J. Sci. Soc. Thailand, **1** (1975), 226–232

THE EQUILIBRIUM CONDENSATE FRACTION IN SUPERFLUID HELIUM

FREDERICK W. CUMMINGS

Department of Physics, University of California, Riverside, California, U.S.A. 92502

KITT VISOOTTIVISETH

Department of Physics, Khon Kaen University, Khon Kaen, Thailand.

(*Received* 20 *October* 1975)

Summary

This paper reports the temperature dependence of the condensate fraction in ^{4}He II determined from Cummings' proposal which involves only measurement of the liquid structure factor.

Introduction

It is widely believed that the ratio of the (bulk) condensate to the total density (ρ_c/ρ) is in the range of 0.08 to 0.25 at T=O°K for ^{4}He II, based on theoretical estimates[1–4]. An experimental confirmation of this theoretical estimates is feasible. One of us[5] has presented a method for determining the condensate density fraction for all temperatures based only on the knowledge of the equilibrium pair distribution function g(r, T) as a function of temperature just above T_λ and below.

Theoretical Considerations

We will briefly review first the method which has been proposed by Cummings[5] and then extend it. From the conditions[6] required of the second order reduced density matrix Ω_2 and the form[6] for Ω_2, we can write it as

$$\begin{aligned}\Omega_2(r) &= \rho^2 g(r,T)\\ &= g_1^2(r)\rho_c^2 + 2g_2^2(r)\rho_c\rho_d + 2g_2^2(r)\rho_c\,|\Lambda_1(r)| + \Lambda_2(r)\end{aligned} \tag{1}$$

which is valid when $r > r_1 \simeq 4.5$ Å the point where the first order reduced density matrix[2] Ω_1 (r) becomes equal to the condensate density ρ_c (see fig. 1), and where g assume the value 1 for the second time. The function Λ_2 must satisfy all conditions analogous to the conditions required of Ω_2. Thus $\Lambda_2(0)=0$, and $\Lambda_2(r)=\rho_d^2$ when $r>r_2$, where r_2 is several times larger than r_1. Therefore $|\Lambda_2(r)|$ may be defined as

$$|\Lambda_2(r)| = \rho_d^2\,\tilde{g}(r,T) \tag{2}$$

The bulk "depletion" density ρ_d is defined via

$$|\Lambda_1(r)| = \rho_d h(r) \tag{3}$$

where h(r) approaches zero when $r > r_1$ (it should be noted that the characteristic length of Λ_2 exceeds)that of Λ_1). The screening factors, $\$_1(r)$ and $\$_2(r)$, have been introduced in the expression (1) because of the "core" condition[6] required of Ω_2.

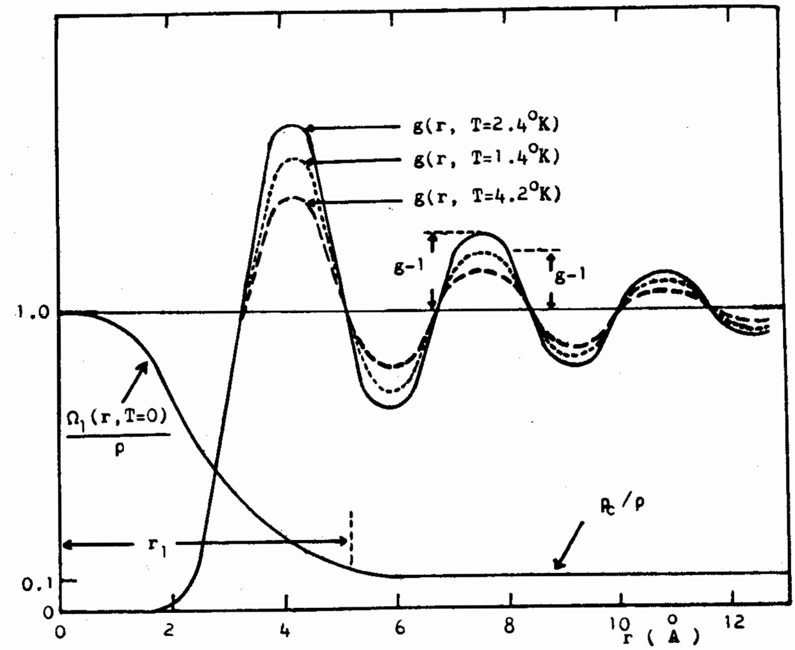

FIG.1 The r-dependence of g(r,T), and Ω_1/ρ ; observe that $\Omega_1(r)$ attains its asymptotic value of ρ_c at r ≈ 4.5 Å (cf. ref.5).

