RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2006.32.021

ScienceAsia 32 (2006): 21-24

On the Generalized Ultra-hyperbolic Heat Kernel Related to the Spectrum

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Received 11 Apr 2005
Accepted 20 Jul 2005

ABSTRACT: In this paper, we study the equation

$$\frac{\partial}{\partial t}u(x,t)=c^2\,\square^k u(x,t)$$

with the initial condition $u(x,0)=f(x)$ for $x\in\mathbb{R}^n$ - the n -dimensional Euclidean space. The operator $\square^k$ is named the ultra-hyperbolic operator iterated k -times, defined by

$$\square^k=\left(\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_p^2}-\frac{\partial^2}{\partial x_{p+1}^2}-\cdots-\frac{\partial^2}{\partial x_{p+q}^2}\right)^k$$

$p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n, u(x,t)$ is an unknown function for $(x,t)=(x_1,\ldots,x_n,t)\in\mathbb{R}^n\times(0,\infty), f(x)$ is a positive integer, and c is a positive constant.

We obtain the solution of such equation which is related to the spectrum and the kernel which is so called the generalized ultra-hyperbolic heat kernel.

Moreover, such the generalized ultra-hyperbolic heat kernel has interesting properties and also related to the the kernel of an extension of the heat equation.

KEYWORDS: ultra-hyperbolic, heat kernel, spectrum.

INTRODUCTION

It is well known that for the heat equation

$$\frac{\partial}{\partial t}u(x,t)=c^2\Delta u(x,t) \tag{1.1}$$

with the initial condition $u(x,0)=f(x)$ where $\Delta=\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_n^2}$ is the Laplacian operator and $(x,t)=(x_1,\ldots,x_n,t)\in\mathbb{R}^n\times(0,\infty)$, we obtain the solution

$$u(x,t)=\frac{1}{(4c^2\pi t)^{n/2}}\int_{\mathbb{R}^n}f(y)e^{-|x-y|^2/4c^2t}dy$$

or the solution in the convolution form

$$u(x,t)=E(x,t)*f(x) \tag{1.2}$$

where

$$E(x,t)=\frac{1}{(4c^2\pi t)^{n/2}}e^{-|x|^2/4c^2t} \tag{1.3}$$

and the symbol $*$ designates as the convolution.

The equation (1.3) is called the heat kernel, where $|x|^2=x_1^2+x_2^2+\cdots+x_n^2$ and $t>0$, see [ref. 2, p208-209].

Moreover, we obtain $E(x,t)\to\delta$ as $t\to 0$ where δ is the Dirac-delta distribution.

We can extend (1.1) to the equation

$$\frac{\partial}{\partial t}u(x,t)=c^2\,\square u(x,t) \tag{1.4}$$

with the initial condition

$$u(x,0)=f(x) \tag{1.5}$$

where $\square$ is the ultra-hyperbolic operator, that is

$$\square=\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_p^2}-\frac{\partial^2}{\partial x_{p+1}^2}-\cdots-\frac{\partial^2}{\partial x_{p+q}^2}.$$

Then we obtain

$$u(x,t)=E(x,t)*f(x) \tag{1.6}$$

as a solution of (1.4) which satisfies (1.5) where $E(x,t)$ is the kernel of (1.4) and is defined by

$$E(x,t)=\frac{(i)^q}{(4c^2\pi t)^{n/2}}\exp\left[-\frac{\sum_{i=1}^{p}x_i^2-\sum_{j=p+1}^{p+q}x_j^2}{4c^2t}\right] \tag{1.7}$$

where $p+q=n, i=\sqrt{-1}$ and $\sum_{i=1}^{p}x_i^2>\sum_{j=p+1}^{p+q}x_j^2$, see [ref.1, pp. 215-225].

Moreover, we obtain $E(x,t)\to\delta$ as $t\to 0$,where δ is the Dirac-delta distribution. In addition, we studied the ultra-hyperbolic heat kernel which is related to the spectrum, see [ref. 2, pp. 19-28].

