RESEARCH ARTICLES

SYMMETRIC WAVES OF A TWO-LAYER FLUID WITH FREE SURFACE OVER AN UNEVEN BOTTOM

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ABSTRACT

In this paper we study steady two dimensional waves in a two-layer fluid bounded above by a free surface and below by a horizontal rigid boundary with a small obstruction. Two critical speeds for the waves are obtained. Near the smaller critical speed, the derivation of the usual forced KdV equation (FKdV) fails when the coefficient of the nonlinear term in the FKdV vanishes. To overcome this difficulty, a new equation called a Steady Modified KdV equation with forcing term (SFMKdV) governing interfacial wave forms is obtained by a refined asymptotic method. By using SFMKdV we find the traveling solition-like solutions and symmetric wave solutions for different choices of parameters. Existence theorems are proved and numerical results of this equation are presented.

1. INTRODUCTION

This paper concerns the symmetric wave solutions between two immiscible, inviscid, and imcompressible fluids of different but constant densities in the presence of small elliptic obstruction of compact support at the rigid bottom when the effect of gravity is considered (Fig. 1). We assume that the upper boundary is a free surface and the two dimensional obstruction is moving along the lower rigid boundary at a constant speed. By choosing a coordinate system moving with the object, the fluid motion becomes steady. Two critical speeds are obtained, near either one of which an FKdV for steady flow can be derived and has been studied extensively in [1] and [2]. Forbes [3], Belward and Forbes [4], Sha and Vanden-Broeck [5], and Moni and King [6] studied numerically steady flow of a two layer fluid over a bump or a step bounded by a free surface and a rigid boundary. An asymptotic approach for the case of a rigid upper boundary was developed without surface tension by Shen [7] on the basis of FKdV theory, and with surface tension by Choi *et al.* [8]. The case of free upper boundary was studied with surface tension by Choi *et al.* [9] asymptotically on the basis of EKdV theory. In the case considered here, when the wave speed is near the smaller critical speed for internal wave, the nonlinear term in the FKdV may vanish and the derivation of FKdV fails. To overcome this difficulty, a refined asymptotic method is used to derive the Steady Modified KdV equation with forcing term (SFMKdV) in the following form:

$$(A\eta_2^2 + B)\eta_{2x} + C\eta_{2xxx} + Db_x = 0,$$

where A to D are constants depending on several parameters and $b(x)$ is a function with compact support due to the obstruction on the rigid lower boundary. We investigate solutions of the SFMKdV, which represent possible interfacial wave forms.

In section 2, we formulate the problem and develop the asymptotic scheme to derive the SFMKdV. In section 3, existence theorems are proved and numerical solutions of soliton-like solutions and symmetric wave solutions are presented for different values of parameters. The parameters are determined along density ratios of the two fluids, depth ratio of the two fluids, and perturbation of the horizontal velocity at far upstream.

2. FORMULATION AND SUCCESSIVE APPROXIMATE EQUATIONS

We consider steady internal gravity waves between two immiscible, inviscid and incompressible fluids of constant but different densities bounded above by a free surface and below by a horizontal rigid boundary with a small obstruction of compact support. The domains of the upper fluid with a constant density ρ^{*+} and the lower fluid with a constant density ρ^{*-} are denoted by Ω^{*+} and Ω^{*-} respectively (Fig. 1). Assume that the small obstruction is moving with a constant speed C. In reference to a coordinate system moving with the obstruction, the flow is steady and moving with the speed C far upstream. The governing equations and boundary conditions are given by the following Euler equations:

In $\Omega^{*\pm}$,

$$
\begin{aligned}
u^{*\pm}_{x^*} + v^{*\pm}_{y^*} &= 0,\\
u^{*\pm}u^{*\pm}_{x^*} + v^{*\pm}u^{*\pm}_{y^*} &= -p^{*\pm}_{x^*}/\rho^{*\pm},\\
u^{*\pm}v^{*\pm}_{x^*} + v^{*\pm}v^{*\pm}_{y^*} &= -p^{*\pm}_{y^*}/\rho^{*\pm} - g;
\end{aligned}
$$

at the free surface, $y^* = h^{*+} + \eta^*_1$,

$$
\begin{aligned}
u^{*+}\eta^*_{1x^*} - v^{*+} &= 0,\\
p^{*+} &= 0;
\end{aligned}
$$

at the interface, $y^* = \eta^*_2$,

$$
\begin{aligned}
p^{*+} - p^{*-} &= 0,\\
u^{*\pm}\eta^*_{2x^*} - v^{*\pm} &= 0;
\end{aligned}
$$

at the rigid bottom, $y^* = -h^{*-} + b^*(x^*)$

$$
v^{*-} - b^*_x u^{*-} = 0
$$

where $u^{*\pm}$ and $v^{*\pm}$ are horizontal and vertical velocities, $p^{*\pm}$ are pressures, g is the gravitational acceleration constant. We define the following nondimensional variables:

$$
\begin{aligned}
&\varepsilon = H/L << 1,\ \eta_1 = \varepsilon^{-1}\eta^*_1/h^{*-},\ \eta_2 = \varepsilon^{-1}\eta^*_2/h^{*-},\ p^{\pm} = p^{*\pm}/gh^{*-}\rho^{*-},\\
&(x,y) = (\varepsilon x^*, y^*)/h^{*-},\ (u^{\pm},v^{\pm}) = (gh^{*-})^{-1/2}(u^{*\pm},\varepsilon^{-1}v^{*\pm}),\\
&\rho^{+} = \rho^{*+}/\rho^{*-} < 1,\ \rho^{-} = \rho^{*-}/\rho^{*-} = 1,\ \mathrm{U} = \mathrm{C}/(g h^{*-})^{1/2},\\
&h = h^{*+}/h^{*-},\ b(x) = b^*(x^*)(h^{*-}\varepsilon^3)^{-1},
\end{aligned}
$$

where L is the horizontal scale, H is the vertical scale, $b(x) = b^*(x^*)(h^{*-}\varepsilon^3)^{-1}$, h^{*+} and h^{*-} are the equilibrium depths of the upper and lower fluids at $x^* = -\infty$ respectively, and $y^* = -h^{*-} + b^*(x^*)$ is the equation of the obstruction. In terms of the nondimensional quantities, the above

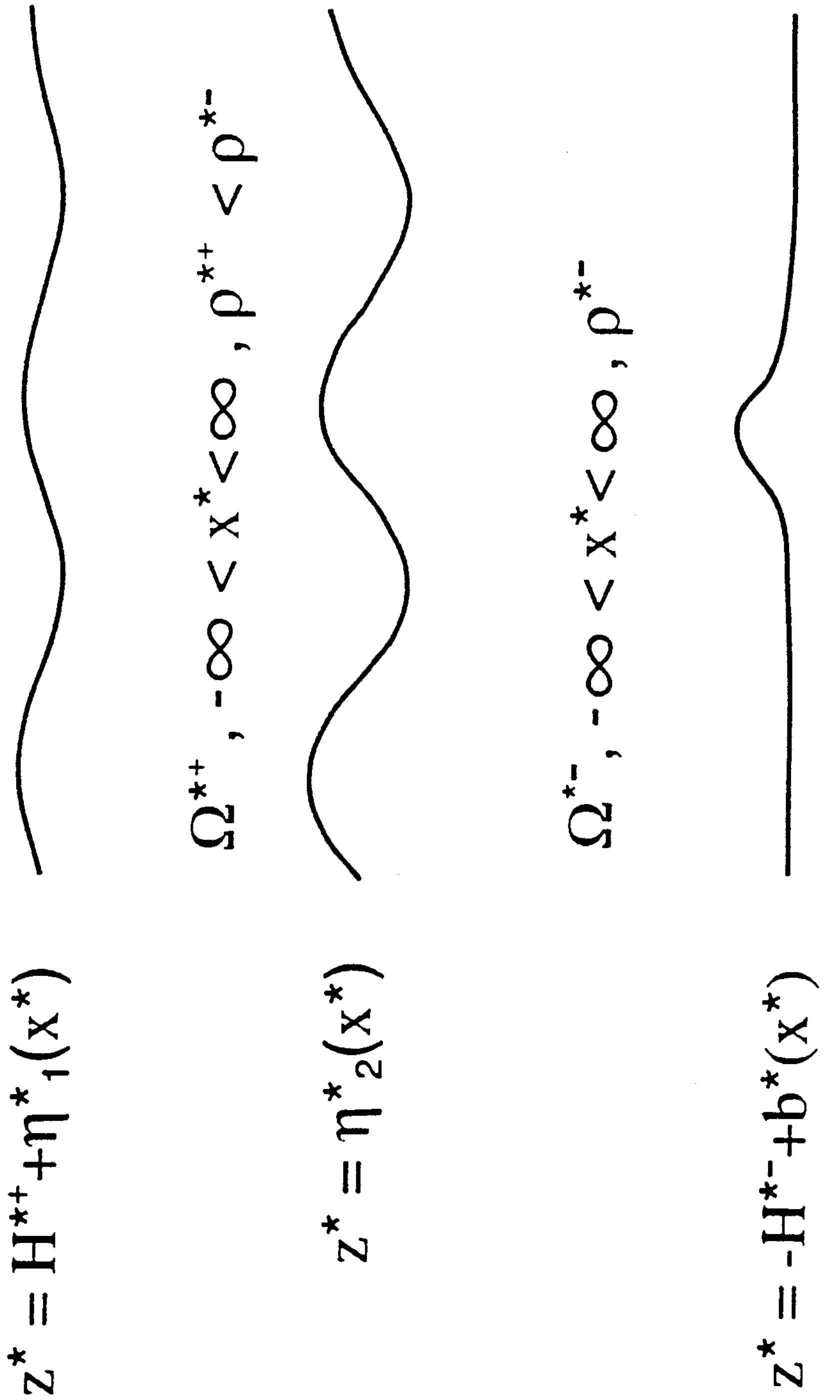

Fig. 1. Fluid Domain.

