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Weakly Nonlinear Solutions of Flows Past An Applied Pressure Distribution

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ABSTRACT: Steady two-dimensional flows due to an applied pressure distribution in water of finite depth are considered. Gravity is included in the free surface condition. A unified asymptotic approach is employed to derive the forced Korteweg-de Vries equation. Existence theorems for symmetric and nonsymmetric solutions are given and proven. Numerical solutions are also provided to supplement these findings.

KEYWORDS: weakly nonlinear, pressure distribution, forced Korteweg-de Vries equation.

INTRODUCTION

The flow past an applied pressure distribution over the free surface of a horizontal stream subject to gravitational force is of considerable importance in ship hydrodynamics.The linearized case when the fluid depth is infinite was discussed in detail by Lamb[1]. In the last three decades there has been progress in the understanding of the nonlinear aspects of this problem. A fully nonlinear problem was considered and reformulated as an integral equation by Schwartz[2]. He showed that, for certain values of the Froude number, nonlinear theory anticipates drag-free solution, while linear theory does not. Vanden-Broeck and Tuck[3] found various families of free-surface pressure distributions that do not generate waves. In the case of finite depth, Von-Kerczek and Salvesen[4] found numerically nonlinear solutions of the problem by using finite difference methods. Recently, Asavanant, *et. al.*[5] reconsidered the fully nonlinear problem and solved it by using the boundary integral technique. Related studies on transient behavior and surface tension effects for the two-dimensional model can be found in Vanden-Broeck[6, 7] and Okita, *et. al.*[8]. The problem of this type can serve as a model of a moving vehicle, such as hovercraft in a canal. It may also be viewed as an inverse method of solution to the classical ship-wave problem.

The purpose of this study is to find weakly nonlinear solutions of flows past an applied pressure distribution in a fluid domain of finite depth. The third order Korteweg-de Vries equation with a forcing term is derived. It is shown that symmetric solutions exist in the supercritical flow regime whereas periodic solutions exist in the subcritical flow regime. Existence theorems are given and proven. Numerical solutions are provided as the confirmation to weakly nonlinear theory.

PROBLEM FORMULATION

Let the motion be two-dimensional, steady, and irrotational, and the fluid be inviscid with constant density. The flow domain is bounded below by a rigid bottom and above by a free surface as shown in Fig 1.

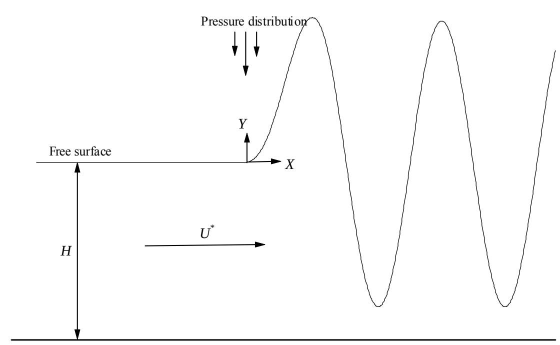


Fig 1. Typical free surface.

We choose Cartesian coordinates with the X-axis along the undisturbed free surface at infinity and the Y-axis directed vertically upwards through the symmetry line (or the midpoint) of the applied pressure distribution. Gravity is acting in the negative Y-direction. As $x^* \to -\infty$, the flow is assumed to approach a uniform stream with constant velocity U^* and constant depth H. The governing equations and boundary conditions are given by the following Euler equations :

$$u^*_{x^*} + v^*_{y^*} = 0$$

$$u^* u^*_{x^*} + v^* u^*_{y^*} = \frac{-p^*_{x^*}}{\rho}$$

$$u^* v^*_{x^*} + v^* v^*_{y^*} = \frac{-p^*_{y^*}}{\rho} - g$$

at the bottom $y^* = -H$

$v^* = 0$,

at the surface $y^* =$

$u^* \eta^*_{x^*} - v^* = 0$

$p^* = b^*(x^*)$ with compact support

where u^* and v^* are horizontal and vertical velocities, p^* is pressure, g is the gravitational acceleration, ρ is the density of the fluid and all the subscripts denote derivatives with respect to the corresponding variables. We define the following non-dimensional variables :

