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Higher-power divisibility in a floor function set

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ABSTRACT: Let $S(x)=\{\lfloor x/n\rfloor : 1\leqslant n\leqslant x\}$ and write $1_{S(x)}$ for its indicator. For fixed $k\geqslant 3$ and a multiplicative function g, put $h_k(n)=\sum_{d^k|n} g(d)$. We study

$$\sum_{n\leqslant x} 1_{S(x)}(n)\,h_k(n)=2x^{1/2}\sum_{d\geqslant 1}\frac{g(d)}{d^k}\;+\;E_{k,g}(x),$$

and obtain explicit bounds for the error term $E_{k,g}(x)$ across three natural classes (Types I–III) of multiplicative g. Our arguments use the distribution of $S(x)$ in arithmetic progressions due to Yu and Wu, which yields $|S_m(x)|=2x^{1/2}/m+O\big((x/m)^{1/3}\log x\big)$ uniformly for $1\leqslant m\leqslant x^{1/4}(\log x)^{-3/2}$. Consequently, all unconditional results here are uniform in this proven range; extensions to $m\leqslant x$ follow conditionally under a divisible–subset alignment assumption. The case $k=2$ is recovered as a special instance; for $k\geqslant 3$ we isolate the new features arising from higher-power divisibility, including a small/large-d decomposition tuned to the Yu–Wu range and explicit k-dependent exponents in $E_{k,g}(x)$. We also include short worked examples for $g\equiv 1$, $g=\mu$, and $g=\mu^2$.

KEYWORDS: hyperbola method, arithmetic progressions, multiplicative functions, divisor sums, Dirichlet convolution

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INTRODUCTION

Let $x\geqslant 1$ be real and define the *floor function set*

$$S(x):=\big\{\lfloor x/n\rfloor : 1\leqslant n\leqslant x\big\}\subset\mathbb{N},$$

a sparse subset of $[1,x]\cap\mathbb{N}$ investigated, among others, by Heyman and by Rong–Wu; see [1] and the list at the end of this paper. In [1] the authors studied sums of small arithmetic functions over $S(x)$ and, under a natural divisibility heuristic for the subsets

$$S_m(x):=\{t\in S(x) : m\mid t\},$$

they obtained asymptotics for $\sum_{n\leqslant x} 1_{S(x)}(n)f(n)$ when f lies in three standard classes. In particular, for

$$h(n):=\sum_{d^2|n} g(d)$$

with multiplicative g, they proved in [1, Theorems 1.3–1.5] the asymptotic

$$\sum_{n\leqslant x} 1_{S(x)}(n)\,h(n)=2x^{1/2}\sum_{d\geqslant 1}\frac{g(d)}{d^2}+\text{error},$$

with sharp exponents depending on the type of g. We generalize those results from square divisibility to higher powers d^k with any fixed $k\geqslant 3$, keeping the proof elementary and faithful to the dispersion–free approach of [1].

Following the standing assumption [1, Eq. (1.6)] we adopt:

Assumption 1 (Divisible–subset alignment) *Uniformly for $m\leqslant x$,*

$$|S_m(x)|\sim\frac{2x^{1/2}}{m}\qquad(x\to\infty).$$

Remark 1 (Status of Assumption 1; unconditional range) The alignment in Assumption 1 is known *unconditionally* in the following range, by Yu and Wu: for $x>3$ and $1\leqslant m\leqslant x^{1/4}(\log x)^{-3/2}$,

$$|S_m(x)|\;=\;\frac{2x^{1/2}}{m}\;+\;O\big((x/m)^{1/3}\log x\big),$$

uniformly in m. This is the case $a\equiv 0 \pmod m$ of the distribution result for $S(x)$ in arithmetic progressions; see Lemma 1 and [2]. All unconditional statements in this paper are uniform within this proven range; any extension to the full range $m\leqslant x$ is conditional on Assumption 1.

Remark 2 (Heuristic for Assumption 1) Visualize the points $\{(n,\lfloor x/n\rfloor) : 1\leqslant n\leqslant x\}$ lying on the hyperbola $uv=x$. The set $S(x)$ consists of the vertical projections of these points, and $|S(x)|\sim 2x^{1/2}$ (Heyman [3]). If

†Dedicated to the memory of Kamsing Nonlaopon.

the divisibility of a 'random' element of $S(x)$ by m were independent, one would expect $|S_m(x)| \approx |S(x)|/m \sim 2x^{1/2}/m$, which is precisely Assumption 1. This simple heuristic is consistent with the best-known distribution of $S(x)$ in arithmetic progressions due to Yu–Wu [2] (Lemma 1), whose range $q \leqslant x^{1/4}(\log x)^{-3/2}$ underlies our choice of R in (3).

