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MATHEMATICAL MODELLING OF COUPLED RIVER-GROUNDWATER FLOWS

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ABSTRACT

We present an adaptive grid refinement technique to simulate coupled river-groundwater flows.

New numerical results based on a splitting algorithm, a domain decomposition and an algebraic mapping are analyzed and applied to real-world river-groundwater systems.

INTRODUCTION

Mathematical models of a coupled river-groundwater flow, based on the numerical solution of the Saint-Venant[1] and the Boussinesq[2] equations, have been extensively studied in recent years[3-15].

For this non-standard initial-boundary value problem, we propose a numerical approach based on the splitting algorithm[5] and on a decomposition technique combined with numerically generated curvilinear grids.

We obtain a non-monotone, nine-point, divergent, self-adjoint and positive definite numerical scheme for the Boussinesq equation in a curvilinear coordinate system and a family of seven-point conditionally monotone schemes. The property of monotonicity provides a stable global splitting algorithm in terms of some geometrical parameters related to the corresponding coordinate transformation.

Our technique is highly efficient in regions of large gradients of the solution (for instance near rivers).

BASIC EQUATIONS

In this section, we present a mathematical formulation of a coupled river-groundwater problem based on a system of PDE's.

First, we introduce the parabolic version of the Saint-Venant equations, i.e.,

$$w_t + Q_s = d, \tag{1}$$

$$h_s - I + (n^2 Q|Q|) / (w^2 R^{4/3}) = 0, \tag{2}$$

where t is the time coordinate, $t \in (O,T)$, T the simulation time, s the space coordinate along a river, $s \in G$, where G is a set of river curves at a plane (x,y), $\zeta \equiv \zeta(s,t) = h+z$ the water surface level, $h=h(s,t)$ the water depth, $z=z(s)$ the bottom level, $w=w(s,h)$ the cross-sectional area of a river flow, $Q=Q(s,t) = wV$ the discharge, $V=V(s,t)$ the velocity, $R=R(s,h)$ the hydraulic radius, $I\equiv-(z)_x$ the slope of a bottom, n the Manning coefficient and $d\equiv d(s,t)$ the lateral discharge.

Next, we introduce the Boussinesq equation

$$\mu H_t = \text{div}\,(K\ \text{grad}\ H) + \varepsilon. \tag{3}$$

Here, $\text{div}\equiv\partial/\partial x + \partial/\partial y$, $\text{grad}\equiv(\partial/\partial x, \partial/\partial y)$ and $x,y \in D$, where D is the groundwater region, $\mu\equiv\mu(x,y)$ the storage coefficient, $H\equiv H(x,y,t)$ the groundwater level, $K=K_s(H-Z)$ if $Z \le H \le Z0$, $Z\equiv Z(x,y)$ the confining groundwater bed level, $Z_0\equiv Z_0(x,y)$ the ground level, $K_s\equiv K_s(x,y)$ the hydraulic conductivity and $\varepsilon\equiv\varepsilon(x,y,t)$ an influx.

Let $\alpha\equiv\alpha(t)$, $\beta\equiv\beta(t)$ be the coefficients of the hydraulic connection between river and groundwater flows.

Define

$$Q^H = \begin{cases} \alpha\ (\zeta - H), & \text{if}\quad H \ge z, \\ \beta\ (\zeta - z), & \text{if}\quad H < z, \end{cases} \tag{4}$$

$$d(s,t) = -Q^H, \qquad \varepsilon(x,y,t) = Q^H, \tag{5}$$

where Q^H denotes the river-groundwater exchange

Mass conservation and continuity of ζ, at internal nodes of the intersection of river branches, is represented by the adjoint conditions

$$\sum_{J \in i_*} Q_J = 0,\ \zeta_J = \zeta_*, J \in i_*, \tag{6}$$

where i_* is a set of branches with the common internal node J and where ζ_* denotes the water level in the corresponding node.

It is also necessary to specify the boundary conditions at the boundary Γ of a groundwater region , at inflow and outflow nodes of the river system, as well as the initial conditions for H and ζ at $t=0$.