Now, the purpose of this work is to study the equation

$$\frac{\partial}{\partial t}u(x,t) = c^2 \square^k u(x,t) \qquad (1.8)$$

with the initial condition

$$u(x,0) = f(x), \text{ for } x \in \mathbb{R}^n \qquad (1.9)$$

where the operator $\square^k$ is named the ultra-hyperbolic operator iterated k-times defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k, \qquad (1.10)$$

$p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n$, $u(x,t)$ is an unknown function for $(x,t)=(x_1,\dots,x_n,t)\in\mathbb{R}^n\times(0,\infty)$, $f(x)$ is the given generalized function, k is a positive integer and c is a positive constant.

We obtain $u(x,t)=E(x,t)*f(x)$ as a solution of (1.8) which satisfies (1.9) where

$$E(x,t) = \frac{1}{(2\pi)^{n/2}}\int_\Omega \exp\left[c^2 t\left(\sum_{j=p+1}^{p+q}\xi_j^2 - \sum_{i=1}^{p}\xi_i^2\right)^k + i(x,\xi)\right]d\xi \qquad (1.11)$$

and $\Omega\subset\mathbb{R}^n$ is the spectrum of $E(x,t)$ for any fixed $t>0$. The function $E(x,t)$ is called *the generalized ultra-hyperbolic heat kernel* iterated k-times or the elementary solution of (1.8). And all properties of $E(x,t)$ will be studied in details.

Now, if we put $k=1$ and $q=0$ in (1.8) and (1.11) then (1.8) and (1.11) reduce to (1.1) and (1.3) respectively.

Preliminaries

Definition 2.1 *We say* $f\in L^1(\mathbb{R}^n)$ *if*

$$\int_{\mathbb{R}^n}|f(x)|\,dx<\infty.$$

For $f\in L^1(\mathbb{R}^n)$, we define its Fourier transform at a point $\xi\in\mathbb{R}^n$ as

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-i(\xi,x)}f(x)dx \qquad (2.1)$$

where $\xi=(\xi_1,\xi_2,\dots,\xi_n)$ and $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, $(\xi,x)=\xi_1x_1+\xi_2x_2+\cdots+\xi_nx_n$ and $dx=dx_1\dots dx_n$.

Also, the inverse of Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}\hat{f}(\xi)d\xi. \qquad (2.2)$$

Definition 2.2 *The spectrum of the kernel* $E(x,t)$ *of* (1.11) *is the bounded support of the Fourier transform* $\widehat{E(\xi,t)}$ *for any fixed* $t>0$.

Definition 2.3 *Let* $\xi=(\xi_1,\xi_2,\dots,\xi_n)\in\mathbb{R}^n$ *and denote by*

$$\Gamma_+ = \left\{\xi\in\mathbb{R}^n : \xi_1^2+\cdots+\xi_p^2-\xi_{p+1}^2-\cdots-\xi_{p+q}^2>0 \text{ and } \xi_1>0\right\}$$

the set of an interior of the forward cone, and $\overline{\Gamma}_+$ denotes the closure of Γ_+.

Let Ω be spectrum of $E(x,t)$ defined by definition 2.2 and $\Omega\subset\overline{\Gamma}_+$. Let $\widehat{E(\xi,t)}$ be the Fourier transform of $E(x,t)$ *and define*

$$\widehat{E(\xi,t)} = \begin{cases}\frac{1}{(2\pi)^{n/2}}\exp\left[c^2t\left(\sum_{j=p+1}^{p+q}\xi_j^2-\sum_{i=1}^{p}\xi_i^2\right)^k\right] & \text{for } \xi\in\Omega\\ 0 & \text{for } \xi\notin\Omega\end{cases} \qquad (2.3)$$

Lemma 2.4 *Let* L *be the operator defined by*

$$L = \frac{\partial}{\partial t} - c^2\square^k \qquad (2.4)$$

where $\square^k$ is the ultra-hyperbolic operator iterated k-times, defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k,$$

$p+q=n$ *is the dimension of* $\mathbb{R}^n$, *k is a positive integer,* $(x_1,\dots,x_n)\in\mathbb{R}^n$, $t\in(0,\infty)$, *and* c *is a positive constant. Then we obtain*

$$E(x,t) = \frac{1}{(2\pi)^{n/2}}\int_\Omega \exp\left[c^2 t\left(\sum_{j=p+1}^{p+q}\xi_j^2 - \sum_{i=1}^{p}\xi_i^2\right)^k + i(x,\xi)\right]d\xi \qquad (2.5)$$

as a elementary solution of (2.4) *in the spectrum* $\Omega\subset\mathbb{R}^n$ for $t>0$.

