_w^T \left(zI - F\right)^{-1} g_w\right) P^{-1}(z).$$

On the other hand, from (48),

$$\hat{\psi}(k+1) - \hat{\psi}(k) = \hat{\theta}(k+1) - \hat{\theta}(k) = \frac{\alpha}{K_1} \hat{\xi}(k) e(k), \tag{63}$$

where

$$\hat{\xi}(k) = \begin{bmatrix} \xi_1(k) \\ M^T \bar{\xi}_2(k) + \hat{\xi}_2(k) \\ r(k) \end{bmatrix}.$$

It should be noted that the relation (48) implies that

$$G(z) = \left(k_w + c_w^T \left(zI - F\right)^{-1} g\right) W^{-1}(z). \tag{64}$$

Combining (62) with (63) yields

$$\begin{aligned} \hat{\psi}(k+1) - \hat{\psi}(k) &= \frac{\alpha}{K_1} \hat{\xi}(k) e(k) \\ &= \frac{\alpha}{K_1} \hat{\xi}(k) \left(G(z) + K_1\right)^{-1} \hat{\xi}(k)^T \hat{\psi}(k) \\ \psi(k+1) &= \psi(k) + \frac{\alpha}{K_1} \xi(k) \left(G(z) + K_1\right)^{-1} \xi(k)^T \psi(k) \\ \psi(k+1) &= \left(I - \xi(k) L_1(z) \xi(k)^T\right) \psi(k), \end{aligned} \tag{65}$$

where

$$L_1(z) := \left(G(z) + K_1\right)^{-1} \frac{\alpha}{K_1}. \tag{66}$$

which is the same form as (3). According to Theorem 2, the difference equation (65) is asymptotically stable, if $L_1(z)$ given by (66) is s.p.r., K_1 is chosen such that $G(z) + K_1$ is s.p.r. Such K_1 always exists from Definition 2 of s.p.r. (See Remark following Lemma 1). If $G(z) + K_1$ is s.p.r., so is $L_1(z)$. Thus, the following fundamental result has been established.

Theorem 4. Under the assumptions (A1w)-(A5w), the feedback error learning method is converging, i.e. the controller K_2 converges to $W(z)P^{-1}(z)$.

Robust Discrete-Time Feedback Error Learning (RDTFEL)

Although DTFEL provides a near perfect tracking to a given input signal, same as another adaptive control systems, it is very sensitive to noise. Therefore, improving DTFEL robustness is the main interest of control problems. To develop such a characteristic, the additions of anti-fluctuator is needed. The ability of anti-fluctuator to smoothen the vigourous vibration from the internal disturbance is a very useful candidate.

Anti-Fluctuator (AF)

The job of Anti-Fluctuator (AF) is to suppress the internal disturbance of DTFEL. The internal noise occur frequently from temperature-sensitive components in the motor or in circuitries. AF, initially proposed by K. Ohishi, et al.[11], is an extension to the control system that aims to cancel or reduce the effect of the disturbance at the current iteration in advance. This can be done by extracting the value of disturbance from the previous iteration and subtract it with the input signal to the plant, P. Therefore, when the disturbance occurs in a plant, the extracted disturbance from the previous iteration cancels the new disturbance.

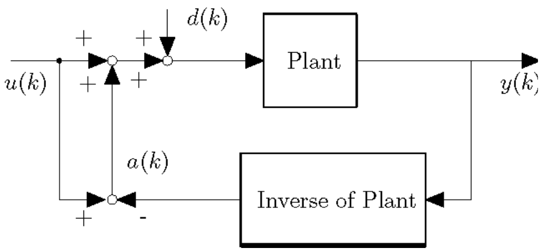

Fig 3. Block Diagram of an Anti Fluctuator.

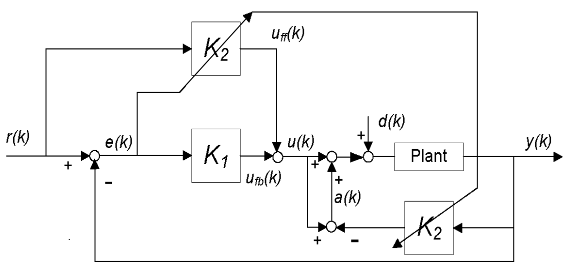

Fig 4. Addition of Anti Fluctuator to FEL Diagram.