equations become in $\Omega^{\pm}$,

$$u_x^{\pm} + v_y^{\pm} = 0, \tag{1}$$

$$u^{\pm}u_x^{\pm} + v^{\pm}u_y^{\pm} = -p_x^{\pm}/\rho^{\pm}, \tag{2}$$

$$\varepsilon^2 u^{\pm}v_x^{\pm} + \varepsilon^2 v^{\pm}v_y^{\pm} = -p_y^{\pm}/\rho^{\pm} - 1; \tag{3}$$

at $y = h + \varepsilon\eta_1$,

$$p^+ = 0 \tag{4}$$

$$\varepsilon u^+\eta_{1x} - v^+ = 0; \tag{5}$$

at $y = \varepsilon\eta_2$,

$$\varepsilon u^-\eta_{2x} - v^- = 0; \tag{6}$$

$$\varepsilon u^+\eta_{2x} - v^+ = 0; \tag{7}$$

$$p^+ - p^- = 0; \tag{8}$$

at $y = -1 + \varepsilon^3 b(x)$,

$$v^- = \varepsilon^3 u b_x, \tag{9}$$

where $b(x)$ has a compact support.

In the following, we use a unified asymptotic method to derive the equations for $\eta_1(x)$ and $\eta_2(x)$. We assume that $u^{\pm}$, $v^{\pm}$, and $p^{\pm}$ are functions of x, y near the equilibrium state $u^{\pm} = u_0$, $v^{\pm} = 0$, $p^+ = -\rho^+y + \rho^+h$ and $p^- = -\rho^-y + \rho^+h$, where u_0 is a constant, and possess asymptotic expansions:

$$\begin{aligned}(u^{\pm}, v^{\pm}, p^{\pm}) &= (u_0, 0, -\rho^{\pm}y + \rho^+h) + \varepsilon(u_1^{\pm}, v_1^{\pm}, p_1^{\pm}) \\ &\quad + \varepsilon^2(u_2^{\pm}, v_2^{\pm}, p_2^{\pm}) + \varepsilon^3(u_3^{\pm}, v_3^{\pm}, p_3^{\pm}) + O(\varepsilon^4).\end{aligned} \tag{10}$$

By inserting (10) into (1) to (4) and (7) to (9) and arranging the resulting equations according to the powers of ε, it follows that $(u_0, 0, -\rho^{\pm}y + \rho^+h)$ are the solutions of the zeroth order system of equations and the equations of the order ε are as follows:

$$u_{1x}^{\pm} + v_{1y}^{\pm} = 0, \tag{11}$$

$$u_0 u_{1x}^{\pm} = -p_{1x}^{\pm}/\rho^{\pm}, \tag{12}$$

$$p_{1y}^{\pm} = 0; \tag{13}$$

at $y = h$,

$$p_1^{\pm} + \eta_1 p_{0y}^{+} = 0; \tag{14}$$

at $y = 0$,

$$(p_1^+ - p_1^- + \eta_2(p_{0y}^+ - p_{0y}^-) = 0; \tag{15}$$

$$u_0\eta_{2x} - v_1^+ = 0; \tag{16}$$

at $y = -1$,

$$v_1^- = 0. \tag{17}$$

Hereafter for the sake of convenience we shall use ρ to denote ρ^+ and set ρ^- equal to 1. From (13), $p^{\pm}_1$ are functions of x only. $p^+_1 = \rho\eta_1$ by (14) and $p^-_1 = \rho\eta_1 + \eta_2(1 - \rho)$ by (15). We can find $v^{\pm}_1$ by using (11), (12), (15), and (17) so that

$$v_1^+ = y(\eta_{1x}/u_0) + u_0\eta_{2x}, \qquad v_1^- = (y+1)(\rho\eta_{1x} + (1-\rho)\eta_{2x})/u_0 . \tag{18}$$

$u^{\pm}_1$ are also derived from (11)

$$u_1^+ = -\eta_1/u_0, \qquad u_1^- = (-\rho\eta_1 - (1-\rho)\eta_2)/u_0, \tag{19}$$

where we assume $\eta_1(x=-\infty) = \eta_2(x=-\infty) = 0$, $u_1^{\pm}(x=-\infty) = 0$.

Similarly, we can find $p^{\pm}_2$, $v^{\pm}_2$, $u^{\pm}_2$, $p^{\pm}_3$, $v^{\pm}_3$, $u^{\pm}_3$ in terms of η_1 and η_2 without using the kinematic conditions (5) and (6). From (5) and (6), and the asymptotic expansions of u^- and v^-, we have

at $y = h$,

$$\begin{aligned} &u_0\eta_{1x} - v_1^+ + \varepsilon(u_1^+\eta_{1x} - \eta_1 v_{1y}^+ - v_2^+) \\ &+\varepsilon^2(u_2^+\eta_{1x} + \eta_1\eta_{1x}u_{1y}^+ - v_{1yy}^+\eta_1^2 - \eta_1 v_{2y}^+ - v_3^+) + O(\varepsilon^3) = 0, \end{aligned} \tag{20}$$

and at $y = 0$,

$$\begin{aligned} &u_0\eta_{2x} - v_1^- + \varepsilon(u_1^-\eta_{2x} - \eta_2 v_{1y}^- - v_2^-) \\ &+\varepsilon^2(u_2^-\eta_{2x} + \eta_2\eta_{2x}u_{1y}^- - v_{1yy}^-\eta_2^2 - \eta_2 v_{2y}^- - v_3^-) + O(\varepsilon^3) = 0. \end{aligned} \tag{21}$$