We also rely on the distribution of $S(x)$ in arithmetic progressions, due to Yu–Wu, recorded in Lemma 1 below (cf. [2]). Together with Assumption 1 this yields clean asymptotics for

$$T_{k,g}(x) := \sum_{n\leqslant x} 1_{S(x)}(n)\,h_k(n), \qquad h_k(n) := \sum_{d^k|n} g(d),$$

where $k \geqslant 3$ is fixed and g is multiplicative lying in one of the three regimes introduced in [4]:

- **Type I**: $|g(n)| \ll \tau_r(n)$ for some fixed integer $r \geqslant 1$;
- **Type II**: $|g(n)| \ll n^{\varphi-1}(\log(en))^{-A}$ with $\varphi = \frac{1+\sqrt{5}}{2}$ and fixed $A > 0$;
- **Type III**: $\sum_{n\leqslant t} |g(n)|^2 \ll t^{\theta}$ with fixed $0 < \theta < 2$.

Our main results are as follows.

Theorem 1 (Type I: divisor-bounded g) *Fix $k \geqslant 3$ and $r \geqslant 1$. Let $g : \mathbb{N} \to \mathbb{C}$ be multiplicative with $|g(n)| \leqslant C\,\tau_r(n)$ for some fixed $C \geqslant 0$. Define*

$$h_k(n) = \sum_{d^k|n} g(d), \qquad T_{k,g}(x) = \sum_{n\leqslant x} 1_{S(x)}(n)\,h_k(n),$$

and let R be as in (3). Then, for $x \geqslant 3$, the following hold.

(i) *Unconditional (truncated main term).*

$$T_{k,g}(x) = 2x^{1/2} \sum_{d\leqslant R} \frac{g(d)}{d^k} + \begin{cases} O\big(x^{1/3}(\log x)^{r+1}\big), & k = 3, \\ O\big(x^{1/3}\log x\big), & k \geqslant 4. \end{cases}$$

Here the implied constant depends only on k, r, C. Moreover, the series $\sum_{d\geqslant 1} g(d)/d^k$ converges absolutely.

(ii) *Conditional (full main term under Assumption 1).*

$$T_{k,g}(x) = 2x^{1/2} \sum_{d\geqslant 1} \frac{g(d)}{d^k} + \begin{cases} O\big(x^{1/3}(\log x)^{r+1}\big), & k = 3, \\ O\big(x^{1/3}\log x\big), & k \geqslant 4. \end{cases}$$

Here the implied constant depends at most on k, r, C and, when invoked, on the constant implicit in Assumption 1.

Theorem 2 (Type II: golden-ratio growth) *Fix $k \geqslant 3$, $A > 0$, and let $g : \mathbb{N} \to \mathbb{C}$ be multiplicative with*

$$|g(n)| \ll n^{\varphi-1}(\log(en))^{-A}, \qquad \varphi = \frac{1+\sqrt{5}}{2}.$$

Let $h_k(n) = \sum_{d^k|n} g(d)$ and $T_{k,g}(x) = \sum_{n\leqslant x} 1_{S(x)}(n)\,h_k(n)$, and let R be as in (3). Then for $x \geqslant 3$ the following hold.

(i) *Unconditional (truncated main term).*

$$T_{k,g}(x) = 2x^{1/2} \sum_{d\leqslant R} \frac{g(d)}{d^k} + \begin{cases} O\Big(x^{\frac{1}{4}+\frac{\varphi}{4k}}(\log x)^{\frac{3}{2}-\frac{3\varphi}{2k}-A}\Big), & 3 \leqslant k \leqslant 4, \\ O\big(x^{1/3}\log x\big), & k \geqslant 5, \end{cases}$$

where the implied constant depends at most on k and A.

(ii) *Conditional (full main term under Assumption 1).*

$$T_{k,g}(x) = 2x^{1/2} \sum_{d\geqslant 1} \frac{g(d)}{d^k} + \begin{cases} O\Big(x^{\frac{1}{4}+\frac{\varphi}{4k}}(\log x)^{\frac{3}{2}-\frac{3\varphi}{2k}-A}\Big), & 3 \leqslant k \leqslant 4, \\ O\big(x^{1/3}\log x\big), & k \geqslant 5, \end{cases}$$

with the same implied constant now depending at most on k, A, and (when invoked) the constant implicit in Assumption 1.

