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The Weak Law of Large Numbers and the Accompanying Distribution Function of Random Sums

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ABSTRACT In this paper, we give necessary and sufficient conditions for random sums satisfying the weak law of large numbers and give conditions for convergence of random sums. The random variables considered in this paper are not to have finite variance.

KEYWORDS: law of large numbers, accompanying distribution.

INTRODUCTION

For every n, let $X_{n1}, X_{n2},\ldots$ be a sequence of independent random variables and Z_n a random index independent of the sequence (X_{nk}), $k = 1, 2,\ldots$. Put

$$S_n^{(k)} = X_{n1} + X_{n2} + \ldots + X_{nk}$$

and

$$S_{Z_n} = X_{n1} + X_{n2} + \ldots + X_{nZ_n}.$$

Many authors[1-5] have investigated the limiting behavior of the distribution functions of the random sums S_{Z_n}. The aim of our investigation is to extend the theory of limit distribution function of sums of independent random variables, especially the weak law of large numbers and the accompanying distribution functions to the case of random sums.

A sequence of random variables (X_n) **converges in probability** to the random variable X,

if $P(|X_n - X| \geq \varepsilon) \to 0$ as $n \to \infty$ for every fixed $\varepsilon > 0$.

In this case we use the notation $X_n \xrightarrow{P} X$ as $n \to \infty$.

A sequence (X_n) is called **stable** if there exists a sequence of constants (A_n) such that $X_n - A_n \xrightarrow{P} 0$ as $n \to \infty$. It is well-known that, if (X_n) is stable, the constant A_n may be taken to be the medians m_n of X_n.

A double sequence of random variables (X_{nk}), k = 1, 2,…, k_n obeys **the weak law of large numbers** if the sequence of the sums

$$S_n^{(k_n)} = X_{n1} + X_{n2} + \ldots + X_{nk_n}$$

is stable.

A double sequence (X_{nk}), k = 1, 2,… obeys **the weak law of large numbers with respect to** (w.r.t) (Z_n) if there exists a double sequence of constants (A_{nk}) k = 1,2,… such that

$$S_{Z_n} - A_{nZ_n} \xrightarrow{P} 0 \text{ as } n \to \infty.$$

We shall say that (X_{nj}) is **random infinitesimal** w.r.t (Z_n) if

$$\max_{1\leq k\leq Z_n} P(|X_{nk}| \geq \varepsilon) \xrightarrow{P} 0 \text{ for every } \varepsilon > 0.$$

For the random sums $X_{n1} + X_{n2} + \ldots + X_{nZ_n} - A_{nZ_n}$, the **random accompanying distribution function** of the random sums is defined to be the distribution function whose logarithm of its characteristic function is $E\left[\psi_{Z_n}(t)\right]$ where

$$\log\psi_k(t) = -iA_{nk}t + it\sum_{j=1}^{k}\alpha_{nj} + \sum_{j=1}^{k}\int(e^{itx}-1)dF_{nj}(x+\alpha_{nj})$$

and $\alpha_{nj} = \int_{|x|<\tau} x\,dF_{nj}(x)$ and $\tau > 0$ is a constant.

THE WEAK LAW OF LARGE NUMBERS OF RANDOM SUMS

The following theorems are well-known conditions under which the weak law of large numbers holds.[6, 7]

Theorem 2.1A The following statements are equivalent.

1. (X_{nk}) , k = 1, 2,…, k_n obeys the weak law of large numbers.

2. $$\sum_{k=1}^{k_n}\int_{|x|>1} dF_{nk}(x+m_{nk}) \to 0 \text{ as } n \to \infty$$

   and $$\sum_{k=1}^{k_n}\int_{|x|\leq 1} x^2 dF_{nk}(x+m_{nk}) \to 0 \text{ as } n \to \infty.$$

3. $$\sum_{k=1}^{k_n}\int_{-\infty}^{\infty}\frac{x^2}{1+x^2}\,dF_{nk}(x+m_{nk}) \to 0 \text{ as } n \to \infty.$$

4. $\sum_{k=1}^{k_n} \int_{|x|\geq\varepsilon} dF_{nk}(x+m_{nk}) \to 0$ as $n \to \infty$

for every $\varepsilon > 0$ and

$$\sum_{k=1}^{k_n} \{ \int_{|x|<1} x^2 dF_{nk}(x+m_{nk}) - (\int_{|x|<1} x dF_{nk}(x+m_{nk}))^2 \} \to 0 \text{ as } n \to \infty.$$