Further details are given in [3-6,14].

DECOMPOSITION. SPLITTING TECHNIQUE

In this section, we present the main steps of the numerical algorithm.

Our numerical procedure is based on a partition of a region D into subdomains and on a generation (in every subdomain) of a local curvilinear coordinate system adapted to the shape of both the boundary and river system.

An efficiency-estimate of this approach (with regard to problems of river-groundwater flows) is given in[13-14]. A comparison with rectangular grids shows an accuracy increase of about 30%.

The first stage of the numerical procedure involves the solution of Eq.(3) with the time step $\Delta\tau$, by an iterative implicit nine-point scheme and a block version of the SOR method.

At the second stage, we solve the system of the Eq's (1)-(2), (6), with the time-step $\Delta\tau=\Delta\tau / L$ (L an integer), by an implicit four-point scheme and by a special version of the sweep method[15].

The global iterative algorithm is based on the convergence criterion

$\left\| Q^{H,ng+1} - Q^{H,ng} \right\| \le \varepsilon$ where ng is an iteration number.

ALGEBRAIC METHOD OF THE CURVILINEAR GRID GENERATION

Next, we propose a modification of well-known algebraic methods formerly developed for and applied to geometric problems of computer-aid design[16-18].

Let $R_k(\xi)$, $k=0,\ldots,N_\eta$, $T_p(\eta)$, $p=0,\ldots,N_\xi$, be the boundary and inner coordinate curves at the "physical space" (x,y) to be transformed into the "computational space". (ζ,η) (Cf. Fig.1).

Define the coordinate transformation $F(\xi,\eta) \equiv (x(\xi,\eta), y(\xi,\eta))$ by

$$F(\xi,\eta) = \sum_p T_p(\eta)\psi_p(\xi) + \sum_k R_k(\xi)\varphi_k(\eta) - \sum_p \sum_k T_r(\eta_k)\psi_p(\xi)\varphi_k(\eta), \qquad (6)$$

where $\psi_p(\xi)$, $\varphi_k(\eta)$ are the blending (mixing) functions[16] satisfying $\psi_p(\xi_i)=\delta_{p,i}$, $\varphi_k(\eta_j)=\delta_{k,j}$, $p,i=0,\ldots,N_\xi$, $k,j=0,\ldots,N_h$ and ξ_p,η_k the corresponding coordinate lines $\xi=$ const, $\eta=$ const.

Consequently $F(\xi,\eta)$ satisfies the conditions

$$F(\xi,\eta_k) = R_k(\xi), \quad k=0,\ldots,N_\eta$$

$$F(\xi_p,\eta) = T_p(\eta), \quad p=0,\ldots,N_\xi .$$

Finally, we invoke the calibration parameters $\varepsilon_{1,p}$, $\varepsilon_{2,p}$ and we define $\psi_p(\xi)$, $\varphi_k(\eta)$ by

$$\varphi_p(\xi) = \begin{cases} \phi\left((\xi-\xi_p)/\varepsilon_{1,p}\right), & \text{if } \xi \ge \xi_p , \\ \phi\left((\xi-\xi_p)/\varepsilon_{2,p}\right), & \text{if } \xi < \xi_p , \end{cases}$$

where

$$\phi(t)=\begin{cases}(1-t)^2(1+2t) , & \text{if } 0\le t\le 1 ,\\ 0 , & \text{if } t\ge 1 ,\\ \phi(-t) , & \text{if } t<0.\end{cases}$$

Fig.2 shows an example of the grid generation technique.

NUMERICAL SCHEME FOR THE BOUSSINESQ EQUATION

In this section, we construct a set of self-adjoint, positive definite, conditionally monotone operators and a divergent finite-difference approximation of the Boussinesq equation (in a curvilinear coordinate system) based on the weak formulation of Eq.(3), i.e.,

$$\iint_D \mu H_t \upsilon dxdy = \oint_{G\cap T} (\sigma_1 H+\sigma_2)d\iota - \iint_D K\mathrm{grad}(H)\mathrm{grad}(\upsilon)dxdy = 0, \qquad (7)$$

where $H, \upsilon \in W^1_2(D)$, $H_t \in L_2((0,T)\times D)$, $d\iota$ is an element of G or and σ_1, σ_2 are determined from the boundary conditions or by the Eq's (4)-(5).