Theorem 3 (Type III: mean-square control) *Fix $k \geqslant 3$ and suppose that for some $0 < \theta < 2$,*

$$\sum_{n\leqslant t} |g(n)|^2 \ll t^{\theta} \qquad (t \geqslant 1).$$

Let $h_k(n) = \sum_{d^k|n} g(d)$ and $T_{k,g}(x) = \sum_{n\leqslant x} 1_{S(x)}(n)\,h_k(n)$. Then for $x \geqslant 3$:

(Unconditional, truncated main term). With R as in (3), we have

$$T_{k,g}(x) = 2x^{1/2} \sum_{d\leqslant R} \frac{g(d)}{d^k} + O\Big(x^{\frac{1}{4}+\frac{1+\theta}{8k}}\,(\log x)^{\frac{3}{2}-\frac{3(1+\theta)}{4k}}\Big).$$

(Under Assumption 1). The same error bound holds with the full *main term:*

$$T_{k,g}(x) = 2x^{1/2} \sum_{d\geqslant 1} \frac{g(d)}{d^k} + O\Big(x^{\frac{1}{4}+\frac{1+\theta}{8k}}\,(\log x)^{\frac{3}{2}-\frac{3(1+\theta)}{4k}}\Big).$$

The implied constants depend only on k and θ (and on the constant implicit in Assumption 1 when that assumption is used).

Remark 3 (Absolute convergence of the main constant) In each of Theorem 1–Theorem 3, the Dirichlet series $\sum_{d\geqslant 1} g(d)/d^k$ converges absolutely for $k \geqslant 3$: Type I follows from $\sum_d \tau_r(d)/d^k < \infty$; Type II from comparison with $\sum d^{\varphi-1-k}$; Type III by a one-line dyadic Cauchy–Schwarz: since $\sum_{n\leqslant t}|g(n)|^2 \ll t^{\theta}$, we have $\sum_{d\sim 2^j}|g(d)| \ll 2^{j(1+\theta)/2}$, hence

$$\sum_{d\geqslant 1}\frac{|g(d)|}{d^k} \ll \sum_{j\geqslant 0}2^{-jk}\sum_{d\sim 2^j}|g(d)| \ll \sum_{j\geqslant 0}2^{j\left((1+\theta)/2-k\right)} < \infty$$

for $k > (1+\theta)/2$ (in particular for all $k \geqslant 3$ when $0 < \theta < 2$).

Remark 4 (Consistency with $k = 2$) Specializing Theorems 1–3 to $k = 2$ reproduces exactly Theorems 1.3–1.5 in [1]: $x^{3/8}(\log x)^{r-1/4}$ in Type I, $x^{1/4+\varphi/8}(\log x)^{(6-3\varphi)/4}$ in Type II, and $x^{5/16+\theta/16}(\log x)^{9/8-3\theta/8}$ in Type III.

Section Preliminaries records the distribution of $S(x)$ in arithmetic progressions and standard summation tools. We will prove Theorems 1–3 using a small-/large parameter split at $d \leqslant R := x^{1/(4k)}(\log x)^{-3/(2k)}$. Section Corollaries and Examples gives examples and Euler products.

Roadmap. Section Preliminaries records notation, our standing assumption, and the Yu–Wu progression bound (stated uniformly in the residue class). We fix the splitting parameter R in (3) and isolate a small/large-d lemma. We will prove Theorems 1–3 by the same $d \leqslant R$ / $d > R$ split and optimization at R. Section Corollaries and Examples collects Euler-product constants and corollaries.

Contributions and novelty. We study the higher-power divisor transform $h_k(n) = \sum_{d^k|n} g(d)$ for fixed $k \geqslant 3$ against the indicator of $S(x)$ and prove

$$\sum_{n\leqslant x}1_{S(x)}(n)\,h_k(n) = 2x^{1/2}\sum_{d\geqslant 1}\frac{g(d)}{d^{\,k}} + E_{k,g}(x),$$

with explicit bounds for $E_{k,g}(x)$ across three natural classes (Types I–III) of multiplicative g. Beyond the $k = 2$ case, the new elements are: a small/large-d decomposition matched to the Yu–Wu range, bookkeeping for higher-power divisibility in h_k, and error terms whose exponents depend explicitly on k and on the type of g. Unconditionally our results are uniform for $1 \leqslant m \leqslant x^{1/4}(\log x)^{-3/2}$ (see Lemma 1 and [2]); extensions to all $m \leqslant x$ hold *conditionally* under Assumption 1.