5. There exists a sequence of constants (A_n) such that the distribution function of the $S_n^{(k_n)} - A_n$ converges weakly to the degenerate distribution function

$$D(x) = \begin{cases} 0 \text{ if } x < 0, \\ 1 \text{ if } x \geq 0. \end{cases}$$

Theorem 2.2A If a double sequence (X_{nk}), $k = 1, 2, \ldots, k_n$ obeys the weak law of large numbers, then the sequence of distribution functions of the sums $S_n^{(k_n)}$ converges weakly to the degenerate distribution function with unit-jump at c for some constant c.

In the following, we extend Theorem 2.1A–2.2A to the case of random sums.

Theorem 2.1B The following statements are equivalent.

1. (X_{nk}), $k = 1, 2, \ldots$ obeys the weak law of large numbers w.r.t (Z_n).

2. $\sum_{k=1}^{Z_n} \int_{|x|>1} dF_{nk}(x+m_{nk}) \xrightarrow{P} 0$ as $n \to \infty$ .....(B-1)

and

$$\sum_{k=1}^{Z_n} \int_{|x|\leq 1} x^2 dF_{nk}(x+m_{nk}) \xrightarrow{P} 0 \text{ as } n \to \infty. \quad \text{.....(B-2)}$$

3. $\sum_{k=1}^{Z_n} \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dF_{nk}(x+m_{nk}) \xrightarrow{P} 0$ as $n \to \infty$.

4. $\sum_{k=1}^{Z_n} \int_{|x|\geq\varepsilon} dF_{nk}(x+m_{nk}) \xrightarrow{P} 0$ as $n \to \infty$

for every $\varepsilon > 0$ and

$$\sum_{k=1}^{Z_n} \{ \int_{|x|<1} x^2 dF_{nk}(x+m_{nk}) - (\int_{|x|<1} x dF_{nk}(x+m_{nk}))^2 \} \xrightarrow{P} 0 \text{ as } n \to \infty.$$

5. There exists a double sequence of constants (A_{nk}), $k = 1, 2, \ldots$ such that a sequence of the distribution functions of the random sums $S_{Z_n} - A_{nZ_n}$ converges weakly to the degenerate distribution function.

Theorem 2.2B If a double sequence (X_{nk}) , $k = 1, 2, \ldots$ obeys the weak law of large numbers w.r.t. (Z_n), then the sequence of distribution functions of S_{Z_n} converges weakly to the distribution function $F(x)$ $= \int_0^1 D_{a_q}(x)dq$ where D_{a_q} is the degenerate distribution function with unit - jump at a_q.

In proving the theorems, the notion of q-quantiles of the random variable Z_n plays an important role. The q-quantiles of Z_n is the function $\ell_n : (0,1) \to N$ defined by

$\ell_n(q) = max\{k \in N / P(Z_n < k) \leq q\}$.

Clearly ℓ_n is non-decreasing in q and

$P(Z_n < \ell_n(q)) \leq q < P(Z_n \leq \ell_n(q))$. Next we give lemmas that will be basic in the proofs of main theorems.

Lemma 2.1 ([5], p.336) For every n, let (a_{nk}), $k = 1, 2, \ldots$ be a non-decreasing sequence of non-negative real numbers and $a \geq 0$. Then

(i) if $a_{nZ_n} \xrightarrow{P} a$ as $n \to \infty$, then $a_{n\ell_n(q)} \to a$ as $n \to \infty$ for every $q \in (0,1)$ and

(ii) if $a_{nl_n(q)} \to a$ as $n \to \infty$ for a.e.q, then $a_{nZ_n} \xrightarrow{P} a$ as $n \to \infty$.

Lemma 2.2 ([8], p.63) $X_n \xrightarrow{P} X$ as $n \to \infty$ if and only if every subsequence (X_{n_k}) contains a sub-subsequence $(X_{n_{k_r}})$ such that $X_{n_{k_r}} \to X$ a.e. as $r \to \infty$.

Proof of Theorem 2.1B

To prove 1. $\to$ 2., we assume that there exists a double sequence of constants (A_{nk}), $k = 1, 2, \ldots$ such that $S_{Z_n} - A_{nZ_n} \xrightarrow{P} 0$ as $n \to \infty$.