Transformation of Eq (7) in terms of the new variables ξ,η reveals that

$$\iint_\Delta \mu JH_t\, \upsilon d\xi\, d\eta + \oint_{\gamma\cap S}(\chi_1 H+\chi_2)d\vartheta +$$
$$+\iint_\Delta K[A\csc(\varphi)H_\xi \upsilon_\xi - \mathrm{ctg}(\varphi)(H_\xi \upsilon_\eta + H_\eta \upsilon_\xi) + A^{-1}\csc(\varphi)H\upsilon]d\xi d\eta = 0, \qquad (8)$$

where J is the Jacobian of the transformation, φ an angle between the local base vectors $1_\xi \equiv 1_\xi(x,y), 1_\eta \equiv 1_\eta(x,y)$ and $A = |1_\xi| / |1_\eta|$.

The correspondence between the "physical" and "computational" space is shown in Fig.1.

Let $\Delta\tau$ be a time step, $\Delta\xi$, $\Delta\eta$ space steps in the transformed region Δ and $\Delta\vartheta$ a space step along a boundary S . The uniform grids in the transformed region, in the boundary of this region and in the river curves are respectively denoted by Δ_h, S_h and γ_h.

Applying the symmetric approximant to the rectangular cell $\Delta_{i,j} \in \Delta_h$, i.e.,

$$\iint_{\Delta_{i,j}} K\mathrm{grad}(H)\mathrm{grad}(\upsilon)\, d\xi\, d\eta \approx 0.25 \sum_{\alpha,\beta=0}^{1} I_{i,j}^{\alpha,\beta} ,$$

yields

$$\sum_{\Delta_h}[\upsilon_{i,j}S_{i,j}\mu_{i,j}(H)_{t,i,j} + 0.25\sum_{\alpha,\beta=0}^{1} I_{i,j}^{\alpha,\beta}]\Delta\xi\,\Delta\eta + \sum_{\gamma_h\cap S_h}(\chi_1 H+\chi_2)\Delta\vartheta = 0, \qquad (9)$$

where $I_{i,j}^{\alpha,\beta}$ involves the points $(x_{i-2\alpha+1}, y_j)$, (x_i, y_j) , $(x_i, y_{j-2\beta+1})$ and

$$(H)_{t,i,j} = (H_{i,j}^{n+1} - H_{i,j}^{n}) / \Delta\tau , H_{i,j}^{n} = H(x_i, y_j, t_n).$$

The condition $\upsilon_{i,j} = 1$, $\upsilon_{k,p} = 0$ for $i \ne k$, $j \ne p$ yields the divergent scheme:

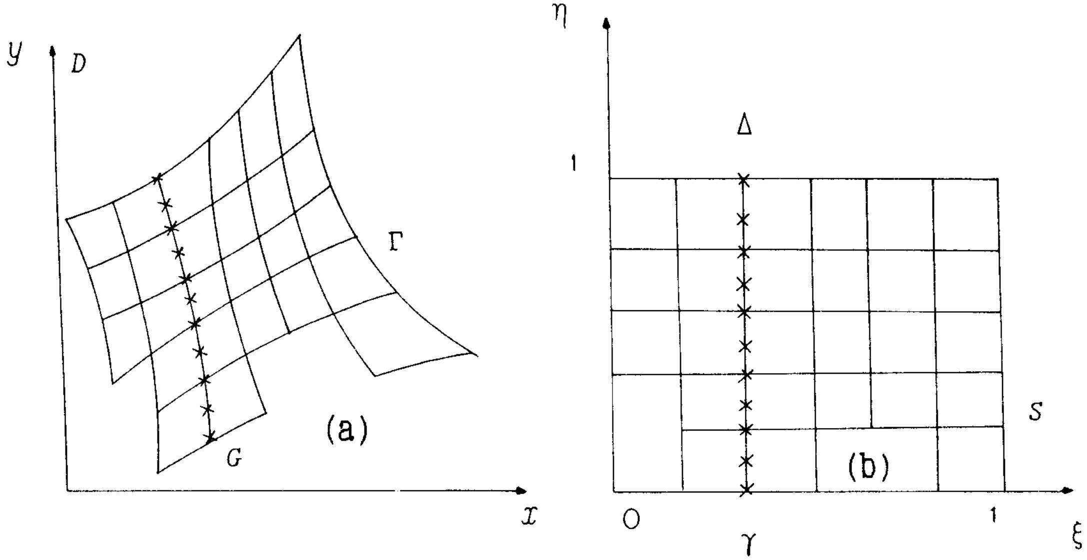

Fig. 1. Coordinate tranformation, (a) "physical space", (b) "Computational space".

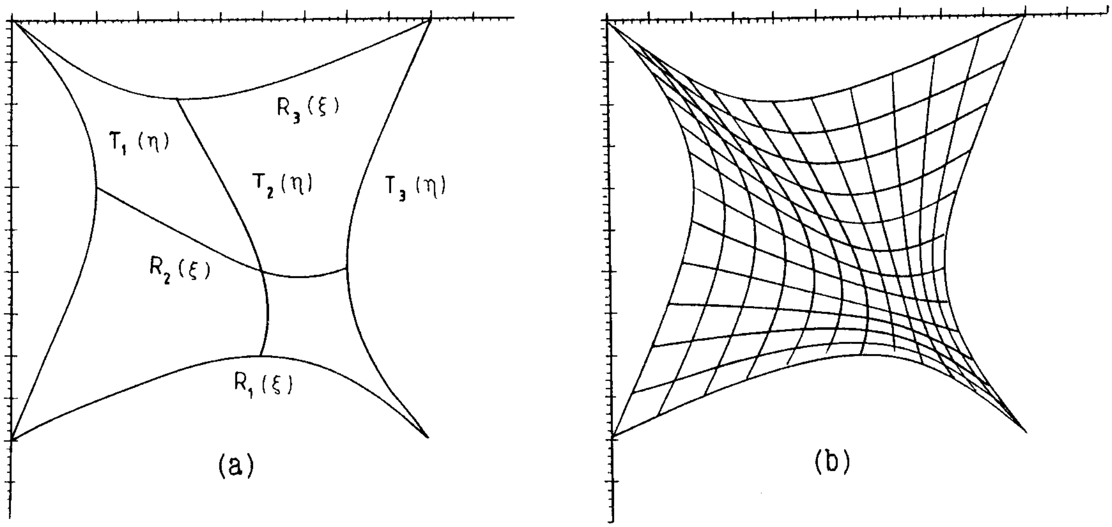

Fig. 2. Algebraic mapping technique, (a) initial region, (b) final grid.

$$S_{i,j}\,\mu_{i,j}\,(H)_{t,i,j} = [(h^{\eta} W_{\xi})^{n+1}_{i+1/2,j} - (h^{\eta} W_{\xi})^{n+1}_{i-1/2,j}]\,/\,\Delta\xi \; +$$

$$[(h^{\xi} W_{\eta})^{n+1}_{i,j+1/2} - (h^{\xi} W_{\eta})^{n+1}_{j,j-1/2}]\;/\,\Delta\eta + (\omega_{1i,j} H^{n+1}_{i,j} + \omega_{2\,i,j})\Delta\vartheta$$

where $S_{i,j}$ is the area of the cell $\Delta_{i,j}$ and $(h^{\eta} W_{\xi})_{i+1/2,j}$, $(h^{\xi} W_{\eta})_{i,j+1/2}$ are the finite-difference approximations of the projections of the vector-function W=-Kgrad(H) on the vectors l_{ξ}, l_{η}, with components

$$(W, l_{\xi}) = K|l_{\xi}|(H_{\xi}\,/\,|l_{\eta}|\csc(\varphi) - H_{\eta}\,/\,|l_{\xi}|\mathrm{ctg}(\varphi)),$$

$$(W, l_{\eta}) = K|l_{\eta}|(H_{\eta}\,/\,|l_{\xi}|\csc(\varphi) - H_{\xi}\,/\,|l_{\eta}|\mathrm{ctg}(\varphi)).$$

The third term in Eq.(3) corresponds to an approximation of the boundary conditions and internal sources (4)-(5).