Then we make use of these equations to find the equations of the free surface $\eta_1(x)$ and the interface $\eta_2(x)$. By substituting u_0, $u^{\pm}_1$, $v^{\pm}_1$, $u^{\pm}_2$, $v^{\pm}_2$, $v^{\pm}_3$ into (20) and (21) and eliminating η_1, we obtain

$$\begin{aligned} &(u_0 - \rho c_1/u_0 - (1-\rho)/u_0)\eta_{2x} + \varepsilon(E\eta_2\eta_{2x} + E_2\eta_{2x}) \\ &+ \varepsilon^2(F_1\eta_2^2\eta_{2x} + F_2\eta_{2x} + F_3\eta_{2xxx} + F_4 b_x) \\ &+ O(\varepsilon^3) = 0, \end{aligned} \tag{22}$$

where if we let $c_1 = (2u_0^2 - (1-\rho))/(\rho + u_0^2 - h)$, $D_1 = u_0/(\rho + u_0^2 - h)$, $\lambda = u_2^{\pm}(-\infty)$, and $R = \rho c_1 + 1 - \rho$, then

$$E = -(R^2 + 2Ru_0^2)u_0^{-3} - \rho D_1((hc_1^2 - R^2)u_0^{-4} + (2c_1^2 - 2R - 2c_1)u_0^{-2}),$$

$$\begin{aligned} F_1 = &-\rho D_1 u_0^{-1}((3c_1^3 - 3c_1^2 + R^2/2)u_0^{-3} + (3hc_1^3/2 - 3R^3/2)u_0^{-5} \\ &+ 3D_1(\rho u_0^{-1} + \rho R u_0^{-3})((3R/2 + c_1 - c_1^2)u_0^{-1} + (R^2/2 - hc_1^2/2)u_0^{-3}) \\ &- 3R^2u_0^{-3}/2 - 3R^3u_0^{-5}/2, \end{aligned}$$

$$F_2 = \lambda((-\rho D_1 u_0^{-1})(2 + Ru_0^{-2} - c_1 - hc_1u_0^{-2}) + (1 + Ru_0^{-2})),$$

$$\begin{aligned} F_3 = &(-\rho D_1 u_0^{-1})(-c_1(\rho h^2/2 + \rho/3)u_0^{-1} - (u_0^2\rho h + (1-\rho)/3)u_0^{-1} \\ &+ (c_1(\rho h^3/3)/u_0\rho) + u_0 h^2/2) \\ &- c_1(\rho h^2/2) + \rho/3)u_0^{-1} - (u_0^2\rho h + (1-\rho)/3)u_0^{-1}, \end{aligned}$$

$$F_4 = \rho D_1 - u_0 .$$

3. STEADY MODIFIED KdV EQUATION WITH FORCING (SFMKdV)

From the zeroth order term of (22), we obtain

$$u_0 - (\rho c_1/u_0) - (1-\rho)/u_0 = 0 ,$$

and by the expression for c_1 in (22), it follows that

$$u_0^4 - (1+h)u_0^2 + h(1-\rho) = 0 , \tag{23}$$

and

$$u_0^2 = (1 + h \pm ((1-h)^2 + 4\rho h)^{1/2})/2.$$

We denote the two values of u_0^2 by u_{01}^2 and u_{02}^2 respectively corresponding to the plus and minus signs. Without loss of generality we assume u_{01} and u_{02} are both positive and call them critical speeds, near each of which a nonlinear theory for the motion of the interface has to be developed.

Next we consider the coefficients of $\eta_2\eta_{2x}$ in the first order terms of the equation (22). If E in (22) is not zero, an FKdV can be derived if we assume $b(x) = b^*(x^*)(h^* \cdot \varepsilon^2)^{-1}$ and $x = \varepsilon^{1/2}x^*/h^*$ in nondimensional variables and similar results as in [1] can be obtained. However, E may vanish. First, let us simplify the expression of E,

$$\begin{aligned} E &= -((\rho c_1 + 1 - \rho)^2/u_0^3) - 2((\rho c_1 + 1 - \rho)/u_0) \\ &\quad - \rho D_1[-2((\rho c_1 + 1 - \rho)/u_0) - ((\rho c_1 - \rho + 1)^2/u_0^3) + 2(c_1^2/u_0) \\ &\quad + h(c_1^2/u_0^3) - 2(c_1/u_0)]/u_0 \\ &= 3(u_0\rho)^{-1}(u_0^2 + \rho - h)(\rho(u_0^2 h - u_0^4 - u_0^2 + 1) - u_0^4 + 2u_0^2 - 1), \\ &= 3u_0(1 - u_0^2)(\rho h(u_0^2 + \rho - h))^{-1}(u_0^4 + (1-2h)u_0^2 + h^2 - 1). \end{aligned}$$

where (23) has been used. When u_0 satisfies the equation (23), it is seen that u^2_0 is neither 1 nor h - ρ. Hence $E = 0$ implies $u^4_0 + (1 - 2h)u^2_0 + h^2 - 1 = 0$. Let $u_0 = u_{01}$ or u_{02}. Then

$$u_{01}^4 + (1-2h)u_{01}^2 + h^2 - 1 = 1 + h\rho + (2-h)((1-h)^2 + 4h\rho)^{1/2} , \tag{24}$$

$$u_{02}^4 + (1-2h)u_{02}^2 + h^2 - 1 = 1 + h\rho - (2-h)((1-h)^2 + 4h\rho)^{1/2} , \tag{25}$$

