ScienceAsia 29 (2003): **121-126**

An Inverse Boundary Value Problem using the Extended Kalman Filter

Krishna Murari Neaupane[a]* and Mitsutaka Sugimoto[b]

[a] Civil Engineering Program, SIIT, Thammasat University, Pathumthani 12121,

[b] Soil mechanics Laboratory, Nagaoka University of Technology, Niigata, 940-2135, Japan

* Corresponding author, E-mail: krishna@siit.tu.ac.th

Received 18 Jan 2002
Accepted 18 Oct 2002

ABSTRACT It is necessary to estimate thermal boundary condition in a number of scientific and engineering applications. The research in this paper uses extended Kalman filter coupled with the finite element method to formulate an inverse problem and estimate the thermal boundary known as heat transfer coefficient (HTC). A simple non-linear formulation based on steady-state heat conduction has been incorporated in the Kalman filter loop. From the laboratory experiment, steady state temperatures were measured at predefined locations. The heat transfer coefficient (HTC) was estimated inversely from these measurements.

KEYWORDS: Kalman Filter, heat transfer coefficient, finite element method.

INTRODUCTION

General

For a numerical simulation of a system involving temperature, input parameters are required to be determined in a controlled environment. Mechanical properties such as Young's modulus of elasticity and compressive strength, are determined from established laboratory experiments. However, direct measurements are not always possible. In such cases, a suitable parameter for which standard laboratory procedure is available, is measured. A functional relationship between this measured property and the required parameter is established for back analysis.

Parameter estimation can be visualized as an inverse problem of optimization that deals with the determination of the mechanical system with unknown material properties, geometry sources or boundary conditions, from the knowledge of response to given excitations on its boundary. There are many methods available to solve the parameter estimation problems such as general minimization procedure, weighted least square method, Bayesian decision-theoretic approach and Kalman filtering techniques. This research employs extended Kalman filtering techniques for the inverse estimation.

The heat transfer coefficient (HTC) describes the rate at which heat flows from the source to the surrounding mass and is one of the most important input parameters for a number of scientific and engineering applications including nuclear waste repository and reservoir engineering.[1,2] Therefore, determination of accurate values of the heat boundary condition is essential to improve the quality of a simulation process involving temperature.

Some literature is available on heat and mass transfer in porous media.[3,4] Many researchers consider soil being directly impregnated from the heat radiating from a single source.[5] Buried transmission cables, for example, are covered with backfill. Thermal instability of soil affects the ability of such backfill to dissipate heat.[6] In sanitary landfill, clay liners separate the soil system from the landfill waste. Heat generated from the microbial activity (decomposition of carbohydrate) affects the ground water regime.[7] While temperature can be measured in the field, the heat transfer coefficient is difficult to measure once the site is sealed.

The simplification that there exists no boundary between heat source and the medium to which heat is being transferred, may lead to an easy analytical solution but it is an over-simplification of the real situation. Determination of boundary condition thus becomes indispensable.

Neaupane *et al.* used a constant value of HTC, independent of source temperature, to simulate a low temperature problem. The dependency of source temperature on the boundary condition was largely ignored.[8] This research aims at estimating thermal boundary condition from the temperature measurements and explores in detail the dependency of HTC on source temperature.

The Kalman Filtering Technique

The behavior of a dynamic system is described by state variables. Such variables may not be determined from direct measurements, but are functions of other measurable state variables. These measurements are generally corrupted by random noises and disturbances.

The state variables are to be estimated from these noisy observations.

The Kalman filter is a set of mathematical equations that provides an efficient computational (recursive) solution of the least-squares method. The Kalman filtering technique has been chosen extensively as a tool to solve the parameter estimation problem. The technique is simple and efficient, takes explicit measurement uncertainty incrementally (recursively), and can also take into account *a priori* information, if any.[9,10]

The Kalman filter estimates a process by using a form of feedback control. To be precise, it estimates the process state at some time and then obtains feedback in the form of noisy measurements. As such, the equations for the Kalman filter fall into two categories: time update and measurement update equations. The time update equations project forward (in time) the current state and error covariance estimates to obtain the *a priori* estimates for the next time step. The measurement update equations are responsible for the feedback by incorporating a new measurement into the *a priori* estimate to obtain an improved *a posteriori* estimate. The time update equations are thus predictor equations while the measurement update equations are corrector equations.