It is not hard to demonstrate (Cf.Eq.(9)) that the finite-difference approximation of the elliptic part of Eq.(1) possesses the properties analogcus to those of symmetry and positive definiteness of the initial differential operator.

Finally, observe that the proposed finite-difference operator is not monotone (i.e. L(H) $\geq$ 0 does not imply H $\geq$ 0). A non-symmetric approximant of the integral $\iint K\mathrm{grad}(H)\mathrm{grad}(\upsilon)d\xi\, d\eta$ (for instance $0.5(I^{01}_{i,j} + I^{11}_{i,j})$or $0.5(I^{00}_{i,j} + I^{10}_{i,j})$) yields the family of seven-point finite-difference operators being monotone under the conditions:

$$|\cos(\varphi(x,y))| \; < \; A(x,y) \; < \; 1/|\cos(\varphi(x,y))|\,,$$

$$0 < \varphi\;(x,y)\; < \pi.$$

The above conditions provide stability of the global splitting algorithm[5].

Observe that the non-monotonicity of the nine-point operator is practically "weak" without any impact on the stability of the algorithm, provided that the grid does not contain too small angles.

APPLICATION

The following application illustrates the capability of the proposed method in modelling the impact of an irrigation system on a natural groundwater flow (Fig.3).

We calibrate the model by measurements at the points 41,42,45,47,120,161,163.

The groundwater region is decomposed into four subregions 1,2,3 and 4. In each subregion, we generate a local curvilinear coordinate system and adapt a grid to the shape of the channels (Cf.Fig. 4 and Fig. 5). We apply our numerical procedure with the time steps $\Delta t = 1$ hour and $\Delta t = 1$ day. Internal iterations are interrupted whenever $\|M^{p+1} - M^{p}\| \leq 10^{-4}$, where M=H or M=$\xi$ and p is a number of internal iterations. Restricting the number of global iterations to ng=3, reveals that $\|H^{ng+1} - H^{ng}\| \leq 5\;10^{-4}$, $\|\xi_{\alpha}{}^{ng+1} - \xi_{\alpha}{}^{ng}\| \leq 10^{-4}$, where ξ_{α} is the average value of ξ (s,t), $s \in \gamma_h$, $t \in (t, t + \Delta\tau)$.