Hence $\lim_{n\to\infty} P(S_{Z_n} - A_{nZ_n} \leq x) = \begin{cases} 0 & if\ x < 0, \\ 1 & if\ x > 0. \end{cases}$ ..... (1)

For each n, let $\text{Im}\, Z_n = \{k_{nj} | k_{nj} < k_{n(j+1)}\}$, $q_{nj} = \sum_{k=1}^{k_{nj}} P(Z_n = k)$ and $q_{no} = 0$.

Then for each $q \in [\, q_{n(j-1)}, q_{nj})$ we have $\ell_n(q) = k_{nj}$ and

$$\begin{aligned} P(S_{Z_n} - A_{nZ_n} \leq x) &= \sum_{k_{nj} \in \text{Im}\, Z_n} P(Z_n = k_{nj})\, P(S_n^{(k_{nj})} - A_{nk_{nj}} \leq x) \\ &= \sum_{k_{nj} \in \text{Im}\, Z_n} (q_{nj} - q_{n(j-1)})\, P(S_n^{(k_{nj})} - A_{nk_{nj}} \leq x) \\ &= \int_0^1 P(S_n^{(l_n(q))} - A_{nl_n(q)} \leq x)dq. \quad \text{.....(2)} \end{aligned}$$

From (1) and (2)

$$\lim_{n\to\infty}\int_0^1 P(S_n^{(l_n(q))}-A_{nl_n(q)}\le x)dq=\begin{cases}0 \text{ } if \text{ } x<0,\\ 1 \text{ } if \text{ } x>0.\end{cases}$$

By Fatou's lemma, for every subsequence (n') of (n) we have

$$\int_0^1 \liminf_{n'\to\infty} P(S_{n'}^{l_{n'}(q))}-A_{n'l_{n'}(q)}\le x)dq \quad = 0 \text{ if } x<0$$

and

$$\int_0^1 (1-\limsup_{n'\to\infty} P(S_{n'}^{l_{n'}(q))}-A_{n'l_{n'}(q)}\le x)dq = 0 \text{ if } x>0.$$

Hence for a.e.q,

$$\liminf_{n'\to\infty} P(S_{n'}^{(l_{n'}(q))}-A_{n'l_{n'}(q)}\le x)=0 \text{ } if \text{ x}<0 \quad \text{.....(3)}$$

and

$$\limsup_{n'\to\infty} P(S_{n'}^{(l_{n'}(q))}-A_{n'l_{n'}(q)}\le x)=1 \text{ } if \text{ x}>0. \quad \text{.....(4)}$$

To prove (B-1) and (B-2), by Lemma 2.2 we let (n') be any subsequence of (n). From (3) and (4), we have a sub-subsequence (n'') of (n') such that $(S_{n''}^{(l_{n''}(q))}-A_{n''l_{n''}(q)})$ converges weakly to the degenerate distribution function. By Theorem 2.1 A ,$(X_{n''k})$, k = 1, 2,..., $\ell_n(q)$ obeys the weak law of large numbers and

$$\sum_{k=1}^{l_{n''}(q)}\int_{|x|>1} dF_{n''k}(x+m_{n''k})\to 0 \text{ as } n''\to\infty \quad \text{.....(5)}$$

and

$$\sum_{k=1}^{l_{n''}(q)}\int_{|x|\le 1} x^2 dF_{n''k}(x+m_{n''k})\to 0 \text{ as } n''\to\infty \quad \text{.....(6)}$$

for a.e.q. By Lemma 2.1(ii) and (5), we have

$$\sum_{k=1}^{Z_{n''}}\int_{|x|>1} dF_{n''k}(x+m_{n''k})\xrightarrow{P} 0 \text{ as } n''\to\infty.$$

From this fact and Lemma 2.2, (B-1) holds. Similarly,we can show that (B-2) holds.

To prove 2. → 1. ,we assume the conditions (B-1) and (B-2) hold. By Lemma 2.1, for every $q \in (0,1)$ we have

$$\sum_{k=1}^{l_n(q)}\int_{|x|>1} dF_{nk}(x+m_{nk})\to 0 \text{ as } n\to\infty \quad \text{.....(7)}$$

and

$$\sum_{k=1}^{l_n(q)}\int_{|x|\le 1} x^2 dF_{nk}(x+m_{nk})\to 0 \text{ as } n\to\infty. \quad \text{.....(8)}$$