Equation (24) tells us that E does not vanish if we take u_{01} as a critical speed. Suppose both sides of (24) vanish. Then real u^2_{01} imples $h < 5/4$ and the right hand side of (24) is greater than zero. This is a contradiction. Thus the only possible case for $E = 0$ is that the critical speed u^2_0 is equal to u^2_{02}, and it is easy to show that $E = 0$ if $u^2_0 = u^2_{02}$, and

$$1 + h\rho = (2-h)((1-h)^2 + 4h\rho)^{1/2}. \tag{26}$$

With the conditions (21) and (25), we obtain a Steady FMKdV,

$$F_1\eta_2^2\eta_{2x} + F_2\eta_{2x} + F_3\eta_{2xxx} + F_4 b_x = 0 \tag{27}$$

where

$$\begin{aligned} F_1 &= 3u_0(4\rho + 3h - u_0^2)\,, \\ F_2 &= \lambda(2(1+h)u_0^2 - 4h(1-\rho))u_0^{-2}\,, \\ F_3 &= u_0^{-1}(h(1+h) - u_0^2(h^2 + 1 + 3\rho h)\,, \\ F_4 &= u_0(h - u_0^2)\,. \end{aligned}$$

The coefficients F_1 to F_4 here are the simplified forms of F_1 to F_4 in the previous section by using (23). The sign of F_3F_1 determines the existence of solutions of (27). In the following sections, we assume $F_3F_1 > 0$ and the case for $F_3F_1 < 0$ is considered in subsequent study [10].

3.1 Symmetric solition-like waves

We assume the speed $U^{\pm}$ of the fluid at $x = -\infty$ are the same and given by $U = u_0 + \lambda\varepsilon^2 + O(\varepsilon^3)$ and consider (27) $F_1/F_3 > 0$ and $F_2/F_3 < 0$. (27) can be rewritten as

$$\eta_{2xxx} = -A_1\eta_2^2\eta_{2x} + A_2\eta_{2x} + A_3b_x\,, \tag{28}$$

where $A_1 = F_1/F_3 > 0$, $A_2 = -F_2/F_3 > 0$, $A_3 = -F_4/F_3$. Here λ is a parameter determining the flow regime, eg. $\lambda > 0$ and $\lambda < 0$ represent the supercritical and subcritical cases respectively. Where $b_x \equiv 0$, (28) has soliton solutions whose value is 0 at $x = \pm\infty$ for $A_2 \geq 0$:

$$\eta_2(x) = \pm(6A_2/A_1)^{1/2}\operatorname{sech}((A_2)^{1/2}x), \tag{29}$$

For $A_2 \leq 0$, there is no soliton solution. The solutions in (29) are obtained as in the classical case by taking the limit of elliptic functions in the periodic solutions of (28) for $b_x = 0$ when the wave length tends to infinity. Next we consider (28) when $b_x \neq 0$ but of compact support.

We look for a solution $\eta_2(x)$ such that $A_2 > 0$ and

$$\lim_{|x|\to\infty}(d/dx)^j\eta_2(x) = 0 \qquad j = 0, 1, 2.$$

Integrating (28) from $-\infty$ to x, it follows that

$$A_2\eta_2 - \eta_{2xx} = A_1\eta_2^3/3 - A_3b(x)\,, \quad \infty < x < \infty\,. \tag{30}$$

(30) can be converted to the following integral equation:

$$\eta_2(x) = \int_{-\infty}^{\infty} K(x,\xi)(A_1\eta_2^3(\xi)/3 - A_3b(\xi))\,d\xi\,,$$

where $K(x,\xi) = \exp(-\sqrt{A_2}\,|\,x-\xi\,|)/(2\sqrt{A_2})$ is a Green function of $A_2K(x,\xi) - K_{xx}(x,\xi) = \delta(x,\xi)\,, -\infty < x < \infty$.

Define

$$\begin{aligned} T(\eta_2) &= \int_{-\infty}^{\infty} K(x,\xi)(A_1\eta_2^3(\xi)/3 - A_3b(\xi))\,d\xi\,, \\ \|u\| &= \|u\|_\infty = \sup_{x\varepsilon\Re}|\eta_2(x)|, \\ H &= \{u\,|u\,\varepsilon\, C(\Re), \|\exp(\sqrt{A_2}\,|x|)u\| < \infty\}, \end{aligned}$$

$$B_M = \{u \mid u \in H, \|u\| \le M, 0 < M < \infty\}.$$

Then clearly H is a complete metric space and B_M is a closed ball in H, and the following theorem can be proved by Contraction Mapping Theorem[8].