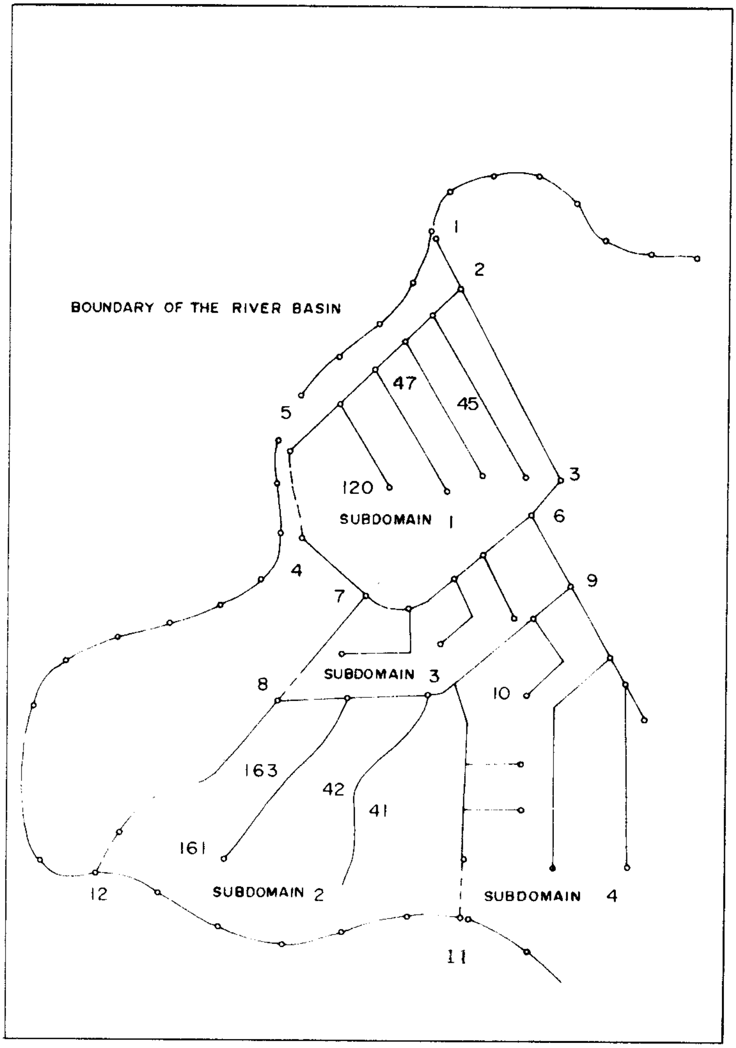

Fig. 3. Scheme of the irrigation system.

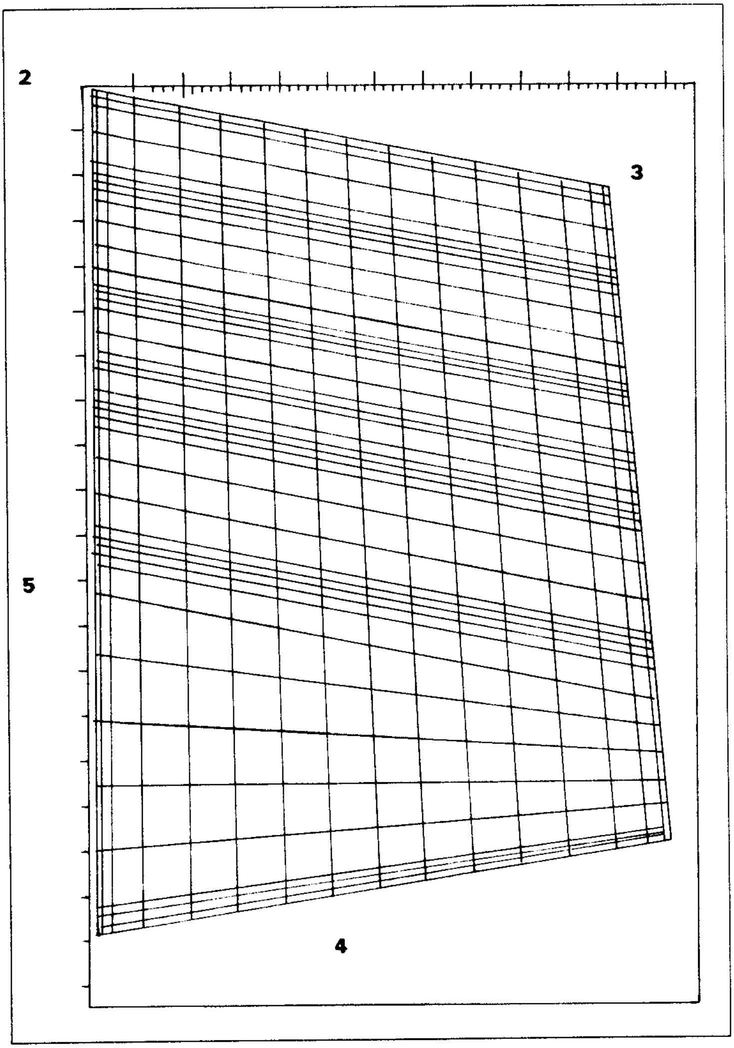

Fig. 4. Curvilinear grid for subdomain 1.