$$, \ p = \frac{p^*}{\rho g H}, \ (x,y) = \left(\frac{\varepsilon^{\frac{1}{2}} x^*}{H}, \frac{y^*}{H} \right),$$

$$(u,v) = \left(\frac{u^*}{\sqrt{gH}}, \frac{\varepsilon^{\frac{1}{2}} v^*}{\sqrt{gH}} \right), \ b(x) = \frac{b^*(x^*)\varepsilon^{-2}}{\rho g H}, \ \varepsilon = \left(\frac{H}{L} \right)^2 << 1$$

where H and L are horizontal and vertical length scales. In terms of this new set of variables, the governing equations and boundary conditions become

$$u_x + v_y = 0$$

$$uu_x + vu_y = -p_x$$

$$\varepsilon(uv_x + vv_y) = -p_y - 1$$

at the bottom y =-1

$v = 0$,

at the surface $y = \varepsilon\eta$

u

$p = \varepsilon^2 b(x)$ with compact support

where $b(x)$ has a compact support and $F = \frac{U^*}{\sqrt{gH}}$ represents the upstream Froude number.

Let us assume that u, v, p and η possess asymptotic expansions of the form

and put $F = 1 + \varepsilon\lambda$. After substituting the asymptotic forms of u, v, p and into the dimensionless governing equations and boundary conditions, the formal perturbation procedure is used to collect terms of like powers of ε. Without using the free surface kinematic condition, we can express approximations of u and v in terms of and its derivatives. The final relation is obtained by imposing the condition at y= . Here, we expand u and v using Taylor's approximations about y=0. This gives

$$-\left[\left(v_0 + \varepsilon v_{0y}\eta + \varepsilon^2 v_{0yy}\frac{\eta^2}{2!} + \ldots\right) + \varepsilon\left(v_1 + \varepsilon v_{1y}\eta + \varepsilon^2 v_{1yy}\frac{\eta^2}{2!} + \ldots\right)\right.$$

$$\left. + \varepsilon^2\left(v_2 + \varepsilon v_{2y}\eta + \varepsilon^2 v_{2yy}\frac{\eta^2}{2!} + \ldots\right) + \ldots\right] = 0 .$$

Collecting terms of order ε^2, we obtain

$$3\eta\eta_x - 2\lambda\eta_x + \frac{1}{3}\eta_{xxx} + b_x(x) = 0 \qquad (1)$$

This is known as the forced Korteweg-de Vries equation. In order to compare with previous work, we chose the distribution of pressure to be in the form of

$$b(x) = \begin{cases} 0 & \text{for } |x| \geq 1 \\ \in \exp\left(\frac{1}{x^2 - 1}\right) & \text{for } |x| < 1 \end{cases}$$

where $\in$ is a constant. Here $b(x)$ is a smooth function with compact support.

Based on the flow characteristics, we consider two separate cases: (supercritical) and (subcritical). Numerical solutions for both cases are obtained by using the shooting method and the Runge-Kutta method. In the case of supercritical flow ( ), our numerical results show that the flow is always symmetric with respect to the symmetry line of the pressure distribution without the presence of waves. There are two different families of solutions when . One family is a perturbed solution of uniform stream, whereas the other is a perturbation of a solitary wave. When , there exists only one family of solutions for all values of up to zero. In the case of subcritical flow ( ), a train of nonlinear waves is generated behind the applied pressure distribution while the flow upstream satisfies the radiation condition. As decreases, there are critical values of at which the wave amplitude vanishes.

Symmetric Solutions : The case of

We look for a solution of (1) such that

, j = 0, 1, 2 and $\lambda > 0$. Integrating (1)

from to x, we find

6 . (2)

It can easily be shown that the above equation is equivalent to an integral equation:

$$\eta(x) = \int_{-\infty}^{\infty} K(x,\xi)\left(\frac{9}{2}\eta^2(\xi) + 3b(\xi)\right)d\xi .$$

Here the kernel $K(x,\xi) = \dfrac{e^{-\sqrt{6\lambda}|x-\xi|}}{2\sqrt{6\lambda}}$ is the Green's function satisfying

$$6\lambda K(x,\xi) - K_{xx}(x,\xi) = \delta(x-\xi), -\infty < x,\xi < \infty .$$