Theorem 1. (30) has a solution in $C^3(\Re)$ which decays exponentially at $|x| = \infty$ if A_2 is sufficiently large.

We have shown that (28) has an exponentially decaying solution as x tends to $\pm\infty$. In the following we use numerical computation to find symmetric soliton-like solutions of (28) when the obstruction $b(x)$ is given by $b(x) = R(1 - x^2)^{1/2}$ for $|x| \le 1$ and $b(x) = 0$ for $|x| > 1$, where R is a given constant.

Let

$$\eta_2(x) = \pm(6A_2/A_1)^{1/2}\operatorname{sech}((A_1)^{1/2}(x - x_0)), \tag{31}$$

where x_0 is a phase shift. To find a solution in $|x| \le 1$, we need only consider (30) in $-1 \le x \le 0$ subject to $(\eta'_2(x))^2 = -A_1\eta^4_2/6 + A_2\eta^2_2$ at $x = -1$ and $\eta'_2(x) = 0$ at $x = 0$. This problem can be solved numerically by a shooting method and the phase shift x_0 is determined by (31) for $x = -1$. There are three parameters involved in this analysis: the depth ratio h, the perturbation of the horizontal velocity at far upstream λ and the density ratio ρ. The numerical results are given in Fig. 2 and Fig. 3. Since solutions for different values of h and ρ are qualitatively similar, we choose $h = 0.98$ and $r = 0.25$ in all calculations. Four typical soliton-like solutions are shown in Fig. 2. Fig. 3 shows the relation between the value of soliton-like solution at $x = 0$ as a function of λ. In both numerical results, we assume $R = 1$.

We remark that the shooting method for two-point boundary value problem is simple. The differential equation is solved as an initial value problem in some form over the given domain for a succession of trial values of η which are adjusted till the boundary conditions at both ends can be satisfied at once. The simplest way to do is to shoot from one end to the other, that is to say we choose η such that the left-end boundary is satisfied. The second trial for shooting is done with the corrected value of η which is adjusted according to the miss-distance from the first shooting. We repeat the process until η satisfies the right-end boundary condition.

3.2 SYMMETRIC WAVES WITH ZERO BEHIND AND AHEAD OF THE OBSTRUCTION

Similar to section 3.1, we consider the equation

$$\eta_{2xxx} = -A_1\eta_2^2\eta_{2x} + A_2\eta_{2x} + A_3b_x, \tag{32}$$

where $A_1 = F_1/F_3 > 0$, $A_2 = -F_2/F_3 > 0$, $A_3 = -F_4/F_3$. Integrating (32) from $-\infty$ to x, we obtain

$$\eta_{2xx} = -A_1\eta_2^3/3 + A_2\eta_2 + A_3b(x), \tag{33}$$

where $b(x)$ is assumed to have compact support and $\eta_2(-\infty) = 0$. We assume $\eta_2 \equiv 0$ in $(-\infty, x_-)$ where $[x_-, x_+]$ is the support of the obstruction. We can show that the solution of (33) exists and is bounded with initial values $\eta_2(x_-) = \eta_{2x}(x_-) = 0$. [8] In the following, we use numerical computation to find symmetric wave solution of (33) which is zero behind and