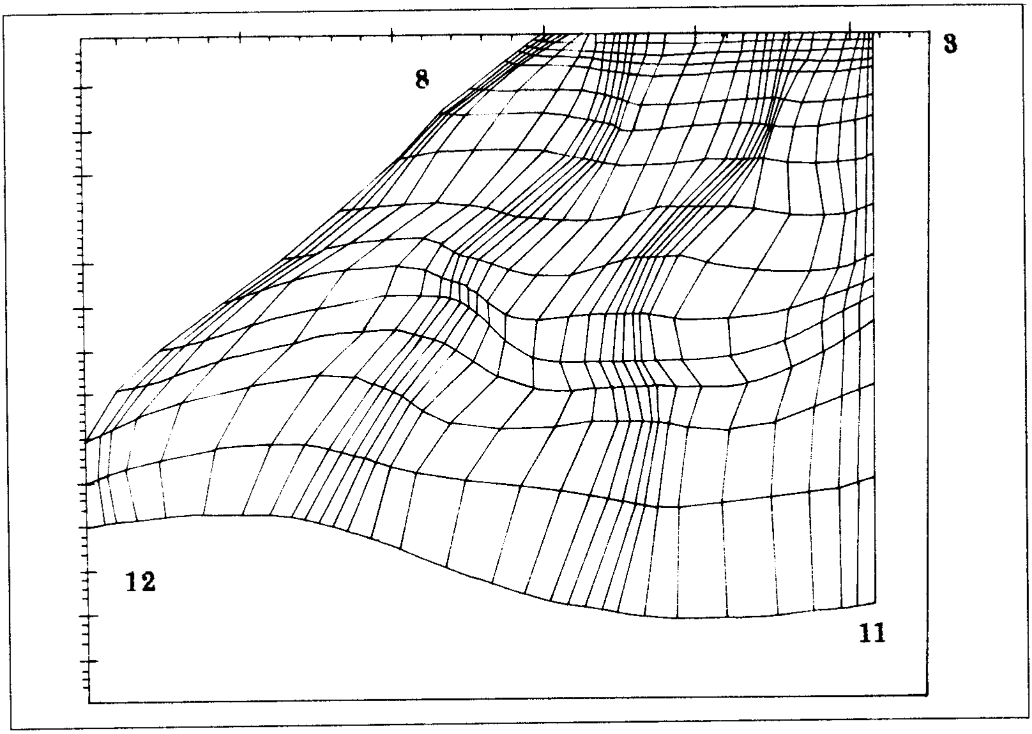

Fig. 5. Curvilinear grid for subdomain 2.