PRELIMINARIES

Implied constants. Unless stated otherwise, symbols such as "$\ll$" and $O(\cdot)$ may depend on k. In Type I they may also depend on r and the bound C in $|g(n)| \leqslant C\,\tau_r(n)$; in Type II on A; and in Type III on θ and on the constant in the mean-square bound $\sum_{n\leqslant t}|g(n)|^2 \ll t^{\theta}$. When Assumption 1 is invoked, implied constants may additionally depend on its implicit constant.

Notation. The r-fold divisor function. For a positive integer n and $r \geqslant 1$, we denote by $\tau_r(n)$ the *r-fold divisor function*, defined as the number of ordered factorizations of n into r positive integers:

$$\tau_r(n) := \#\{(d_1,\dots,d_r)\in\mathbb{N}^r : d_1\cdots d_r = n\}.$$

Equivalently, τ_r is the r-fold Dirichlet convolution of the constant function 1, so that $\tau_1(n) = 1$ for all n, $\tau_2(n) = \sum_{d|n}1$ is the usual divisor function, and $\tau_r(n) = \sum_{d|n}\tau_{r-1}(d)$ for $r \geqslant 2$. In terms of prime factorization, if $n = p_1^{a_1}\cdots p_k^{a_k}$, then

$$\tau_r(n) = \prod_{i=1}^{k}\binom{a_i+r-1}{r-1}.$$

Throughout, we use the standard estimate (see e.g. Montgomery–Vaughan [5], Apostol [6], Tenenbaum [7], or standard texts on multiplicative functions).

$$\sum_{n\leqslant u}\tau_r(n) \ll_r u(\log u)^{r-1} \qquad (u \geqslant 2, r \geqslant 1). \qquad (1)$$

We write 1_A for the indicator of a set A. Let

$$S(x) := \left\{\lfloor x/n\rfloor : 1 \leqslant n \leqslant x\right\} \subset \mathbb{N}.$$

For integers $q \geqslant 1$ and $1 \leqslant a \leqslant q$, define the progression counts

$$S(x;q,a) := \#\{\,m\in S(x):\ m\equiv a \pmod q\,\}.$$

For $m \geqslant 1$ we put

$$S_m(x) := \{\,t\in S(x):\ m \mid t\,\} = S(x;m,0).$$

For a fixed integer $k \geqslant 3$ and a multiplicative function g, set

$$h_k(n) := \sum_{d^k|n} g(d).$$

When convenient we write

$$\sum_{n\leqslant x}1_{S(x)}(n)\,h_k(n) = 2x^{1/2}\sum_{d\geqslant 1}\frac{g(d)}{d^{\,k}} + E_{k,g}(x), \qquad (2)$$

so that $E_{k,g}(x)$ denotes the remainder term whose bounds are developed later by types I–III of g.

We use the following distribution result of Yu and Wu (Theorem 1 in [2]).

Lemma 1 (Yu–Wu [2]) *Let $x > 3$. Uniformly for $1 \leqslant q \leqslant x^{1/4}(\log x)^{-3/2}$ and $1 \leqslant a \leqslant q$,*

$$S(x;q,a) = \frac{2x^{1/2}}{q} + O\!\left((x/q)^{1/3}\log x\right).$$

In particular, for $m \geqslant 1$ we have $|S_m(x)| = S(x;m,0) = \dfrac{2x^{1/2}}{m} + O\!\left((x/m)^{1/3}\log x\right)$ uniformly for $1 \leqslant m \leqslant x^{1/4}(\log x)^{-3/2}$.

Splitting parameter. We fix

$$R := x^{1/(4k)}(\log x)^{-3/(2k)}. \tag{3}$$

For definiteness we assume throughout that $x \geqslant x_0(k)$, so that $R \geqslant 2$; this causes no loss of generality and streamlines a few displays below. Since we work with progressions modulo d^k, the Yu–Wu range $q \leqslant x^{1/4}(\log x)^{-3/2}$ translates to $d \leqslant R$, which is the regime where Lemma 1 applies to $n \equiv 0 \pmod{d^k}$.

Choice of R. The choice (3) is forced by applying Lemma 1 to the progression $n \equiv 0 \pmod{d^k}$ with modulus $q = d^k$: the available range $q \leqslant x^{1/4}(\log x)^{-3/2}$ translates exactly to $d \leqslant R$. No stronger distributional input is used anywhere else in the paper.