From (7), (8) and Theorem 2.1A, a double sequence (X_{nk}), k = 1, 2,..., $\ell_n(q)$ obeys the weak law of large numbers for every q and there exists a sequence (A_{nk}), k = 1, 2,... such that

$$S_n^{(l_n(q))}-A_{nl_n(q)}\xrightarrow{P} 0 \text{ as } n\to\infty.$$

So for every $q \in (0,1)$ and $\varepsilon > 0$, we have

$$P(|S_n^{(l_n(q))}-A_{nl_n(q)}|\ge\varepsilon)\to 0 \text{ as } n\to\infty.$$

Hence $P(|S_{Z_n}-A_{nZ_n}|\ge\varepsilon)$ which is equal to $\int_0^1 P(|S_n^{(l_n(q))}-A_{nl_n(q)}|\ge\varepsilon)\,dq$ converges to 0. That is 1. true.

To prove 1. ↔ 3. and 1. ↔ 4. we use the same technique and the fact

$$\int_{|x|<1} x^2 dF_{nk}(x+m_{nk})-(\int_{|x|<1} x dF_{nk}(x+m_{nk}))^2\ge 0.$$

To prove 1. → 5. we assume a double sequence (X_{nk}), k = 1, 2... obeys the weak law of large numbers w.r.t (Z_n). By (B-1) and (B-2) and Lemma 2.1(i), we have (7) and (8) hold for every q ∈ (0,1). By Theorem 2.1A there exists a double sequence (A_{nk}), k = 1, 2,... such that $(S_n^{(l_n(q))}-A_{nl_n(q)})$ converges weakly to the degenerate distribution function. Hence for each $q \in (0,1)$,

$$\lim_{n\to\infty} P(S_n^{(l_n(q))}-A_{nl_n(q)}\le x)=\begin{cases}0 \text{ } if \text{ } x<0,\\ 1 \text{ } if \text{ } x>0.\end{cases}$$

which implies $\lim_{n\to\infty}\int_0^1 P(S_n^{(l_n(q))}-A_{nl_n(q)}\le x)dq=\begin{cases}0 \text{ } if \text{ } x<0\\ 1 \text{ } if \text{ } x>0.\end{cases}$

So $(S_{Z_n}-A_{nZ_n})$ converges weakly to the degenerate distribution function.

To prove 5. → 1. assume that

$$\lim_{n\to\infty} P(S_{Z_n}-A_{nZ_n}\le x)=\begin{cases}0 \text{ } if \text{ } x<0,\\ 1 \text{ } if \text{ } x>0.\end{cases} \quad \text{.....(9)}$$

From (9) and the fact that

$$P(|S_{Z_n}-A_{nZ_n}|\ge\varepsilon)=P(S_{Z_n}-A_{nZ_n}\ge\varepsilon)+P(S_{Z_n}-A_{nZ_n}\le-\varepsilon)$$
$$=1-P(S_{Z_n}-A_{nZ_n}<\varepsilon)+P(S_{Z_n}-A_{nZ_n}\le-\varepsilon)$$

we have $S_{Z_n}-A_{nZ_n}\xrightarrow{P} 0$ as $n\to\infty$. #

Proof of Theorem 2.2B

Assume that (X_{nk}), k = 1, 2,.. obeys the weak law of large numbers w.r.t. (Z_n). By Theorem 2.1B and Lemma 2.1(i), we have (8) and (9) for every $q \in (0, 1)$. By Theorem 2.1A and Theorem 2.2A, for every $q \in (0,1)$ the sequence of distribution functions of the sums $(S_n^{l_n(q))})$ converges weakly to D_{a_q}, where D_{a_q} is the degenerate distribution function with unit-jump at a_q. By Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} P(S_{Z_n} \le x) = \int_0^1 \lim_{n\to\infty} P(S_n^{(l_n(q))} \le x)dq$$

$$= \int_0^1 D_{a_q}(x)dq. \qquad \#$$

The random accompanying distribution function of random sums

In this section we generalize neceesary and sufficient condition for convergence of sums of independent random variables to the case in which the number of term in the sums are random. This is done by using the concepts of random infinitesimal and random accompanying distribution function of random sums.