Mathematical Basis

Finite Element Formulation

The case presented here is a simple steady state problem of heat conduction. The governing equation is a two dimensional linear heat flow equation, described by:

$$K\phi = F \tag{1}$$

where ϕ is the temperature distribution. The conductance matrix [K] is given by:

$$[K] = \int_V k\left(\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x} + \frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}\right)dV + \int_S \alpha_c [N]^T[N]\,ds \tag{2}$$

where k and α_c are conductivity and coefficient of heat transfer, respectively; V and S are volume and surface integrals, respectively, and N is the shape function. The matrix [F] is given by:

$$[F] = \int_V Q[N]^T dV - \int_S q_o[N]^T\,ds + \int_S \alpha_c T_c [N]^T\,ds \tag{3}$$

where Q represents the total heat generation and q_o is the rate of heat conduction (heat flux) in the domain.[11] Once the temperature distribution in the domain is known, the remaining problem is to find boundary conditions from the known values of ϕ.

Extended Kalman Filter

The standard Kalman filter addresses the general problem of trying to estimate $x \in \Re$ of a dynamic system governed by a linear stochastic difference equation. The non-linear relationships between the process to be estimated and the measurement is addressed by what is known as an *extended Kalman filter*. In this filtering technique, the estimation around the current estimate is linearized using the partial derivative of the process and measurement functions. The state vector $x \in \Re^n$ is governed by a non-linear stochastic equation

$$x_{i+1} = f(x_i, u_i, w_i) \tag{4}$$

with a measurement $z \in \Re^m$ that is

$$z_i = h(x_i, v_i) \tag{5}$$

where the non-linear function f relates the state at time i to the step $i+1$. It includes as parameters any driving function u_i and the zero-mean process noise w_i. The non-linear function h in the measurement equation relates the state x_i to the measurement z_i. Since individual values of the noise w_i and v_i are not known for each time step, the state and measurement vectors are approximated without them as in

$$\tilde{x}_{i+1} = f(\hat{x}_i, u_i, 0) \tag{6}$$

and

$$\tilde{z}_i = h(\hat{x}_i, 0) \tag{7}$$

where $\hat{x}_i$ is the *a posteriori* estimate of the state, $\tilde{x}_{i+1}$ and $\tilde{z}_i$ are the approximate state and measurement vectors, respectively.

Details on the extended Kalman Filter (*EKF*) equation can be found elsewhere but the end results are presented here for completeness.[12]

- **Time update equations -** The time update equations project the state and covariance estimates from time step i to time step i+1

$$\hat{x}^-_{i+1} = f(\hat{x}_i, u_i, 0) \tag{8}$$

$$P^-_{i+1} = A_i P_i A_i^T + W_i Q_i W_i^i \tag{9}$$

where A_i and W_i are the process Jacobians at step i, and Q_i is the process noise covariance at step i.

- **Measurement update equations -** The measurement update equations correct the state and covariance estimates with the measurement z_i.

$$K_i = P_i^- H_i^T \left(H_i P_i^- H_i^T + V_i R_i V_i^T\right)^{-1} \tag{10}$$

$$\hat{x}_i = \hat{x}_i^- + K\left(z_i - h\left(\hat{x}_i^-, 0\right)\right) \qquad (11)$$

$$P_i = \left(I - K_i H_i\right) P_i^- \qquad (12)$$

Coupling FEM and EKF

The boundary value problem described in this paper requires that the heat boundary be estimated from the temperature measurement. Fig 1 illustrates how finite element formulation is coupled with the Kalman filter loop to achieve this goal. First, it is necessary to assume an *a priori* (guess) value of the heat transfer coefficient (α_c). The temperature distribution over the entire domain can now be estimated using finite element formulation (equation 1). The result from the finite element code is then compared with that from the experiment. This comparison is done for the locations at which the measurements are already known. The Kalman filter is used to obtain a new set of *posteriori* estimates of heat transfer coefficients using error minimization process. The assumed value of heat boundary is then updated until the desired error goal is met. The Kalman filter thus acts as an error minimization filter in the loop and finite element formulation acts as a estimator tool.