We now define

$$T(\eta) = \int_{-\infty}^{\infty} K(x,\xi)\left(\frac{9}{2}\eta^2(\xi) + 3b(\xi)\right)d\xi.$$

$$\|u\| = \|u\|_\infty = \sup_{x\in\Re} |u(x)|$$

$$H = \{u \mid u \in C(\Re); \|e^{\sqrt{6\lambda}|x|}u\| < \infty\} .$$

Clearly, H is a metric space and is complete. We give another definition

$$B_M = \{u \mid u \in H, \|u\| \le M, 0 < M < \infty\}.$$

It follows from the definitions that for any $\eta \in B_M$

$$\|T(\eta)\| \le M \text{ whenever } \frac{9}{2}M + \frac{3\|b\|}{M} \le 6\lambda.$$

Since the radiation condition must be satisfied in the far field, we claim that $T(\eta)$ decays so rapidly that for $\eta \in B_M$. This can be shown for $x \ge 0$, as follows

$$\sup_{x\in\Re}|\exp(\sqrt{6\lambda}x)T(\eta)(x)| = \frac{1}{2\sqrt{6\lambda}}\left|\int_{-\infty}^{\infty}\exp(\sqrt{6\lambda}x - \sqrt{6\lambda}\,|x-\xi|)\left(\frac{9}{2}\eta^2(\xi)+3b(\xi)\right)d\xi\right|$$

$$= \frac{1}{2\sqrt{6\lambda}}\left|\int_{-\infty}^{x}\exp(\sqrt{6\lambda}\xi)\left(\frac{9}{2}\eta^2(\xi)+3b(\xi)\right)d\xi\right.$$

$$\left.+\int_{x}^{\infty}\exp(\sqrt{6\lambda}(2x-\xi))\left(\frac{9}{2}\eta^2(\xi)+3b(\xi)\right)d\xi\right|$$

$$= \frac{1}{2\sqrt{6\lambda}}\left|\int_{-\infty}^{x}\left\{\frac{9}{2}\exp(\sqrt{6\lambda}\xi - 2\sqrt{6\lambda}\,|\,\xi\,|)\left(\eta(\xi)\exp(\sqrt{6\lambda}\,|\,\xi\,|)\right)^2\right.\right.$$

$$\left.+\exp(\sqrt{6\lambda}\xi)3b(\xi)\right\}d\xi$$

$$+\int_{x}^{\infty}\left\{\frac{9}{2}\exp(\sqrt{6\lambda}(2x-\xi) - 2\sqrt{6\lambda}\,|\,\xi\,|)\left(\eta(\xi)\exp(\sqrt{6\lambda}\,|\,\xi\,|)\right)^2\right.$$

$$\left.\left.+\exp(\sqrt{6\lambda}(2x-\xi))3b(\xi)\right\}d\xi\right|$$

$$\le \left|\sup_{x\in\Re}(\eta(x)\exp(\sqrt{6\lambda}\,|\,x\,|))^2\right|\left[\int_{-\infty}^{x}\exp(-\sqrt{6\lambda}\,|\,\xi\,|)d\xi\right.$$

$$\left.+\int_{x}^{\infty}\exp(\sqrt{6\lambda}(2x-3\xi))d\xi\right]\frac{9}{4\sqrt{6\lambda}}$$

$$+\frac{1}{2\sqrt{6\lambda}}\left|\int_{-\infty}^{x}\exp(\sqrt{6\lambda}\xi)3b(\xi)d\xi + \int_{x}^{\infty}3b(\xi)\exp(\sqrt{6\lambda}(2x-\xi))d\xi\right|$$

$$\le \frac{3}{4\lambda}\left|\left(1 - \frac{\exp(-\sqrt{6\lambda}x)}{3}\right)\right|\sup_{x\in\Re}(\eta(x)\exp(\sqrt{6\lambda}\,|\,x\,|))^2$$

$$+\frac{1}{2\sqrt{6\lambda}}\int_{\operatorname{supp}(b(x))} N\exp(\sqrt{6\lambda}\xi)d\xi$$

$$< \infty ,$$

when $N =$. Since $\eta \in H$, it follows that . Similarly, we can also see that $\sup_{x<0}\exp(-\sqrt{6\lambda}x)\,|\,T(\eta)(x)\,| < \infty$. Thus, $T(\eta)$ possesses exponentially decaying behavior for $|x|$ large. We are now ready to state the existence theorem for symmetric solutions of (2).