Mathematical Analysis

The traditional DTFEL scheme is shown in Figure 1. The idea behinds this control method is to learn and adapt $K_2(z)$ to the real inverse of the plant ($P^{-1}(z)$). The diagram of the implementation of anti-fluctuator to DTFEL is shown in Figure 3. The process of extraction involves the plant output as shown in the following mathematical derivation.

$$Y(z)=P(z)[U(z)+D(z)+A(z)] \quad (67)$$

$$A(z)=-P^{-1}(z)Y(z)+U(z)A(z) \quad (68)$$

Substitute (68) into (67),

$$Y(z)=P(z)\left[U(z)+D(z)+\left(-P^{-1}(z)Y(z)+U(z)\right)\right]$$
$$Y(z)=P(z)\left[2U(z)+D(z)-P^{-1}(z)Y(z)\right]$$
$$Y(z)=2P(z)U(z)+P(z)D(z)-Y(z)$$
$$2Y(z)=2P(z)U(z)+P(z)D(z).$$
$$Y(z)=P(z)\left(U(z)+\frac{D(z)}{2}\right).$$

It is obviously seen that AF can reduce the effect of the disturbance to the output by comparing this to the case without AF, where the relationship between output and disturbance is

$$Y(z)=P(z)(U(z)+D(z)).$$

In another words, AF increases the system robustness. Note that AF can be modified by increasing the gain of the signal passing through the inverse of plant in feedback path.

Remark for special case of zero-input condition $U(z) = 0$, the relationship between output and disturbance of the system with AF becomes

$$\frac{Y(z)}{D(z)}=\frac{1}{2}P(z),$$

while that of system without AF is

$$\frac{Y(z)}{D(z)}=P(z).$$

Systems Integration

To integrated AF to DTFEL systems, the use of learning of inverse of a plant, in DTFEL feedforward controller, is applied. That is the inverse of plant in AF is identical to feedforward controller in DTFEL as shown in Figure 4.

Results and Discussion

In this section, the simulation results are illustrated to demonstrate the effectiveness of the theoretical results obtained in this study. Three main simulations have been done in order to guarantee the mathematical analysis.

In the first part, the simulation of the traditional DTFEL with stable and stably invertible plant is demonstrated. The pulse-transfer function of the controlling plant is

$$P(z)=\frac{z+0.2}{z+0.3}.$$

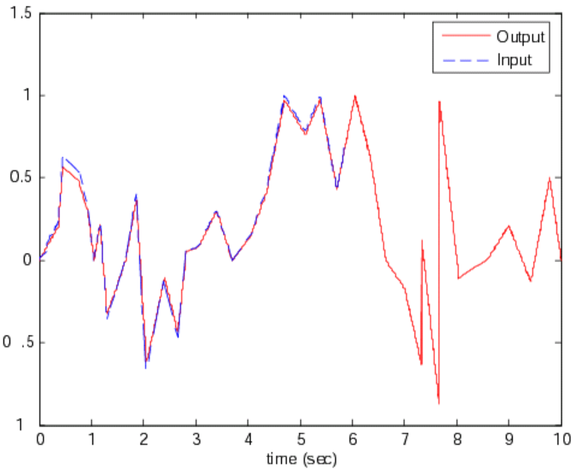

Fig 5. The result of the DTFEL system.

In Figure 5, the tracking performance between the input signal $r(k)$ and the output signal $y(k)$ is shown. The input is represented by a solid line and the output is represented by a dashed line. This figure show the convergence of the signal and the comparison of the tracking performance of the system before adaptation, from 0 second to about 5.7 seconds, with after adaptation, from 5.7 seconds to step 10 seconds. It should be noted that the pulses in the error between 7 seconds and 8 seconds can be considered as the unusual performance of the input. It is interested that the system still be stable. Note also that the learning rate is set to be very low to show the result clearly. In fact, the adaptation rate is very fast.