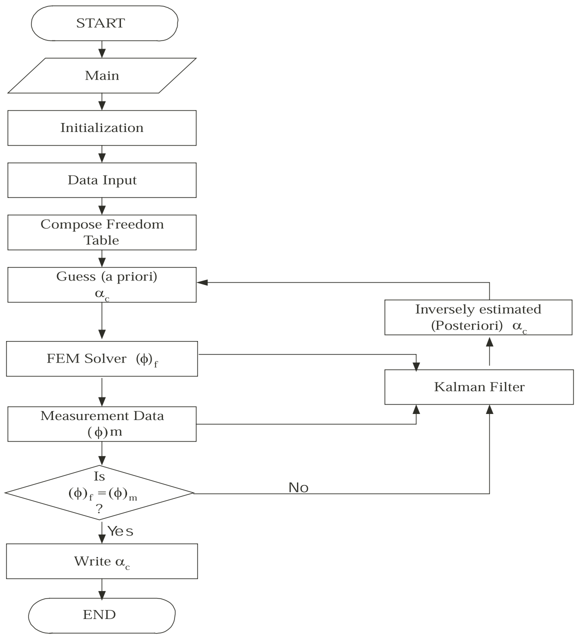

Fig 1. Program flowchart

Experimental Investigation

Material

The material used for the experiment is Shirahama sandstone from the Shirahama district in Japan. The Shirahama sandstone is a fine-grained sandstone. Samples devoid of any visible cracks were taken for freezing experiments. The experiments were performed on fully saturated specimens.

Experimental Procedure

The experimental set up for the freezing tests consists essentially of a number of temperature sensors placed along pre-defined directions at specified distances from the center of the heat source. A prismatic specimen (30 cm x 45 cm x 15 cm in dimension)was prepared with a hole of 4.6 cm at its center. Then the temperature sensors were fixed on the smooth surface of the specimen at pre-defined distances and directions and covered with waterproof resin before it was submerged into water for 3 days to fully saturate it. The complete experimental layout is shown in Fig 2, and the position and orientation of the temperature sensors are shown in Fig 3. The experiments were designed to cover room temperature as well.

The tests were performed on fully saturated specimens, which were confined within a thermostat box so as to closely simulate the adiabatic condition. A circulation type low-temperature thermostat bath (TRL-N30L) was used for source temperature variation from -20°C to 20°C through brine circulation through the center of the circular hole of the specimen. A thin cylinder of copper covered with aluminum foil separated the brine solution from the rock specimen boundary.

Three sets of experiments were performed for each of three different source temperatures. The temperature of the thermo-static bath was maintained at -20°C, -10°C and -5°C, and the specimen was subjected to freezing for 12 hours to achieve as low and stable a temperature as possible. In each case, the temperature of the thermostatic bath was held constant and the specimen was subjected to freezing until a stable temperature was achieved. Once a steady state condition was achieved, temperatures were recorded. An automatic data logger was used for data recording, and measurements were taken every minute.

Results and Discussion

Measurements from the experiment consisted of temperatures at specified radial distances 1, 2, 4 and 8 cm from the heat source. Temperature profile at the radial distance of 1 cm from the heat source shows the steep temperature gradients that gradually approach the steady state condition (Fig 4). After 24 hours of freezing, a steady state condition can be achieved. A summary of these steady state temperatures at three different source inputs is presented in Fig 5.

Results from the experiments were applied to the structured Fortran code. An arbitrary value of α_c =

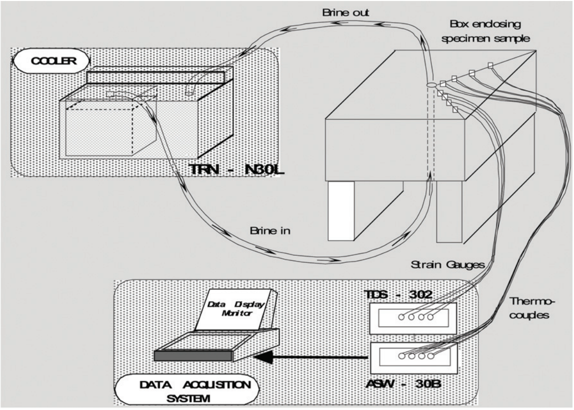

Fig 2. Layout of experimental procedure.