Proof. Let where is the kernel or the elementary solution of operator L and is the Dirac-delta distribution. Thus

$$\frac{\partial}{\partial t}E(x,t) - c^2\square^k E(x,t) = \delta(x)\delta(t).$$

Take the Fourier transform defined by (2.1) to both sides of the equation, we obtain

$$\frac{\partial}{\partial t}\widehat{E(\xi,t)} - c^2t\left(\sum_{j=p+1}^{p+q}\xi_j^2-\sum_{i=1}^{p}\xi_i^2\right)^k\widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}}\delta(t)$$

Thus

$$\widehat{E(\xi,t)} = \frac{H(t)}{(2\pi)^{n/2}}\exp\left[c^2t\left(\sum_{j=p+1}^{p+q}\xi_j^2-\sum_{i=1}^{p}\xi_i^2\right)^k\right]$$

where $H(t)$ is the Heaviside function. Since $H(t)=1$ for $t>1$. Therefore,

$$\widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}}\exp\left[c^2t\left(\sum_{j=p+1}^{p+q}\xi_j^2-\sum_{i=1}^{p}\xi_i^2\right)^k\right]$$

which has been already defined by (2.3). Thus

$$E(x,t) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}\widehat{E(\xi,t)}d\xi$$

$$= \frac{1}{(2\pi)^{n/2}} \int_\Omega e^{i(\xi,x)} \widehat{E(\xi,t)} d\xi$$

where Ω is the spectrum of $E(x,t)$. Thus from (2.3), we have

$$E(x,t) = \frac{1}{(2\pi)^{n/2}} \int_\Omega \exp\left[c^2 t\left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2\right)^k + i(x,\xi)\right] d\xi$$

for $t > 0$

Lemma 2.5 *For all* $t > 0$, *c is a positive constant and all* $x \in \mathbb{R}$ *we have*

$$\int_{-\infty}^{\infty} \exp\left(-c^2\xi^2 t\right) d\xi = \sqrt{\frac{\pi}{c^2 t}} \quad (2.6)$$

and

$$\int_{-\infty}^{\infty} \exp\left(-c^2\xi^2 t + i\xi x\right) d\xi = \sqrt{\frac{\pi}{c^2 t}} \exp\left(-\frac{x^2}{4c^2 t}\right). \quad (2.7)$$

Proof. See [ref. 4, pp.117-118].

Main Results

Theorem 3.1 *Given the equation*

$$\frac{\partial}{\partial t} u(x,t) - c^2 \square^k u(x,t) \quad (3.1)$$

with the initial condition

$$u(x,0) = f(x) \quad (3.2)$$

where $\square^k$ is the ultra-hyperbolic operator iterated k-times, defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k,$$

$p+q=n$ *is the dimension of Euclidean space* $\mathbb{R}^n$, *k is a positive integer,* $u(x,t)$ *is an unknown function for* $(x,t) = (x_1,\ldots,x_n,t) \in \mathbb{R}^n \times (0,\infty)$, $f(x)$ *is the given generalized function*, and c is a positive constant. Then we obtain $u(x,t) = E(x,t) * f(x)$ (3.3) *as a solution of* (3.1) *which satisfies* (3.2) *where* $E(x,t)$ *is given by* (2.5).

Proof. Taking the Fourier transform defined by (2.1) to both sides of (3.1), we obtain

$$\frac{\partial}{\partial t} \hat{u}(\xi,t) = c^2 t\left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2\right)^k \hat{u}(\xi,t).$$

Thus

$$\hat{u}(\xi,t) = K(\xi)\exp\left[c^2 t\left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2\right)^k\right] \quad (3.4)$$

where $K(\xi)$ is constant and $\hat{u}(\xi,t) = K(\xi)$.