Lemma 3.1 Let (X_{nk}), $k = 1, 2, 3, \ldots$ be random infinitesimal. Then

(i) $\max_{1\le k\le Z_n} |\alpha_{nk}| \xrightarrow{P} 0$ *where* $\alpha_{nj} = \int_{|x|<\tau} x dF_{nj}(x)$ *and* $\tau > 0$ is a constant

(ii) $\max_{1\le k\le Z_n} |m_{nk}| \xrightarrow{P} 0$ where m_{nk} is the median of X_{nk}

(iii) $\max_{1\le k\le Z_n} |\beta_{nk}| \xrightarrow{P} 0$ where $\beta_{nk} = \varphi'_{nk}(t) - 1$ and φ'_{nk} is the characteristic function of $X'_{nk} = X_{nk} - \alpha_{nk}$.

Proof.

(i) follows from the fact that

$$|\alpha_{nk}| = |\int_{|x|<\tau} x dF_{nk}(x)|$$

$$\le \int_{|x|\le\varepsilon} |x| dF_{nk}(x) + \int_{\varepsilon<|x|<\tau} |x| dF_{nk}(x)$$

$$\le \varepsilon + \tau P(|X_{nk}| \ge \varepsilon)$$

for sufficiently small ε.

To prove (ii) let $q \in (0,1)$ and $\varepsilon > 0$. By Lemma 2.1(i) we have $\max_{1\le k\le l_n(q)} P(|X_{nk}| \ge \varepsilon) \to 0$. So there exists $n_0 \in N$ such that $\max_{1\le k\le l_n(q)} P(|X_{nk}| \ge \varepsilon) < \frac{1}{2}$ for $n \ge n_0$. We note that if the probability of X lying in some interval is greater than $\frac{1}{2}$, then every median m of X belongs to this interval. So $|m_{nk}| \le \varepsilon$ for every n and k such that $1 \le k \le l_n(q)$ and $n \ge n_0$. Hence for $n \ge n_0$ we have $\max_{1\le k\le l_n(q)} |m_{nk}| \le \varepsilon$ which implies $\max_{1\le k\le l_n(q)} |m_{nk}| \to 0$.

By Lemma 2.2(ii) we have $\max_{1\le k\le Z_n} |m_{nk}| \xrightarrow{P} 0$.

(iii) follows from (i) and the fact that

$$|\varphi'_{nj}(t) - 1| = |\int (e^{itx} - 1)\, dF'_{nj}(x)|$$

$$\le \int_{|x|<\varepsilon} |e^{itx} - 1| dF'_{nj}(x) + 2\int_{|x|\ge\varepsilon} dF'_{nj}(x)$$

$$\le \varepsilon|t| + 2P\left(|X'_{nj}| \ge \varepsilon\right)$$

$$\le \varepsilon|t| + 2[P(|X_{nj}| \ge \frac{\varepsilon}{2}) + P(|\alpha_{nj}| \ge \frac{\varepsilon}{2})].$$

where F'_{nj} is the distribution function of X'_{nj}. #

Lemma 3.2 If there exist A_{nZ_n} such that the sequence of distribution functions of random sums

$$S_{Z_n} = X_{n1} + X_{n2} \ldots + X_{nZ_n} - A_{nz_n}$$

of random infinitesimal (X_{nk}) converges to a limit, then for $q \in (0, 1)$ there exists a constant c_q such that

$$\sum_{k=1}^{l_n(q)} \int \frac{x^2}{1+x^2} dF_{nk}(x + \alpha_{nk}) < c_q.$$

Proof. To prove the lemma, it suffices to show

$$a_n^2 \sum_{k=1}^{l_n(q)} \int \frac{x^2}{1+x^2} dF_{nk}(x + \alpha_{nk}) \to 0,$$

for every sequence (a_n) such that $0 < a_n < 1$ and $a_n \to 0$. Note that, for large n

$$\int \frac{x^2}{1+x^2} dF_{nk}(x + \alpha_{nk})$$

$$\le \int \frac{2x^2}{1+(x+m_{nk}-\alpha_{nk})^2} dF_{nk}(x+m_{nk}) + \int \frac{2(m_{nk}-\alpha_{nk})^2}{1+(x+m_{nk}-\alpha_{nk})^2} dF_{nk}(x+m_{nk})$$

$$\le \int \frac{2x^2}{1+(x+m_{nk}-\alpha_{nk})^2} dF_{nk}(x+m_{nk}) + 2(m_{nk}-\alpha_{nk})^2$$

$$\le \int \frac{2x^2}{1+(x+m_{nk}-\alpha_{nk})^2} dF_{nk}(x+m_{nk}) + 4(\int_{|x|<\tau} (x-m_{nk}) dF_{nk}(x))^2$$