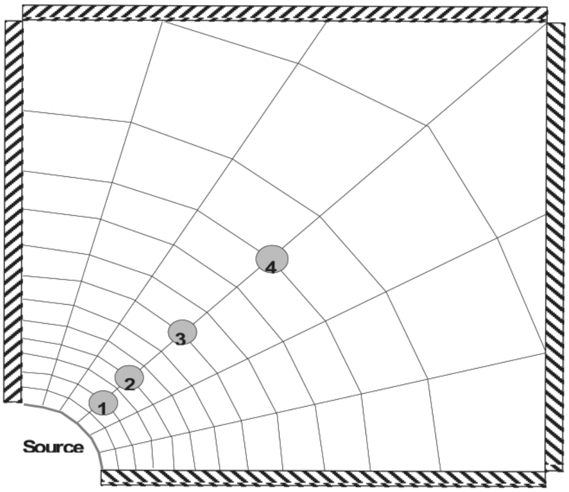

Fig 3. Plan view of specimen showing positions of sensors.

10^{-10} W/cm°C was chosen. The numerical code estimates the temperature at different nodes. Nodes specified by numerals 1, 2, 3 and 4 (Fig 3) for which experimental measurements were already known were chosen and compared with the temperatures obtained from the finite element code. The extended Kalman filter was used to update α_c by error minimization process until the desired goal was achieved.

Fig 5 displays experimental results in the form of steady state temperature for various source temperature inputs. When the source temperature was -20°C, it took 6 iterations to stabilize and reach a steady state value of heat transfer coefficient (Fig 6). For the source input of -10°C, the convergence required 5 iterations. However, in the case of a -5°C source temperature, a steady state condition was obtained in only 2 iterative processes.

The results show that HTC is a function of source temperatures (Fig 7). As the source temperature increases (in a negative direction), the value of HTC increases remarkably. An increase in the source

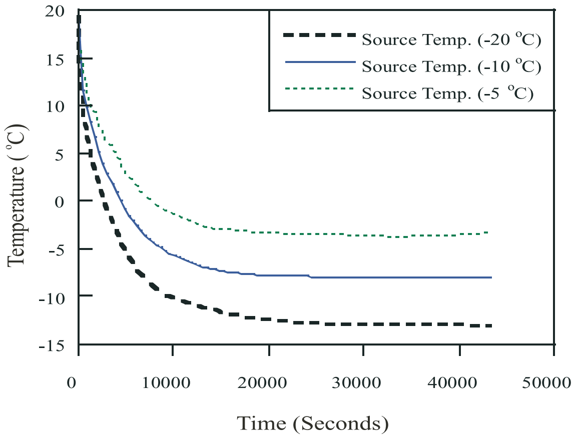

Fig 4. Temperature transfer at a point 1 cm from the heat source.

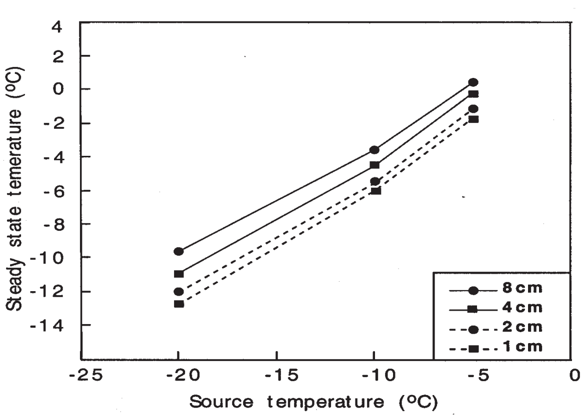

Fig 5. Steady state temperature at predefined locations.