$\tilde{g}(r,T>T_\lambda) = g(r,T)$, and $\rho_c = 0$ above T_λ; $g(r,T<T_\lambda)$—$g(r,T>T_\lambda)$ whenever $g(r,T>T_\lambda) = 1$ (which is the "crossing points" observed by Gordon *et al.*[7], see fig. 1).

In the region $r_1 < r < r_2$, the screening factors, $\$_1(r)$ and $\$_2(r)$, approach unity, and the function h(r) approaches zero. Using equation (1) in this region, we have

$$\Omega_2(r) = \rho^2 g(r,T) = \rho_c^2 + 2\rho_c \rho_d + \rho_d^2 \tilde{g}(r,T),$$

for $r_1 < r < r_2$, which can be rearranged as

$$\frac{\rho_d(T)}{\rho} = \left[\frac{g(r,T) - 1}{\tilde{g}(r,T) - 1}\right]^{\frac{1}{2}} \tag{4}$$

$\tilde{g}(r,T)-1$ represents the structure of the depletion "lumps", and $\tilde{g}(r,T)-1$ may be expected to be equal to $g(r,T\simeq T_\lambda)-1$, which is measurable. Since $\rho=\rho_c+\rho_d$, equation (4) can thus be rewritten as

$$\frac{\rho_c\ (T)}{\rho} = 1 - \left[\frac{g(r,T)\ -\ 1}{g(r,T\ \simeq\ T_\lambda)-1}\right]^{\frac{1}{2}} \tag{5}$$

$$\Omega_1(\vec{x}';\vec{x}'') = \phi^*\ (\vec{x}'')\ \phi(\vec{x}') + \Lambda_1(\vec{x}';\vec{x}'')$$

$(r_1<r<r_2; T<T_\lambda)$. Thus equation (5) gives ρ_c/ρ as a function of the measurable $g(r,T)-1$.

The "off-diagonal long-range order (ODLRO)" as it was called by Yang[8] is due to the appearance of a factorized part in the reduced density matrices. The appearance of the factorized part in Ω_1 for the condensation in superfluid helium system is where $\Lambda_1(\vec{x}';\vec{x}'')\rightarrow 0$; Ω_1, $(\vec{x}';\vec{x}'')\rightarrow\Phi^*(\vec{x}'')\Phi(\vec{x}')$, when $|\vec{x}'-\vec{x}''|\rightarrow\infty$. Therefore, we must not think that one ^{4}He atom is in the condensate and another is not, when the condensation has occurred. Instead each and every atom must be regarded as being in the condensate and partly localized within a distance about the average interatomic spacing (i.e., the range of Λ_1 $(|\vec{x}'-\vec{x}''|)$ is about 4.5 Å). The diffraction pattern from the measurement of the liquid structure factor $S(\vec{k})$ by X-rays of neutron diffraction will be due to the "lumps" of higher density (the depletion part of the density associated with Λ_1) and not from the (structureless) uniform background of condensate (if $\rho=\rho_c$ at $T=0$°K, there is no structure–the case of the ideal Bose gas). The measurement of the relative intensity of the diffraction patterns, at different temperatures below T_λ, will show the diminishing of scattering intensity (lowering of the maxima of $g(r,T<T_\lambda)$) from the depletion part since the "lumps" will have "melted" into the condensate as the temperature is lowered. The total density remains nearly constant.

On the other hand when the temperature is above T_λ the helium atoms are partly localized to within an Angström or so. When the temperature is lowered below T_λ, each helium atom will begin to contribute to a (structureless) uniform condensate density, spreading throughout the volume occupied by the system. But lowering temperature, above T_λ, causes the diminishing thermal motion and increasing localization which have affected experimentally a heightening of the maxima of the pair distribution function $g(r,T>T_\lambda)$. The pair distribution function g is related to the liquid structure factor $S(\vec{k})$ by a Fourier transform. The largest experimental error in $S(\vec{k})$ occurs for very small momentum transfer $\vec{k}$. We expect the maxima of g(r) to be increasingly in error as r is increased. Thus the most reliable value for ρ_c/ρ is expected to come from the second maximum or minimum of g. For $r_1<r<r_2$, equation (5) should be independent of r ideally.