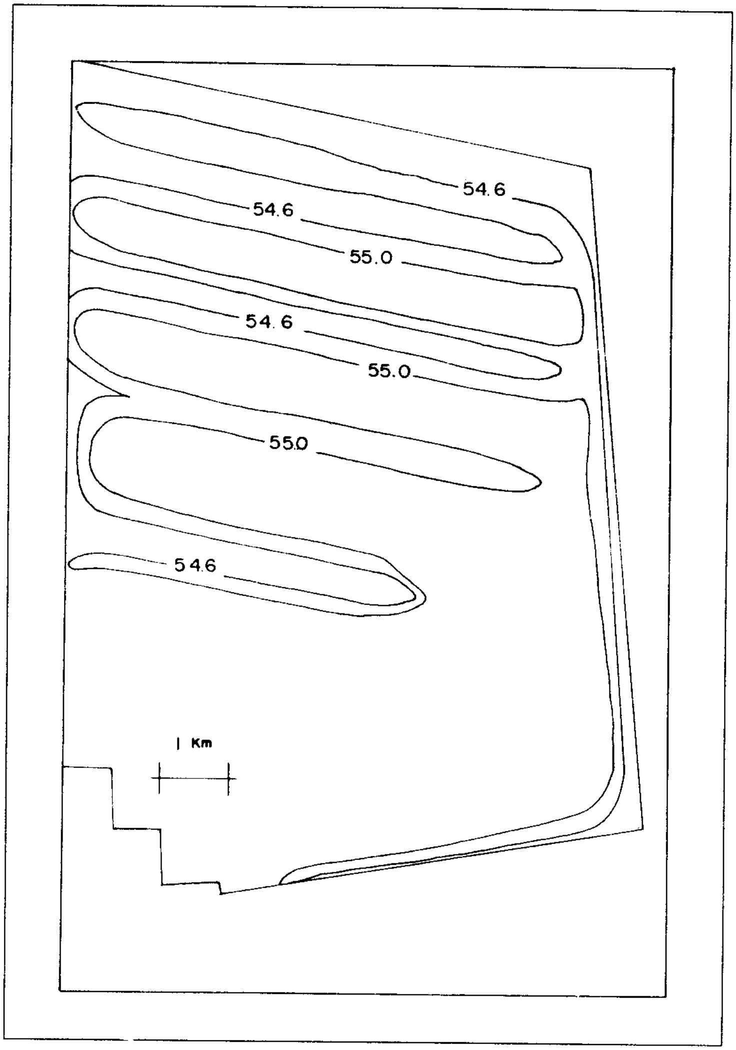

Fig. 6. Contour map of a groundwater level for subdomain 1.

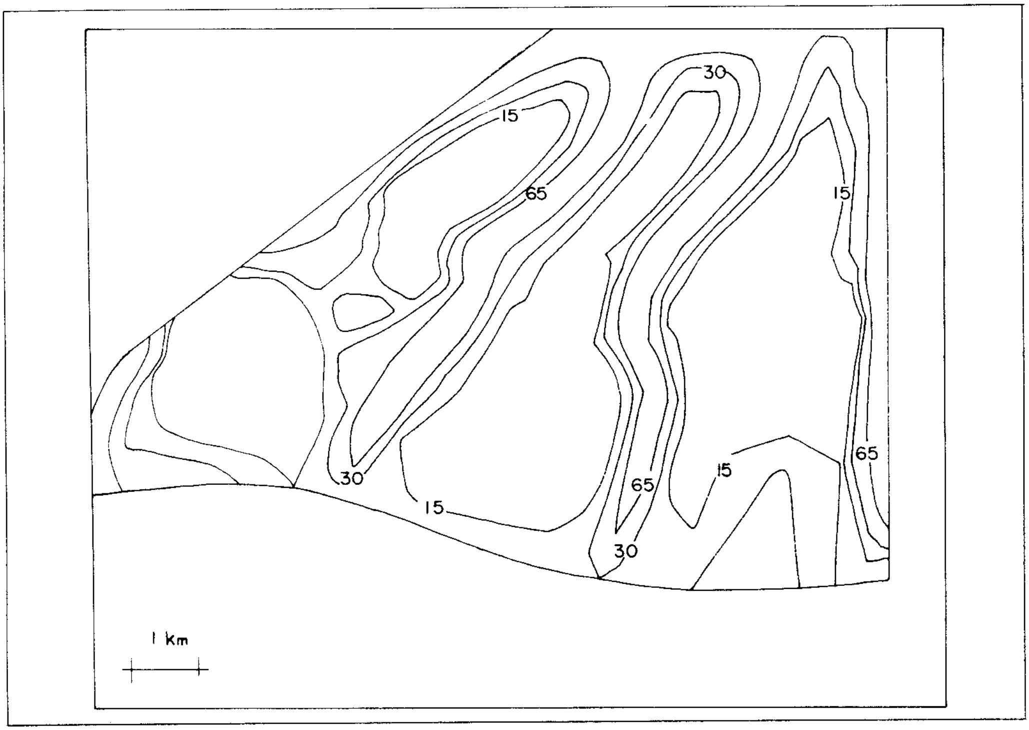

Fig. 7. Contour map of a groundwater level for subdomain 2.

Typical contour maps of the groundwater level, shown in the Fig.6 and Fig.7, demonstrate that the proposed numerical procedure enables us to simulate "a boundary layer" near the drainage channels.

Finally, a comparison between computed and measured groundwater levels displays a negligible deflection of about 5 cm.

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