Theorem 1: , $-\infty < x < \infty$ has a solution which decays exponentially at $|x| =$ if is sufficiently large.

Proof. $\|T(\eta_1) - T(\eta_2)\| \le \sup_{x\in\Re}\left|\frac{9}{2}\int_{-\infty}^{\infty}K(x,\xi)(\eta_1^2(\xi) - \eta_2^2(\xi))d\xi\right|$

$$\le \sup_{x\in\Re}\frac{9}{2}\int_{-\infty}^{\infty}K(x,\xi)|\eta_1 + \eta_2\|\eta_1 - \eta_2|d\xi$$

$$\le \frac{9}{6\lambda}M\|\eta_1 - \eta_2\|.$$

Hence, we can see that T is a contraction mapping if $6\lambda > \max\left\{\frac{9}{2}M + \frac{3\|b\|}{M}, 9M\right\}$ and the integral equation $\eta = T(\eta)$ has the unique solution in . Now

$$\eta_{xx} = \int_{-\infty}^{\infty}K_{xx}(x,\xi)\left(\frac{9}{2}\eta^2(\xi) + 3b(\xi)\right)d\xi$$

$$= \int_{-\infty}^{\infty}6\lambda K(x,\xi)\left(\frac{9}{2}\eta^2(\xi) + 3b(\xi)\right)d\xi - \frac{9}{2}\eta^2(x) - 3b(x)$$

$$= 6\lambda\eta(x) - \frac{9}{2}\eta^2(x) - 3b(x)$$

where $6\lambda K(x,\xi) - K_{xx}(x,\xi) = \delta(x-\xi)$. Hence $\eta \in C^2(\Re)$ and it follows from the right hand side of the above equation that $\eta \in C^3(\Re)$.

In the next section, we consider the other case, i.e., $\lambda < 0$. It is expected that the solutions in this case possess periodic behavior far downstream with an upstream radiation condition.

Unsymmetric (Periodic) Solutions : The case of

Here we consider,

$$. \qquad (3)$$

since $\lambda<0$, we put with $\lambda_0>0$. Equation (3) becomes

$$\eta_{xx}+\frac{9}{2}\eta^2+6\lambda_0\eta=-3b(x). \tag{4}$$

We look for a periodic solution to equation (4) which dies out at the far upstream and oscillates without changing its amplitude at the far downstream. That is to say, η must satisfy the followings:

(i) , i = 0, 1, 2 for $x<x_-=\inf(\operatorname{supp} b)$.

(ii) $\eta\left(x+\dfrac{2\pi}{\sqrt{6\lambda_0}}\right)=\eta(x)$ for $x>x_+=\sup(\operatorname{supp} b)$.

Without loss of generality, we assume that supp(b) $\cap\Re^-=\phi$. Subject to the above requirements, we can convert (3) into an integral equation

$$\eta(x)=\begin{cases}0 & ;x\le 0\\ -\dfrac{1}{\sqrt{6\lambda_0}}\displaystyle\int_0^x \sin\sqrt{6\lambda_0}\,(x-\xi)\left[\frac{9}{2}\eta^2(\xi)+3b(\xi)\right]d\xi & ;x>0.\end{cases}$$

We now define $\eta=S(\eta)$,

with $\|u\|=\sup_{x\in\Re}|u(x)|$.

Clearly, B is a metric space and is complete. We give another definition

$$B_N=\{u\mid u\in B,\|u\|\le N, 0<N<\infty\}$$

which is a closed subset of B.