Now, by (3.2) we have

$$K(\xi) = \hat{u}(\xi,t) = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x) d\xi \quad (3.5)$$

and by the inversion in (2.2), (3.4) and (3.5) we obtain

$$u(x,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \hat{u}(\xi,t) d\xi$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(\xi,x)} e^{-i(\xi,y)} f(y) \times \exp\left[c^2 t\left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2\right)^k\right] d\xi dy.$$

Thus

$$u(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(\xi,x-y)} f(y) \times \exp\left[c^2 t\left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2\right)^k\right] d\xi dy$$

Or

$$u(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y) \times \exp\left[c^2 t\left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2\right)^k + i(\xi,x-y)\right] d\xi dy \quad (3.6)$$

Set

$$E(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp\left[c^2 t\left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2\right)^k + i(\xi,x)\right] d\xi. \quad (3.7)$$

Since the integral of (3.7) is divergent, therefore we choose $\Omega \subset \mathbb{R}^n$ be the spectrum of $E(x,t)$ and by (2.5), we have

$$E(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp\left[c^2 t\left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2\right)^k + i(\xi,x)\right] d\xi$$

$$= \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t\left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2\right)^k + i(\xi,x)\right] d\xi. \quad (3.8)$$

Thus (3.6) can be written in the convolution form

$$u(x,t) = E(x,t) * f(x).$$

Moreover, since $E(x,t)$ exists, then

$$\lim_{t\to 0} E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega e^{i(\xi,x)} d\xi$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(\xi,x)} d\xi$$

$$= \delta(x), \text{ for } x \in \mathbb{R}^n. \quad (3.9)$$

See [ref. 5, p396, Eq.(10.2.19b)].

Thus for the solution $u(x,t) = E(x,t) * f(x)$ of (3.1), then

$$u(x,0) = \lim_{t\to 0} u(x,t)$$

$$= \lim_{t\to 0}\left(E(x,t) * f(x)\right) = \delta * f(x) = f(x)$$

which satisfies (3.2).

In particular, if we put $k = 1$ and $q = 0$ in (3.8), then we obtain

$$E(x,t)=\frac{1}{(2\pi)^n}\int_\Omega \exp\left[-c^2t\left(\sum_{j=1}^n \xi_j^2\right)+i(\xi,x)\right]d\xi$$

$$=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\exp\left[-c^2t\sum_{j=1}^n\xi_j^2+i\sum_{j=1}^n\xi_jx_j\right]d\xi$$

$$=\frac{1}{(2\pi)^n}\prod_{j=1}^n\int_{-\infty}^{\infty}\exp\left[-c^2t\xi_i^2+i\xi_ix_i\right]d\xi$$

$$=\frac{1}{(2\pi)^n}\prod_{j=1}^n\sqrt{\frac{\pi}{c^2t}}\exp\left(-\frac{x_j^2}{4c^2t}\right) \quad \text{by (2.7).}$$

Thus

$$E(x,t)=\frac{1}{(4c^2\pi t)^{n/2}}\exp\left(-\frac{|x|^2}{4c^2t}\right),$$

since $\left(\frac{\pi}{c^2t}\right)^{n/2}\exp\left(-\frac{|x|^2}{4c^2t}\right)=\prod_{j=1}^n\sqrt{\frac{\pi}{c^2t}}\exp\left(-\frac{x_j^2}{4c^2t}\right)$

and $|x|^2=\sum_{j=1}^n x_j^2$.

Therefore, if we put $k = 1$ and $q = 0$ in (3.1) and (3.8) then (3.1) and (3.8) reduce to (1.1) and (1.3), respectively.

Theorem 3.2 *The kernel $E(x,t)$ defined by (3.8) have the following properties :*

(1) $E(x,t)\in C^\infty\left(\mathbb{R}^n\times(0,\infty)\right)$ *- the space of continuous function with infinitely differentiable.*

(2) $\left(\frac{\partial}{\partial t}-c^2\Box^k\right)E(x,)=0$ *for all* $x\in\mathbb{R}^n, t>0$.