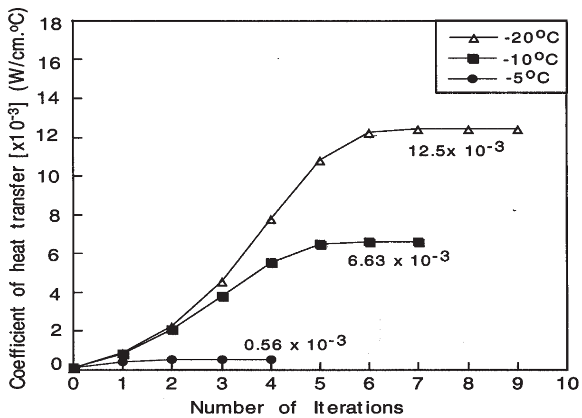

Fig 6. Result from iterative code

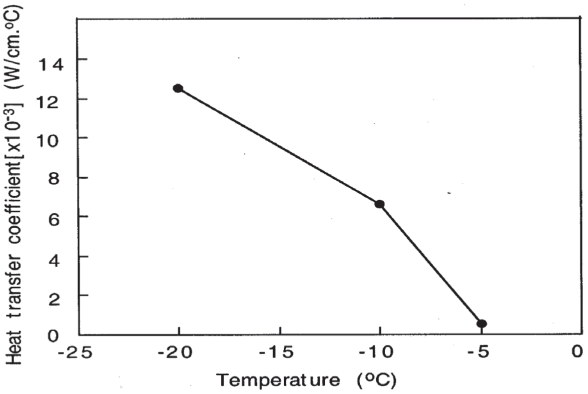

Fig 7. Summary of results showing HTC as a function of source temperature

temperature from -5°C to -10°C leads to an 10 fold increase in the value of HTC, while going from -10°C to -20°C only doubles the HTC value.

Based on these results, we found that it is not suitable to use heat boundary condition independent of source temperature as the value for HTC varies greatly depending on the source input. This variation, however, tends to narrow as the source temperature decreases.

Conclusion

The primary objective of the work presented in this paper was to estimate HTC. This was achieved by means of inverse calculation. The work was centered on one material, the Shirahama sandstone. The Kalman filtering technique was used effectively to estimate HTC from temperature measurements. It is concluded that HTC is a function of source temperature and therefore it is not appropriate to use a single value of HTC irrespective of the degree of heating.

Heat boundary condition is a matter of concern in many engineering and scientific applications. Estimation of the coefficient as described in this paper will improve the accuracy of any simulation related to heat transfer including those in geo-environmental fields such as nuclear waste repository, sanitary landfill and problem of heat transfer from buried high-tension cables.

Acknowledgements

The authors gratefully acknowledge that the article was checked by David Chatham of the English Teaching Unit of Sirindhorn International Institute of Technology, Thammasat University, Thailand.

References

1. Amy MO and Rouset G (1994) Thermo-Hydro-Mechanical Modeling of an Underground Radioactive Wastes Disposal. In: *Numerical Methods in Geotechnical Engineering*, (edited by Smith IM) pp 475-95. AA Balkema, Rotterdam, the Netherlands .
2. Hart RD and John CM (1986) Formulation of a Fully Coupled Thermal-Mechanical-Fluid Flow Model for Non-linear Geologic Systems. *Int. J Rock Mech. Min Sci & Geomech* **23**(**3**), 213-24.
3. Thomas HR and Ferguson WJ (1999) A Fully Coupled Heat and Mass Transfer Model Incorporating Gas Transfer in an Unsaturated Porous Medium. *Computers and Geotechnique* **24**, 65-87.
4. De Vries DA(1987) The Theory of Heat and Moisture Transfer in Porous Media Re-visited. *Int. J Heat and Mass Transfer* **30** (**7**), pp 1343-50,
5. Booker JR and Savvidou C (1985) Consolidation around a Point Heat Source. *Int. J Num. Anal Meth Geomech* **9**, 173-84.
6. Omar NO and James KM (1981) Coupled Heat and Water Flows around Buried Cables. *ASCE Journal of Geotechnical Engineering Divisions* 107, GT11.
7. Neaupane KM and Deeprom K (2001) Heat and Fluid Transfer from Sanitary Landfill in Bangkok Subsoil: A Coupled Theoretic Approach. *Proc of 2nd ANZ GeoEnvironment Conf* 335-40.
8. Neaupane KM Yamabe T and Yoshinaka R (1999) Simulation of a Fully Coupled Thermo-Hydro-Mechanical System in Freezing and Thawing Rock. *Int. J Rock Mech. and Min Sci* **36**, 563-80.
9. Kalman RE (1960) A New Approach to Linear Filtering and Prediction Problem. *J Basic Eng*. **82D**, 35-45.
10. Kalman RE and Bucy RS (1961) New Results in Linear Filtering and Prediction Theory. *J Basic Eng* **83D**, 95-108.
11. Lewis RW, Morgan K, Thomas HR and Seetharamu KN (1996) The Finite Element Method in Heat Transfer Analysis. John Wiley & Sons England.
12. Sorenson SW (1985) Least Squares Estimation: from Gauss to Kalman. In: *Kalman Filtering: Theory and Application* (edited by Sorenson HW), IEEE press, New York.