First we want to show that the map $S(\eta)$ is bounded with periodic behavior when . We observe that, for any $\eta\in B_N$,

$$\begin{aligned}\|S(\eta)\|\le\frac{1}{\sqrt{6\lambda_0}}\sup_{x\in\Re}\Bigg\{&\left|\int_0^{x_+}\sin\sqrt{6\lambda_0}\,(x-\xi)\left[\frac{9}{2}\eta^2(\xi)+3b(\xi)\right]d\xi\right|\\
&+\left|\int_{x_+}^{x_++\frac{2\pi}{\sqrt{6\lambda_0}}}\sin\sqrt{6\lambda_0}\,(x-\xi)\left(\frac{9}{2}\eta^2(\xi)\right)d\xi\right|\\
&+\left|\int_{x_++\frac{2\pi}{\sqrt{6\lambda_0}}}^{x_++\frac{2n\pi}{\sqrt{6\lambda_0}}}\sin\sqrt{6\lambda_0}\,(x-\xi)\left(\frac{9}{2}\eta^2(\xi)\right)d\xi\right|\Bigg\} \quad \text{for some positive integer } n\\
\le\frac{1}{\sqrt{6\lambda_0}}\Bigg\{&\left\|\frac{9}{2}\eta^2+3b\right\|\sup_{x\in\Re}\int_0^{x_+}|\sin\sqrt{6\lambda_0}\,(x-\xi)|\,d\xi\\
&+\left\|\frac{9}{2}\eta^2\right\|\sup_{x\in\Re}\int_{x_+}^{x_++\frac{2\pi}{\sqrt{6\lambda_0}}}|\sin\sqrt{6\lambda_0}\,(x-\xi)|\,d\xi\Bigg\}\\
\le\frac{1}{\sqrt{6\lambda_0}}\Bigg\{&\left[\frac{9}{2}N^2+3\|b\|\right]x_++\frac{9}{2}N^2\frac{2\pi}{\sqrt{6\lambda_0}}\Bigg\}.\end{aligned}$$

That is $\|S(\eta)\|\le N$ provided that;

$$\frac{9}{2}N\left(x_++\frac{2\pi}{\sqrt{6\lambda_0}}\right)+\frac{3\|b\|}{N}x_+\le\sqrt{6\lambda_0}.$$

The periodicity of the map $S(\eta)(x)$, when , can be shown as follows.

$$\begin{aligned}S(\eta)\left(x+\frac{2\pi}{\sqrt{6\lambda_0}}\right)&=-\frac{1}{\sqrt{6\lambda_0}}\int_0^{x+\frac{2\pi}{\sqrt{6\lambda_0}}}\sin\sqrt{6\lambda_0}\left(x+\frac{2\pi}{\sqrt{6\lambda_0}}-\xi\right)\left[\frac{9}{2}\eta^2(\xi)+3b(\xi)\right]d\xi\\
&=-\frac{1}{\sqrt{6\lambda_0}}\int_0^{x+\frac{2\pi}{\sqrt{6\lambda_0}}}\sin\sqrt{6\lambda_0}\,(x-\xi)\left[\frac{9}{2}\eta^2(\xi)+3b(\xi)\right]d\xi\\
&=-\frac{1}{\sqrt{6\lambda_0}}\Bigg\{\int_0^x\sin\sqrt{6\lambda_0}\,(x-\xi)\left[\frac{9}{2}\eta^2(\xi)+3b(\xi)\right]d\xi\\
&\qquad+\int_x^{x+\frac{2\pi}{\sqrt{6\lambda_0}}}\sin\sqrt{6\lambda_0}\,(x-\xi)\left(\frac{9}{2}\eta^2(\xi)\right)d\xi\Bigg\}\\
&=S(\eta)(x)-\frac{1}{\sqrt{6\lambda_0}}\int_{-\frac{\pi}{\sqrt{6\lambda_0}}}^{\frac{\pi}{\sqrt{6\lambda_0}}}\sin\sqrt{6\lambda_0}\,(x-\xi)\left(\frac{9}{2}\eta^2(\xi)\right)d\left(\xi-x-\frac{\pi}{\sqrt{6\lambda_0}}\right)\\
&=S(\eta)(x)\end{aligned}$$