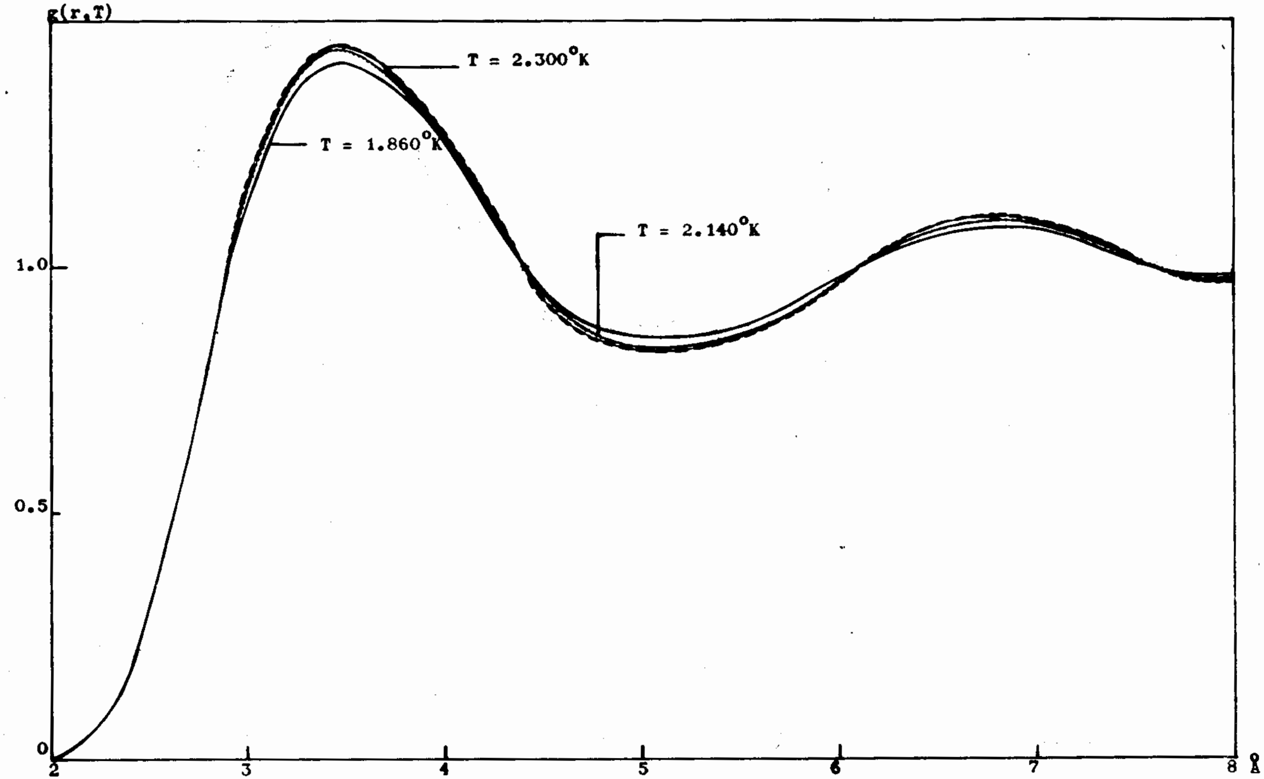

FIG.2 The curves of g(r,T) vs. r , plotted from the data of ref. 9. Show that g assume the value 1 for the second time at $r \simeq 4.5$ Å, and show the second maximum of g.

Calculations

From the work of Mountain and Raveche[9], the experimental data for g are available at 1.860°K, 2.140°K, and density $\rho=0.0230$ atoms$(\text{Å})^{-3}$ (see fig. 2). By using $g(r, T\simeq2.300°K)-1$, in the denominator of the second term, at right hand side of equation (5), it can be estimated from data of (g–1) at T below 2.300°K (second maximum of g) in the following results.

$\rho_c/\rho\simeq0.07$ at $T=1.860°K$,

and $\rho_c/\rho\simeq0.06$ at $T=2.140°K$.

The previous estimate of Cummings[5], which was obtained by the same relation (equation (5)) and the data of Gordon *et al.*[7], is

$\rho_c/\rho\simeq0.10$ at $T=1.4°K$,

which appararently represents an upper limit.

Discussion

Up to now the status of the theory of the equilibrium statistical mechanics of superfluid helium is still not satisfactory. There exists on one hand the Landau-two-fluid model with two well defined quantities the superfluid density ρ_s and the normal-fluid density ρ_n, such that $\rho_n+\rho_s=\rho$. This model gives a very satisfactory description of the equilibrium properties (as well as of the hydrodynamic properties) of superfluid helium.

On the other hand there is the phenomenon of Bose-Einstein condensation in an interacting system of Bose-Einstein particles according to Penrose and Onsager[1] and our present paper. In these theories the two well defined quantities are the condensate density ρ_c and the depletion density ρ_d.

The connection between these two pictures is still not very clear. This is certainly due to the many difficulties encountered in the many-body problem theory of an interacting Bose-Einstein gas. Even in the weak-interaction dilute-gas limit it is very difficult to make rigorous conclusions. One of us[10] has proposed a connection between these two pictures for superfluid helium (^{4}He II). The connection is

$$\rho_c = \left[\alpha/(1-\beta)\right]\rho_s \tag{6}$$

where α and β are defined in ref.6. Hyland and Rowlands[11] have also proposed a connection between the condensate and superfluid densities of ^{4}He II which gives a very sharp rise of the condensate density as T decreases below 0.5°K shown in fig. 3.