(3) $|E(x,t)|\le\frac{2^{2-n}M(t)}{\pi^{n/2}\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}$, *for all* $x\in\mathbb{R}^n, t>0$,

where M(t) is a function of t in the spectrum Ω. Thus $E(x,t)$ is bounded for any fixed $t>0$.

(4) $\lim_{t\to 0}E(x,t)=\delta(x)$ *for all* $x\in\mathbb{R}^n$.

Proof.

(1) From (3.8), since

$$\frac{\partial^n}{\partial t^n}E(x,t)=\frac{1}{(2\pi)^n}\times\int_\Omega\frac{\partial^n}{\partial t^n}\exp\left[c^2t\left(\sum_{j=p+1}^{p+q}\xi_j^2-\sum_{i=1}^p\xi_i^2\right)^k+i(\xi,x)\right]d\xi$$

Thus $E(x,t)\in C^\infty$ for $x\in\mathbb{R}^n, t>0$.

(2) By computing directly, we obtain

$$\left(\frac{\partial}{\partial t}-c^2\Box^k\right)E(x,)=0$$

(3) We have

$$E(x,t)=\frac{1}{(2\pi)^n}\int_\Omega\exp\left[c^2t\left(\sum_{j=p+1}^{p+q}\xi_j^2-\sum_{i=1}^p\xi_i^2\right)^k+i(\xi,x)\right]d\xi$$

$$|E(x,t)|\le\frac{1}{(2\pi)^n}\int_\Omega\exp\left[c^2t\left(\sum_{j=p+1}^{p+q}\xi_j^2-\sum_{i=1}^p\xi_i^2\right)^k\right]d\xi.$$

By changing to bipolar coordinates

And $\xi_1=r\omega_1,\xi_2=r\omega_2,\ldots,\xi_p=r\omega_p$

$\xi_{p+1}=s\omega_{p+1},\xi_{p+2}=s\omega_{p+2},\ldots,\xi_{p+q}=s\omega_{p+q}$

where $\sum_{i=1}^p\omega_i^2=1$ and $\sum_{j=p+1}^{p+q}\omega_j^2=1$.

Thus

$$|E(x,t)|\le\frac{1}{(2\pi)^n}\int_\Omega\exp\left[c^2t\left(s^2-r^2\right)^k\right]r^{p-1}s^{q-1}drdsd\Omega_pd\Omega_q$$

where $d\xi=r^{p-1}s^{q-1}dr\,ds\,d\Omega_pd\Omega_q$, $d\Omega_p$ and $d\Omega_q$ are the elements of surface area of the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. Since $\Omega\subset\mathbb{R}^n$ is the spectrum of $E(x,t)$, we suppose $0\le r\le R$ and $0\le s\le T$ where R and T are constants. Thus we obtain

$$|E(x,t)|\le\frac{\Omega_p\Omega_q}{(2\pi)^n}\int_0^R\int_0^T\exp\left[c^2t\left(s^2-r^2\right)^k\right]r^{p-1}s^{q-1}dr\,ds$$

$$=\frac{\Omega_p\Omega_q}{(2\pi)^n}M(t) \quad \text{for any fixed } t>0$$

$$=\frac{2^{2-n}M(t)}{\pi^{n/2}\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}$$

where $M(t)=\int_0^R\int_0^T\exp\left[c^2t\left(s^2-r^2\right)^k\right]r^{p-1}s^{q-1}dr\,ds$

is a function of t, $\Omega_p=\frac{2\pi^{p/2}}{\Gamma\left(\frac{p}{2}\right)}$ and $\Omega_q=\frac{2\pi^{q/2}}{\Gamma\left(\frac{q}{2}\right)}$.

Thus, for any fixed $t>0$, $E(x,t)$ is bounded.

(4) Obvious by (3.9).

Acknowledgements

The first author is supported by The Royal Golden Jubilee Project Grant No. PHD/02241/2543 and Graduate School of Chiang Mai University. The authors would like to thank the Thailand Research Fund.

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