as required. To ensure the conservation of mass, we require that

and

$$\int_x^{x+\frac{2\pi}{\sqrt{6\lambda_0}}}S^2(\eta)(\varsigma)\sin(\sqrt{6\lambda_0}\,\varsigma)d\varsigma=0 \quad \text{for } x>x_+$$

and $\eta\in B_N$. It is straightforward but rather lengthy to show these results. We therefore present only the idea behind the proof. Rewriting the above integral as

$$\begin{aligned}\int_x^{x+\frac{2\pi}{\Lambda}}S^2(\eta)(\varsigma)\sin\Lambda(x_1-\varsigma)d\varsigma=&-\frac{2}{3}\int_x^{x+\frac{2\pi}{\Lambda}}\eta^2(s)\cos\Lambda(x_1-2s)\int_x^s\eta^2(t)\cos\Lambda t\,dt\,ds\\
&+\frac{2}{3}\int_x^{x+\frac{2\pi}{\Lambda}}\eta^2(s)\sin\Lambda(x_1-2s)\int_x^s\eta^2(t)\sin\Lambda t\,dt\,ds\\
&-2\int_x^{x+\frac{2\pi}{\Lambda}}\eta^2(s)\cos\Lambda x_1\int_x^s\eta^2(t)\cos\Lambda t\,dt\,ds\\
&-2\int_x^{x+\frac{2\pi}{\Lambda}}\eta^2(s)\sin\Lambda x_1\int_x^s\eta^2(t)\sin\Lambda t\,dt\,ds,\end{aligned}$$

where $\Lambda=\sqrt{6\lambda_0}$.

Subject to the requirements on η, we can approximate with its Fourier expansion,

$$\eta^2=\sum_{n=1}^{\infty}\left(E_n\sin 2n\Lambda x+F_n\cos 2n\Lambda x\right).$$

Using integration techniques and orthogonality conditions, we are led to conclude that,

$$\int_x^{x+\frac{2\pi}{\Lambda}} S^2(\eta)(\varsigma)\sin\Lambda(x_1-\varsigma)d\varsigma = 0\,.$$

The conservation of mass is justified after letting $x_1 = 0$ and $\frac{\pi}{2\Lambda}$, respectively.

At this point we can see that

$$\|S(\eta_1)-S(\eta_2)\| = \sup_{x\in\Re}\left|\frac{9}{2\sqrt{6\lambda_0}}\int_0^x \sin\sqrt{6\lambda_0}(x-\xi)(\eta_1^2(\xi)-\eta_2^2(\xi))d\xi\right|$$

$$\le \frac{9}{2\sqrt{6\lambda_0}}\sup_{x\in\Re}\left\{\left|\int_0^{x_+}\sin\sqrt{6\lambda_0}(x-\xi)(\eta_1^2(\xi)-\eta_2^2(\xi))d\xi\right|\right.$$

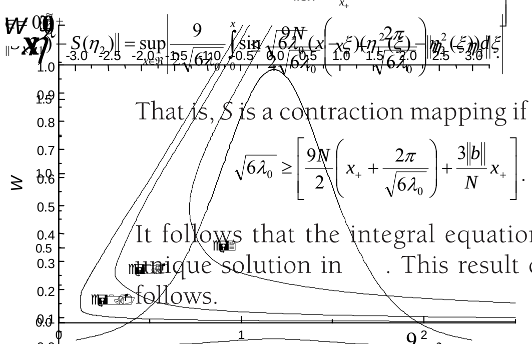
$$+\left|\int_{x_+}^{x_++\frac{2\pi}{\sqrt{6\lambda_0}}}\sin\sqrt{6\lambda_0}(x-\xi)(\eta_1^2(\xi)-\eta_2^2(\xi))d\xi\right|$$

$$\left.+\left|\int_{x_++\frac{2\pi}{\sqrt{6\lambda_0}}}^{x_++\frac{2n\pi}{\sqrt{6\lambda_0}}}\sin\sqrt{6\lambda_0}(x-\xi)(\eta_1^2(\xi)-\eta_2^2(\xi))d\xi\right|\right\}$$

for some positive integer n

$$\le \frac{9}{2\sqrt{6\lambda_0}}\|\eta_1-\eta_2\|\left[\sup_{x\in\Re}\int_0^{x_+}\left|\sin\sqrt{6\lambda_0}(x-\xi)\right|d\xi\right.$$

$$\left.+\sup_{x\in\Re}\int_{x_+}^{x_++\frac{2\pi}{\sqrt{6\lambda_0}}}\left|\sin\sqrt{6\lambda_0}(x-\xi)\right|d\xi\right]$$