Conclusion

In fig. 3 we have shown the temperature dependence of the condensate fractions in ^{4}He II which are computed from the connections between the condensate and superfluid

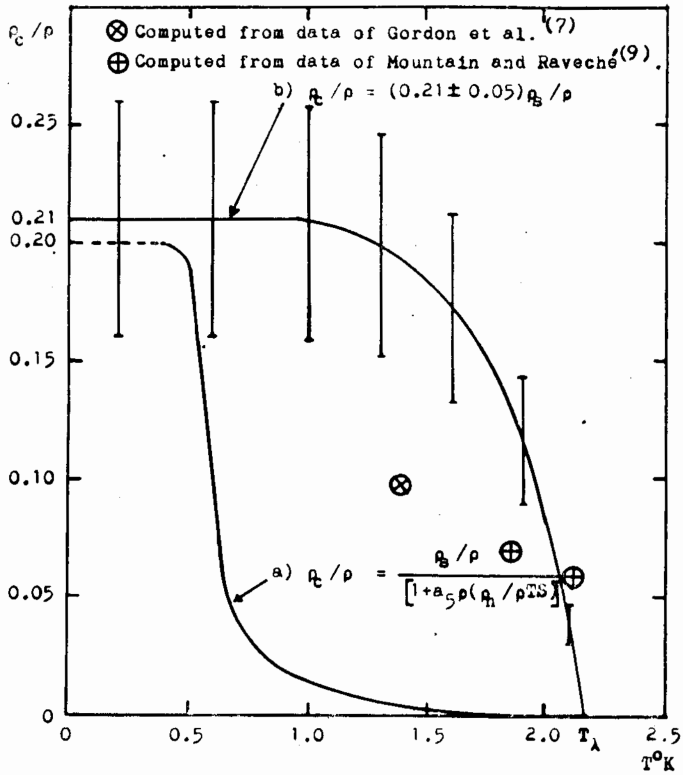

FIG.3 The curves of ρ_c/ρ vs. T :

a) Hyland and Rowlands' curve (cf. ref. 11),

b) One of us (cf. ref.10), by using $\alpha/(1-\beta)$ = (0.21±0.05).

densities comparing to the one computed from Cummings' proposal in this present paper. It seems that experimental result trends to our proposals. However the confirmation of these theoretical estimates needs more experimental result.

Acknowledgement

One of us (K.V.) wishes to thank the National Research Council of Thailand for the financial support through the project "Microscopic Derivation of Second Sound in Helium II". This paper is the second part of the progress report presented to the Council.

References:

1. Penrose, O. and Onsager, L. (1956) *Phys. Rev.* **104,** 576.
2. McMillan, W.L. (1965) *Phys. Rev.* **138,** A442.
3. Schiff, D. and Verlett L. (1967) *Phys. Rev.* **160,** 208.

4. Parry, W.E., and Rathbone, C.R. (1967) *Proc. Phys. Soc.* (*London*) **91,** 273.
5. Cummings, F.W. (1972) in *Statistical Mechanics,* (Rice, S.A., *et al.,* ed.) p.319–333, the University of Chicago Press.
6. Visoottiviseth, K. (1975) *Journal of the Science Faculty of Chiang Mai University, Thailand,* **2**(1), 67.
7. Gordon, W.L., Shaw, C.H., and Daunt, J.G. (1958) *Phys. Chem. Solids* **5,** 117.
8. Yang, C.N. (1962) *Rev. Mod. Phys.* **34,** 694.
9. Mountain, R.D., and Raveche, H.J. (1973) *J. Research Natl. Bur. Stand. A. Phys. Chem.* **77A,** 725.
10. Visoottiviseth, K. (1973) Ph.D. Dissertation, University of California, Riverside, California U.S.A.
11. Hyland, G.J., and Rowlands, G. (1972) *J. Low Temp. Phys.* **7,** 271.

บทคัดย่อ

บทความบทนี้ รายงานการคำนวณหาค่าสัดส่วนของความหนาแน่นควบแน่นต่อความหนาแน่นทั้งหมดของฮีเลียมเหลวชนิดที่สอง ณ อุณหภูมิต่าง ๆ โดยใช้วิธีจากข้อเสนอแนะของคัมมิงส์ ซึ่งเกี่ยวข้องกับการวัดค่าตัวร่วมของโครงสร้างของของเหลวเท่านั้น.