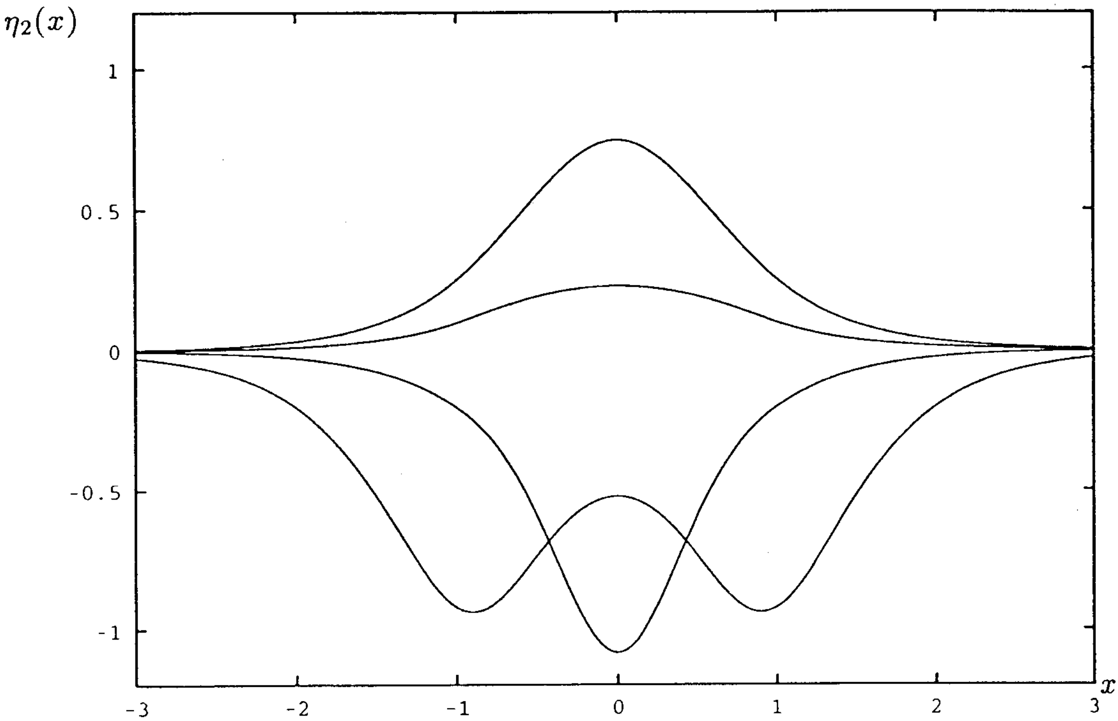

Fig. 2. Four typical soliton-like solution.
h = 0.98, R = 1, λ = -4

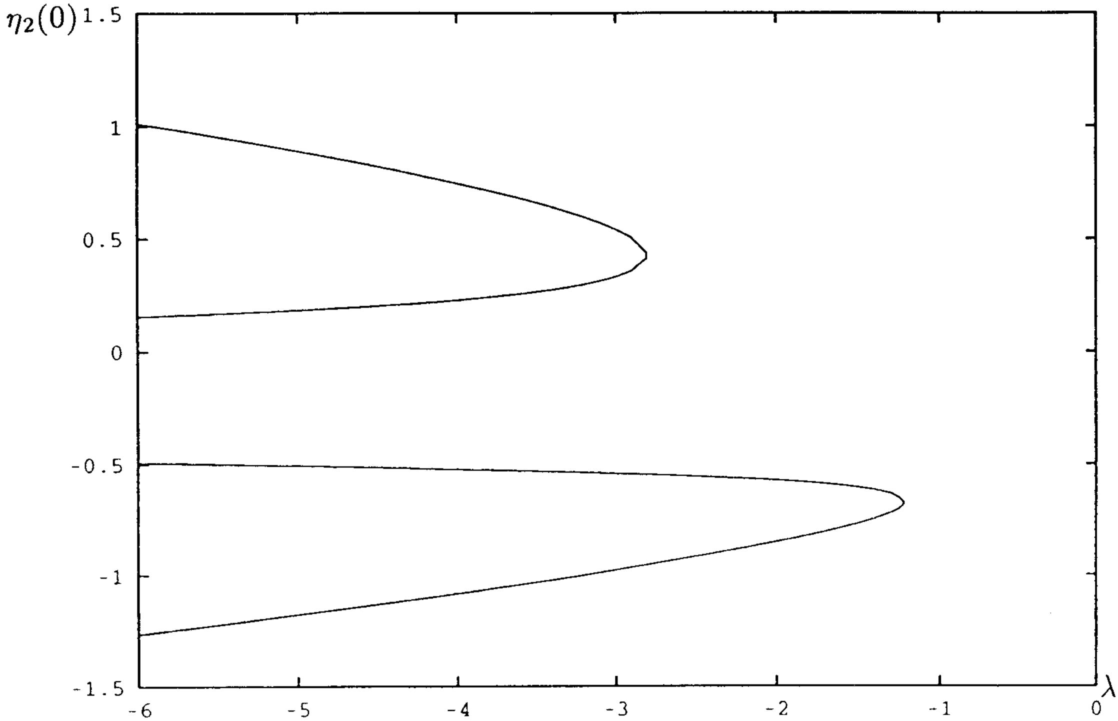

Fig. 3. Relationship between $\eta_2(0)$ and λ
h = 0.98, R = 1

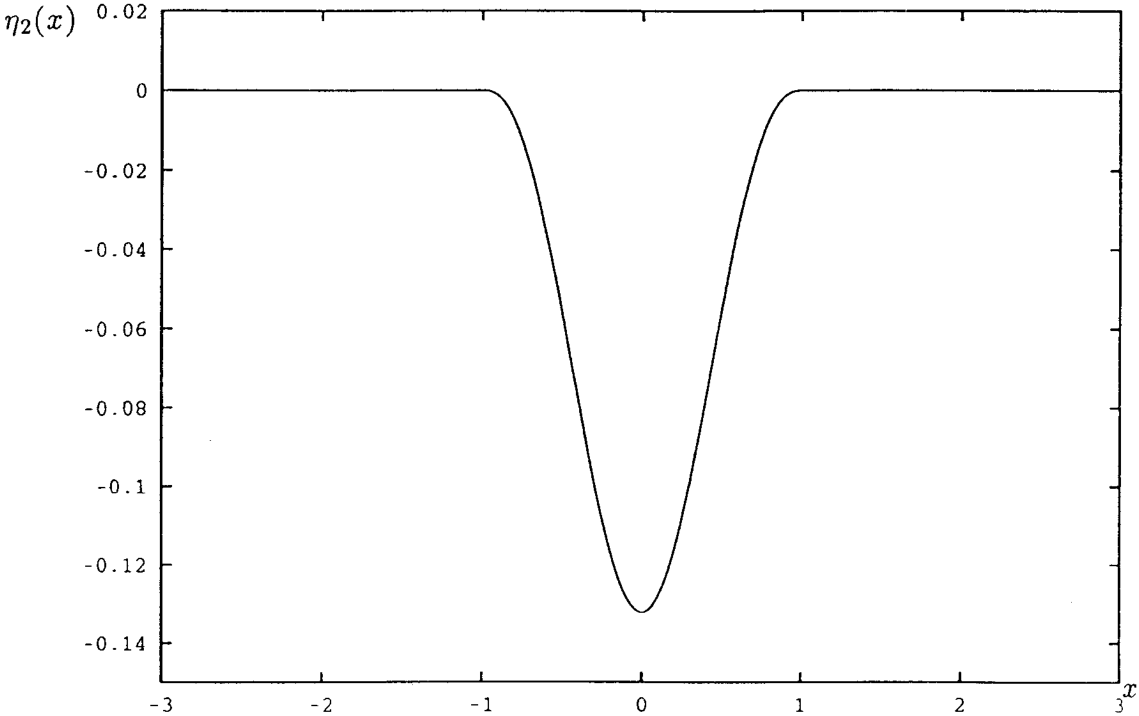

Fig. 4. Symmetric solution with one hump.
$h = 0.98$, $R = 1$, $\lambda = 14.000283$

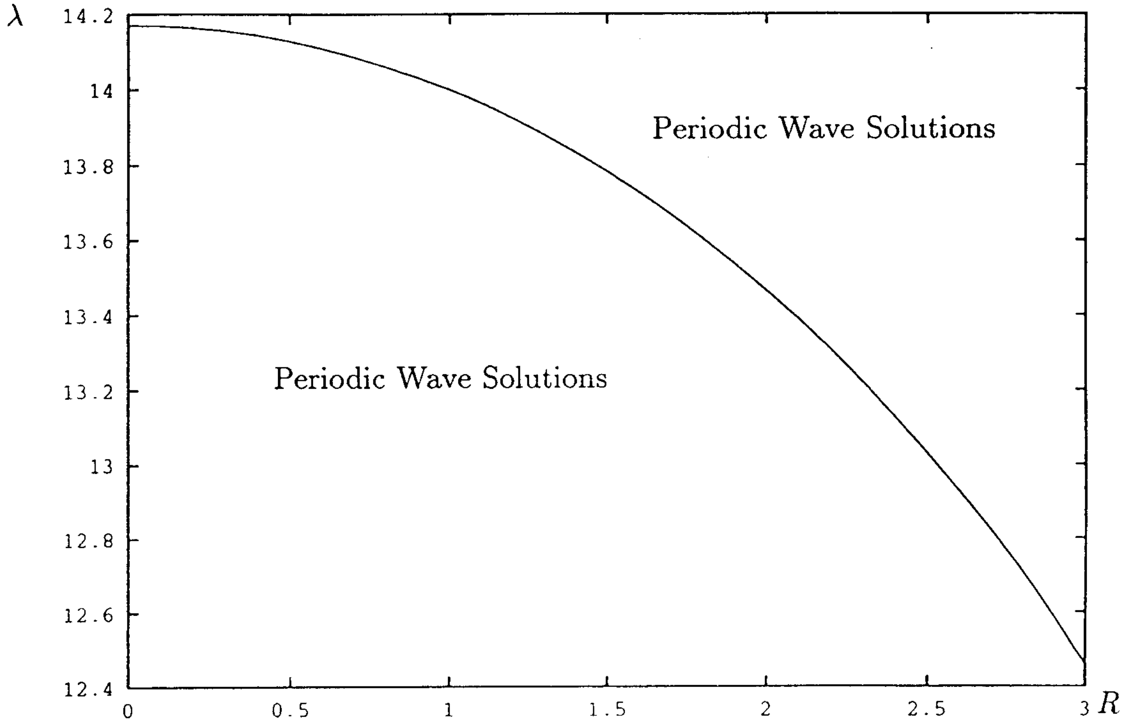

Fig. 5. Solution curve of symmetric solutions with one hump, $h = 0.98$

ahead of the elliptic obstruction. Similar methods as in section 3.1 is used to find the solutions of this problem. To find a solution in $|x| \leq 1$, we need only consider (33) in $-1 \leq x \leq 0$ subject to $\eta'_2(x) = \eta_2(x) = 0$ at $x = -1$ and $\eta_{2x}(x) = 0$ at $x = 0$. The same assumptions as in section 3.1 have been made for the obstruction and the numerical results are shown in Fig. 4 and 5. Fig. 4 shows the symmetric solutions for positive values of λ. The relations between R, which represents the height of the obstruction, and λ are given in Fig. 5. We note that, for a given R, symmetric solution is embedded in periodic solutions.

We have shown that there exist two types of symmetric solutions of the interfacial wave forms both analytically and numerically. First type of solutions depends strongly on the values of λ as shown in section 3.1 and corresponds to symmetric soliton-like solutions. The other type of solutions is the limiting case of the solutions with waves behind the bump and zero ahead. These correspond to symmetric wave solution with one hump.

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