That is, S is a contraction mapping if

$$\sqrt{6\lambda_0} \ge \left[\frac{9N}{2}\left(x_+ + \frac{2\pi}{\sqrt{6\lambda_0}}\right)+\frac{3\|b\|}{N}x_+\right].$$

It follows that the integral equation $\eta = S(\eta)$ has a unique solution in . This result can be stated as follows.

Theorem 2: $6\lambda_0\eta+\eta_{xx} = -\frac{9}{2}\eta^2-3b(x)$ has a solution if $\sqrt{6\lambda_0}$ is sufficiently large.

Numerical Solutions

Here we calculate numerically the solutions in both the supercritical flow regime ($\lambda > 0$) and subcritical flow regime ( ). Our numerical procedure concerns the use of the shooting method for boundary value problem when (supercritical flows) and Runge-Kutta method of 4th order when (subcritical flows). As we shall see, for , there are up to two solutions when (positive forcing) and a unique solution when (negative forcing). Unlike the supercritical solutions, there is at most one solution when regardless of the signature of .

(i) Supercritical Solutions:

We use the shooting method to compute symmetric solutions in the supercritical flow regime for various values of and . It can be seen from equation (2) that if (no pressure forcing), then uniform flow is always a solution for all values of . In addition, one can find the bifurcation of this uniform flow solution when is in the neighborhood of zero, particularly . This solution is known as a *solitary wave* which can be found directly from the unforced KdV equation. That is,

$$\eta(x) = 2\lambda\,\mathrm{sech}^2\left(\sqrt{\frac{3\lambda}{2}}x\right),\ -\infty < x < \infty\,.$$

When , the solutions are characterized by W>0. Here W measures the ratio of maximum (or minimum) elevation on the free surface profile upon which the pressure distribution is applied to the undisturbed level of the free surface. Typical free surface profiles are shown in Fig 2.

Fig 2. Typical free-surface profile for λ = 0.5 and .

In Fig 3, we present numerical values of versus W for various values of . Solutions of this type can be viewed as perturbations of the branch of solutions with that bifurcates from . On these branches of solutions, we find critical value of λ such that, for each , there are no solutions where , and two solutions when $\tilde{\lambda} < \lambda$. This is depicted in Fig 3: the critical value $\tilde{\lambda}$ is the turning point of each curve.

∈ =2

∈ =0.5

∈ =0.1

λ

Fig 3. Relationship between W and λ for various values of .

Fig 2 shows a comparison of flow profiles at the same value of , on both the lower and upper portions of the curves in Fig 3, for .

When the solutions are characterized by . These solutions can be viewed as perturbations of a uniform flow (i.e. they approach the uniform stream as $\in \to 0$ for a fixed value of). From the numerical calculations, it is found that solutions in this case exist for all (see Fig 4).

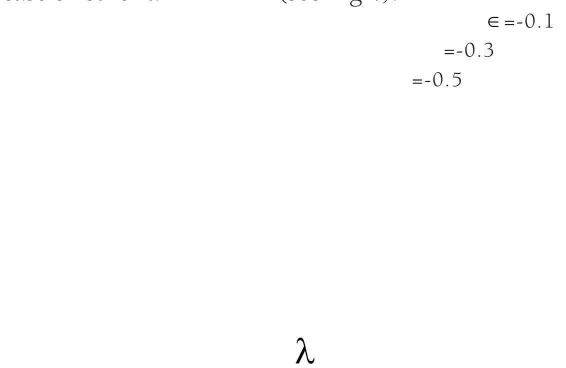


Fig 4. Relationship between *W* and for various values of .

(ii) Subcritical Solutions

For subcritical solutions, we expect a train of waves to be generated behind the applied pressure distribution. We first discuss the case when . Fig 5 shows that the amplitude *A* of the waves, defined as the difference between the levels of the successive crest and trough, decreases as decreases.