$$+ 4(\int_{|x|\ge\tau} m_{nk} dF_{nk}(x))^2$$

$$\le \int \frac{2x^2}{1+(x+m_{nk}-\alpha_{nk})^2} dF_{nk}(x+m_{nk}) + 4(\int_{|x|<\tau} (x-m_{nk}) dF_{nk}(x))^2$$

$$+ 4m_{nk}^2 P(|X_{nk}| \ge \tau)$$

$$\le c\int \frac{x^2}{1+x^2} dF_{nk}(x+m_{nk}) + 4\int_{|x|<\tau} (x-m_{nk})^2 dF_{nk}(x) + 4m_{nk}^2 P(|X_{nk}| \ge \tau)$$

for some constant c (by Lemma 2.1 and Lemma 3.1 (i, ii))

$$\le c\int \frac{x^2}{1+x^2} dF_{nk}(x+m_{nk}) + 4\int_{|x|<2\tau} x^2 dF_{nk}(x+m_{nk}) + 4m_{nk}^2 P(|X_{nk}| \ge \tau)$$

(by Lemma 3.1 (ii)).

Hence

$$a_n^2\sum_{k=1}^{l_n(q)}\int\frac{x^2}{1+x^2}dF_{nk}(x+\alpha_{nk})\le A_n+B_n+C_n$$

where

$$A_n=c\,a_n^2\sum_{k=1}^{l_n(q)}\int\frac{x^2}{1+x^2}dF_{nk}(x+m_{nk})$$

$$B_n=4\,a_n^2\sum_{k=1}^{l_n(q)}\int_{|x|<2\tau}x^2dF_{nk}(x+m_{nk})$$

and $C_n=4\,a_n^2\sum_{k=1}^{l_n(q)}m_{nk}^2\,P(|X_{nk}|\ge\tau)$.

If we can show that $A_n\to 0$, $B_n\to 0$ and $C_n\to 0$, we have the Lemma. Since the distribution functions of S_{Z_n} converge weakly and $a_n\to 0$, $a_n S_{Z_n}\xrightarrow{P}0$. So $a_nS_{Z_n}\xrightarrow{W}D$. ByTheorem 2. 1B and Lemma 2.1(i) we have

$$\sum_{k=1}^{l_n(q)}\int_{-\infty}^{\infty}\frac{a_n^2x^2}{1+a_n^2x^2}dF_{nk}(x+m_{nk})\to 0$$

which implies $A_n\to 0$. In the same way, we can show $B_n\to 0$ and $C_n\to 0$. #

Theorem 3. Assume that (X_{nk}) is random infinitesimal. Then there exists a double sequence of constants (A_{nk}) such that the sequence of distribution functions of random sums

$$S_{Z_n}\;=\;X_{n1}+X_{n2}+\ldots+\;X_{nZ_n}\;-\;A_{nZ_n}$$

converges weakly to a limit, if and only if

1. their random accompanying distribution functions converge weakly to the same limit and
2. $\sum_{k=1}^{l_n(q)}\int\frac{x^2}{1+x^2}d\,F_{nk}(x+\alpha_{nk})$ is bounded for *a.e.q* in (0, 1).

Proof.

$(\to)$ Let $\varphi_{l_n(q)}(t)$ be the characteristic function of $S_n^{(l_n(q))}-A_{nl_n(q)}$. Hence

$$\varphi_{l_n(q)}(t)=\exp(-it\,A_{nl_n(q)})\prod_{k=1}^{l_n(q)}\varphi_{nk}(t).$$

where φ_{nk} is the characteristic function of X_{nk}. Hence

$$\varphi_{l_n(q)}(t)=\exp(-it\,A_{nl_n(q)}+it\sum_{k=1}^{l_n(q)}\alpha_{nk})\prod_{k=1}^{l_n(q)}\varphi'_{nk}(t).$$

Since (X_{nk}) is random infinitesimal, by Lemma 3.1 (iii) and Lemma 2.1(i) we have

$$\max_{1\le k\le l_n(q)}|\beta_{nk}|\to 0. \qquad .....(10)$$

So, for large n we have

$$\log\varphi'_{nk}(t)\;=\;\log(1+\beta_{nk})$$
$$=\;\beta_{nk}-\frac{1}{2}\beta^2_{nk}+\frac{1}{3}\beta^3_{nk}-\ldots..$$

which implies

$$|\log\varphi'_{nk}(t)-\beta_{nk}|\le\frac{1}{2}\left(\frac{|\beta_{nk}|^2}{1-|\beta_{nk}|}\right). \qquad .....(11)$$

