J.Sci.Soc.Thailand, 23 (1997) 57-59

ON POISSON'S SUMMATION FORMULA: A PROBABILISTIC APPROACH

E.J. VANDERPERRE , T. LARSAR AND S.S. MAKHANOV

Faculty of Information Technology, King Mongkut's Institute of Technology. 10520 Bangkok, Thailand.

(Received January 20, 1997)

ABSTRACT

We present a summation formula embedded in the frame - work of Probability Theory. Our result is based on a concatenation of Poisson's summation formula and the Fourier inversion theorem. As an application, we derive a very short demonstration of the transformation formula for the theta function.

INTRODUCTION

A powerful tool of Advanced Analysis is Poisson's summation formula. Poisson's formula, which has many applications, appears in several versions. For our convenience, we state the formula in the following form.

Theorem[1]

Let $\varphi(x), x \in \Re$, be a nonnegative improper Riemann integrable function. Assume that ^{r} is increasing on $(-\infty,0]$ and decreasing on $[0,+\infty)$

Then we have

$$\sum_{n=-\infty}^{+\infty} \frac{\varphi(n+)+\varphi(n-)}{2} = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varphi(t)\ e^{-2\pi\, int} dt,$$

each series being absolutely convergent.

Remark

A summation formula, based on a concatenation of Poisson's formula and the Fourier inversion theorem, embedded in the frame-work of Probability Theory, seems to be overlooked in the Literature.

In order to present our result, we first recall some basic definitions and properties stated for direct reference.

PRELUDE

Let W be an abstract space, B a σ - algebra on Ω and P a probability measure on B. The expectation of a random variable X on (Ω,B) is defined by

$$EX = \int_{\Omega} X(\omega)P(d\omega)$$

Every random variable X induces a probability distribution, denoted by $F_X(x), x \in \Re$, with characteristic function

$$Ee^{iuX} = \int_{-\infty}^{+\infty} e^{iux} dF_X(x), u \in \Re.$$

We recall that X is symmetrically distributed (or just symmetric) if $X \sim -X$, i.e. if X and -X are equal in distribution. Note that X is symmetric if and only if Ee^{iuX} is real for all $u \in \Re$.

If F_X is absolutely continuous (with respect to the Lebesgue measure) then the Radon-Nykodem derivative[2] of F_X is simply denoted by f. The derivative is usually called the probability density (function) of X. Observe that $f(x) = f(-x)$, $x \in \Re$, if and only if X is symmetric. We also need the following inversion formula.

Theorem[1]

Let $g(x)$, $x \in \Re$, be a function of bounded variation and absolutely integrable on $\Re$, with Fourier transform defined by

$$g(u) = \int_{-\infty}^{+\infty} e^{iux} g(x)\ dx \ , \ u \in \Re.$$

Then for all $x \in \Re$,

$$\tfrac{1}{2}\ (g(x+) + g(x-)) \ = \ \frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{+T} e^{-ixu} g(u)\ du.$$

Corollary

If g(u) is absolutely integrable on $\Re$, then $g(x)$ is continuous for all x and

$$g(x) \ = \ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ixu} g(u)\ du.$$

A SUMMATION FORMULA

Our following result is a straightforward concatenation of Poisson's summation formula and the Fourier inversion theorem.

Theorem

Let $\{ X_t, t \geq 0 \}$ be a symmetric stochastic process with state space $\Re$ and probability density $f_t(x), x \in \Re, t > 0$. Suppose that $Ee^{iuX_t}, u \in \Re$, satisfies the conditions of Poisson's theorem. Then we have,

$$2\pi \sum_{n=-\infty}^{+\infty} f_t(2\pi n) = \sum_{n=-\infty}^{+\infty} Ee^{inX_t}.$$

Proof

First, note that for all $u \in \Re$ and $t \geq 0$,

$$Ee^{iuX_t} \ = \int_{-\infty}^{+\infty} e^{iux} f_t(x) dx.$$

Since Ee^{iuX_t} is real, nonnegative and integrable, we obtain by the inversion theorem

$$f_t(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ixu} Ee^{iuX_t} du.$$

Substituting $x = 2\pi n$ (n an integer) into the above formula reveals that

$$\sum_{n=-\infty}^{+\infty} f_t(2\pi n) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{E}e^{iuX_t}e^{-2\pi i n u}du.$$

On the other hand, we have by Poisson's theorem,

$$\sum_{n=-\infty}^{+\infty} \mathrm{E}e^{inX_t} = \sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{E}e^{iuX_t}e^{-2\pi i n u}du.$$

Comparing both results yields our desired summation formula.

APPLICATION

As an application, we use our result to derive a very short (probably the shortest) demonstration of the transformation formula for the so-called theta function defined by

$$\theta(x) = \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 x}, x > 0\cdot$$

Let $\{ X_t, t \geq 0 \}$ be a standard Brownian motion[3,4,5] with density function

$$f_t(x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}} \quad , x \in \Re.$$

Then

$$\mathrm{E}e^{iuX_t} = e^{-t\frac{u^2}{2}}.$$

A straightforward application of our theorem shows that

$$\sqrt{\frac{2\pi}{t}}\sum_{n=-\infty}^{+\infty} e^{-\frac{2\pi^2 n^2}{t}} = \sum_{n=-\infty}^{+\infty} e^{-t\frac{n^2}{2}}.$$

Changing the time scale by means of the transformation $t = 2\pi x, x > 0$, yields the remarkable relation

$$\theta(x) = \frac{1}{\sqrt{x}}\theta(\frac{1}{x}).$$

REFERENCES

1. Apostol T.M. (1993). Mathematical Analysis. Addison - Wesley , Amsterdam.
2. Doob J.L. (1994). Measure Theory. Springer - Verlag , Berlin.
3. Æksendal B. (1992) Stochastic Differential Equations. Springer-Verlag, Berlin.
4. Karr A.F.(1993) Probability. Springer - Verlag , Berlin.
5. Kazamaki N. (1994) Continuous Exponential Martingales and BMO. Springer - Verlag , Berlin.