The wave amplitude ultimately diminishes when the critical value of λ is reached. If we decrease λ further, the wave amplitude increases to its maximum value and then decreases monotonically to zero again at . Table 1 shows some computed values of λ_{*i}. In addition, the free surface, upon which the pressure distribution is applied, deforms into *two humps* when $\lambda \to \lambda_{*2}$ as depicted in Fig 6.

This phenomenon occurs again as λ reaches the

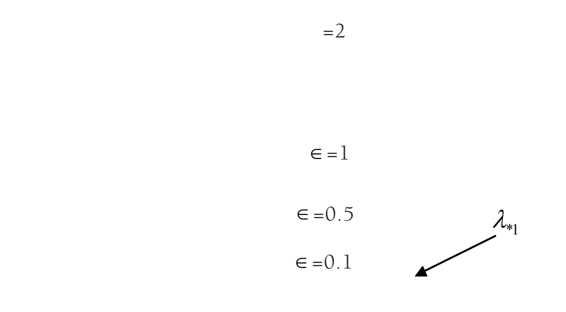


Fig 5. Relationship between amplitude *A* and for various values of .

Table 1. Some computed values of .

i	
1	-4.161
2	-13.161
3	-26.411
4	-43.8

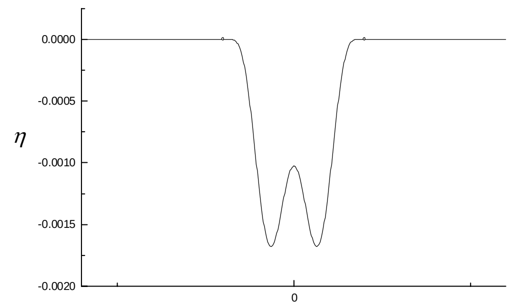


Fig 6. Typical free surface profile for , $\in = 0.1$.

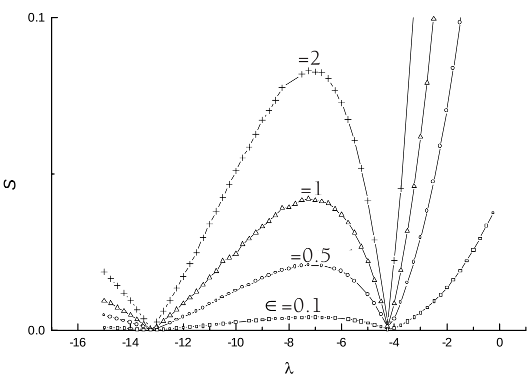


Fig 7. Relationship between steepness *S* and for various values of .

next critical value and so on. In our approximation, the lower bound of λ should be in order for the Froude number *F* to be greater than zero. We conjecture that there are finitely many critical λ_{*i} such that drag-free solutions exist. We would like to add here that there are *n humps* on the free surface for solutions with $\lambda_{*n} < \lambda < \lambda_{*n-1}$. Similar behavior can be found for the steepness (*S*) of the wave, defined as the difference of heights between a crest and a trough divided by the wavelength as shown in Fig 7.

Free surface profiles for the case where $\in < 0$ are found to be similar to those of . Except for the reverse signs of the wave amplitude *A*, overall behaviors of the solutions are qualitatively similar to the case where .

CONCLUSSIONS

In this paper, we have investigated the weakly nonlinear solutions of free-surface flows due to pressure distributions. Existence theorems are given and proved for different flow regimes. Numerical solutions are presented as evidence to support the theorems. It should be noted that our results are found to be qualitatively in good ageement with the fully nonlinear solutions (Asavanant, *et al*[5]). At this preliminary stage of our investigation, we may therefore conclude that, whenever the effect of surface tension is neglected, results from the weakly nonlinear theory are very well accepted and accurate up to $O(\varepsilon^2)$ when ε is the square of scale ratio. However, when surface tension term is included in the Bernoulli equation, interactions between forces due to gravitational acceleration, surface tension and pressure distribution tend to create strange phenomena. Fully nonlinear solutions of the problem are completely different from those reported in this paper and the work is still in progress.

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