For the second part, the DTFEL is extended to control the strictly proper plant with a relative degree of 1. The pulse-transfer function of the controlling plant is

$$P(z)=\frac{1}{z+0.3},$$

and the chosen filter is

$$W(z)=\frac{1}{z+0.2}.$$

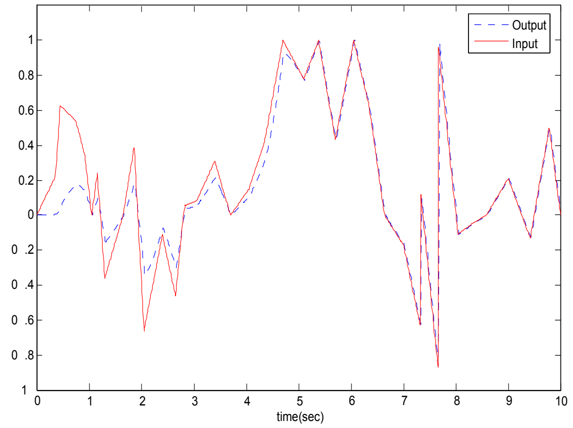

Fig 6. The simulation result of the DTFEL system when the relative degree of Plant is 1.

In Figure 6, the tracking performance between the input signal $r(k)$ and the output signal $y(k)$ is shown. This figure shows the convergence of the signal and the comparison of the tracking performance of the system before adaptation, from 0 to about 5.7 seconds, and after adaptation, from 5.7 to 10 seconds. The learning rate is also set to be very low to show the result clearly.

Finally, the simulation demonstrates the improved controlled system of RDTFEL from the DTFEL alone. The simulation results of DTFEL in two cases are presented in this section; DTFEL with disturbance, and DTFEL with AF (RDTFEL) with disturbance. Each simulation has the same input and the same constant. The result of DTFEL with disturbance depicted by Figure 7 confirms that DTFEL performs poorly under disturbance, as the fluctuated disturbance perturbs the input to the extent that the input is indistinguishable.

The result of the controlled system improved by RDTFEL is shown in Figure 8. Although fluctuation is still obvious, the result of the proposed version of DTFEL has shown a significant improvement in robustness.

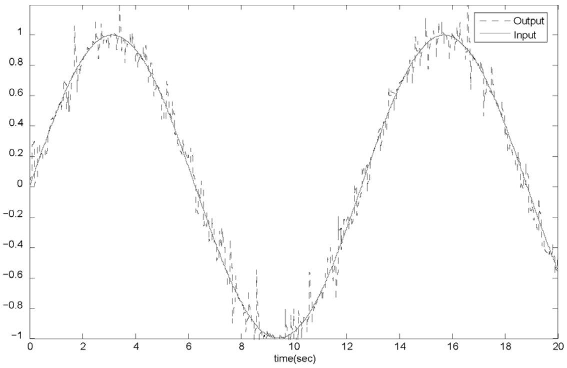

Fig 7. Performance of DTFEL with Disturbance.

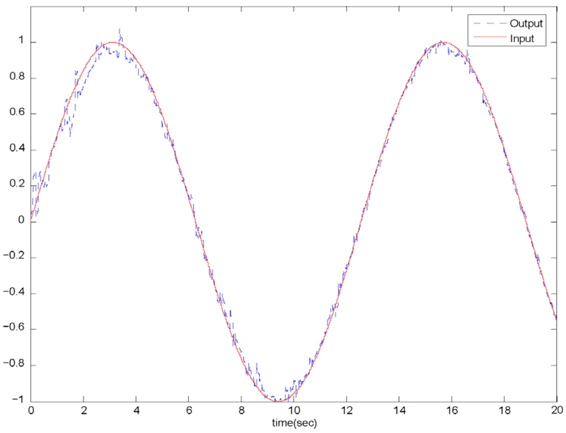

Fig 8. Performance of Robust DTFEL with Disturbance.

Conclusion

In this study, the "Robust Discrete-Time Feedback Error Learning" (RDTFEL) is demonstrated. The mathematics required to analyze the stability of the DTFEL system where the controlling has stable inverse, are studied and proved. Not only restricted to stable and stably invertible plant cases, DTFEL can also be applied to more general systems. This study shows how filter makes DTFEL applicable to control a non-proper plant.

DTFEL proves to be efficient, but its robustness needs to be improved. This study suggests the method for robustness enhancement by providing an additional anti-fluctuator, which makes DTFEL more practical in industries where disturbance is inevitable. The development solves the problem of internal disturbance, from the ability of anti-fluctuator and also smoothen the output signal from DTFEL.

All numerical simulation results using MATLAB® guarantee the mathematical analysis in this study.

Many possible future researches are available. The stability analysis of DTFEL for the plant with time-delay will be the future works. Also, the implementation to the real systems will be done.

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