Hence for large n

$$|\varphi_{l_n(q)}(t)-\psi_{l_n(q)}(t)|\le|\log\varphi_{l_n(q)}(t)-\log\psi_{l_n(q)}(t)|$$
$$\le\frac{1}{2}\sum_{k=1}^{l_n(q)}\frac{|\beta_{nk}|^2}{1-|\beta_{nk}|}\quad\text{(by (11))}$$
$$\le\max_{1\le k\le l_n(q)}|\beta_{nk}|\sum_{k=1}^{l_n(q)}|\beta_{nk}|. \qquad .....(12)$$

Note that

$$|\beta_{nk}|\;=|\varphi'_{nj}(t)-1|$$
$$=|\int(e^{itx}-1)\,dF'_{nj}(x)|$$

$$\le\frac{1}{2}|t|^2\int_{|x|<\tau}x^2dF'_{nk}(x)+2\int_{|x|\ge\tau}dF'_{nk}(x)+|t||\int_{|x|<\tau}xdF'_{nk}(x)|$$

and for large n such that $\max_{1\le k\le l_n(q)}|\alpha_{nk}|<\frac{\tau}{2}$ we have

$$|\int_{|x|<\tau}xdF'_{nk}(x)|$$
$$\le\;|\int_{|x|<\tau}xdF'_{nk}(x)-\int_{|x+\alpha_{nk}|<\tau}xdF'_{nk}(x)|+|\int_{|x+\alpha_{nk}|<\tau}xdF'_{nk}(x)|$$

$$\le\int_{\frac{\tau}{2}<|x|<\frac{3\tau}{2}}xdF'_{nk}(x)+|\int_{|x|<\tau}(x-\alpha_{nk})dF_{nk}(x)|$$
$$\le\frac{3\tau}{2}\int_{|x|>\frac{\tau}{2}}dF'_{nk}(x)+\frac{\tau}{2}\int_{|x|>\frac{\tau}{2}}dF'_{nk}(x)$$

$$=2\tau\int_{|x|>\frac{\tau}{2}}dF'_{nk}(x).$$

Hence

$$|\beta_{nk}|\le\frac{1}{2}|t|^2\int_{|x|<\tau}x^2dF'_{nk}(x)+(2+2\tau|t|)\int_{|x|>\frac{\tau}{2}}dF'_{nk}(x)$$

$$\leq \frac{1}{2}|t|^2(1+\tau^2)\int_{|x|<\tau}\frac{x^2}{1+x^2}dF'_{nk}(x)+2(1+\tau|t|)\frac{(4+\tau^2)}{\tau^2}\int_{|x|>\frac{\tau}{2}}\frac{x^2}{1+x^2}dF'_{nk}(x)$$

$$\leq c(t)\int\frac{x^2}{1+x^2}dF'_{nk}(x) \text{ for some constant } c(t). \quad .....(13)$$

From (12) and (13) we see that

$$|\varphi_{l_n(q)}(t)-\psi_{l_n(q)}(t)|\leq c(t)\max_{1\leq k\leq l_n(q)}|\beta_{nk}|\sum_{k=1}^{l_n(q)}\int\frac{x^2}{1+x^2}dF_{nk}(x+\alpha_{nk}). \quad .....(14)$$

By (14), Lemma 3.1(iii) and Lemma 3.2 we have $|\varphi_{l_n(q)}(t)-\psi_{l_n(q)}(t)|\to 0$

which implies $|E[\varphi_{Z_n}(t)]-E[\psi_{Z_n}(t)]|\to 0$. .....(15)

Since (S_{Z_n}) converges weakly to a limit and the characteristic function of S_{Z_n} is $E[\varphi_{Z_n}(t)]$, by (15) we have the sequence of random accompanying distribution functions converges weakly to the same limit and 2. holds by Lemma 3.2.

($\Leftarrow$) From (10), (14) and the fact $\sum_{k=1}^{l_n(q)}\int\frac{x^2}{1+x^2}dF_{nk}(x+\alpha_{nk})$ is bounded for a.e. q we have $|\varphi_{l_n(q)}(t)-\psi_{l_n(q)}(t)|\to 0$. Using the same technique, we have ($E[\varphi_{Z_n}(t)]$) converges to the same limit. So the converse is proved. #

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