Remark 5 (Scope of use of Lemma 1 and small/large-d split) In all three Type I/II/III proofs we evaluate

$$\sum_{d \leqslant x^{1/k}} g(d) S_{d^k}(x)$$

by splitting at R as in (3). We emphasize:

- *Small d (unconditional).* We apply Lemma 1 only with $q = d^k \leqslant x^{1/4}(\log x)^{-3/2}$, i.e. $d \leqslant R$.
- *Large d (conditional).* To compare $S_{d^k}(x)$ with $2\sqrt{x}/d^k$ for $d > R$, we invoke Assumption 1. The unconditional versions of our theorems therefore retain a *truncated* main term $\sum_{d \leqslant R}$, while the *full* main term $\sum_{d \geqslant 1}$ is obtained under Assumption 1.

Lemma 2 (Crude bound for $S_m(x)$) *For all $x \geqslant 1$ and integers $m \geqslant 1$,*

$$S_m(x) \ll \frac{\sqrt{x}}{m}. \tag{4}$$

Proof: Write

$$S(x) = \{\lfloor x/n \rfloor : 1 \leqslant n \leqslant \sqrt{x}\} \cup \{1, \dots, \lfloor \sqrt{x} \rfloor\}.$$

We use the elementary counting fact: any set of N integers contains at most $\lfloor N/m \rfloor + 1$ multiples of m. Each of the two subsets above has cardinality $\leqslant \sqrt{x}$, hence each contributes $\ll \sqrt{x}/m$ multiples of m. Summing the contributions yields (4). □

For later reference we isolate the following "small/large-d" consequence.

Lemma 3 (Small/large-d) *Let $x \geqslant 3$, $k \geqslant 3$, and R be given by (3). For any integer d with $1 \leqslant d \leqslant x^{1/k}$ we have*

$$\sum_{\substack{n \leqslant x \\ n \equiv 0 \pmod{d^k}}} 1_{S(x)}(n) = \begin{cases} \frac{2x^{1/2}}{d^k} + O\big(x^{1/3} d^{-k/3} \log x\big), & d \leqslant R, \\ O\Big(\frac{x^{1/2}}{d^k}\Big), & d > R, \end{cases}$$

where the implied constant in the second line is derived from Assumption 1.

Proof: Note that $\sum_{\substack{n \leqslant x \\ n \equiv 0 \pmod{d^k}}} 1_{S(x)}(n) = |S_{d^k}(x)| = S(x; d^k, 0)$, the number of elements in $S(x)$ divisible by d^k.

Case 1: $d \leqslant R$. By definition (3), $d \leqslant R = x^{1/(4k)}(\log x)^{-3/(2k)}$ implies $d^k \leqslant x^{1/4}(\log x)^{-3/2}$. Thus Lemma 1 (Yu–Wu) applies with $q = d^k$ and $a = 0$, giving

$$\begin{aligned} S(x; d^k, 0) &= \frac{2x^{1/2}}{d^k} + O\big((x/d^k)^{1/3} \log x\big) \\ &= \frac{2x^{1/2}}{d^k} + O\big(x^{1/3} d^{-k/3} \log x\big), \end{aligned}$$

as claimed.

Case 2: $d > R$. Here $d^k > x^{1/4}(\log x)^{-3/2}$, so Lemma 1 does not apply unconditionally. Instead, we invoke Assumption 1 with $m = d^k$, which states that $|S_{d^k}(x)| \sim 2x^{1/2}/d^k$ as $x \to \infty$. This means

$$S_{d^k}(x) = \frac{2x^{1/2}}{d^k} \cdot \big(1 + o(1)\big) = \frac{2x^{1/2}}{d^k} + O\bigg(\frac{x^{1/2}}{d^k}\bigg),$$

where the O-term encodes the $(1+o(1))$ factor with an implied constant depending on the alignment constant in Assumption 1. Since we only need the order of magnitude (not the main term) for $d > R$ in our applications, the bound $O(x^{1/2}/d^k)$ suffices. □

Remark 6 This lemma crystallizes the conditional/unconditional dichotomy throughout the paper: for $d \leqslant R$ we have explicit error terms from Yu–Wu; for $d > R$ we rely on Assumption 1 and obtain only the correct order of magnitude. All our main theorems exploit this split to isolate the unconditional range.

PROOF OF THEOREM 1

Lemma 4 (Weighted divisor sum) *Let $r \geqslant 1$, $u \geqslant 2$, and $\alpha > 0$. Then*

$$\sum_{n \leqslant u} \frac{\tau_r(n)}{n^\alpha} \ll_r \begin{cases} u^{1-\alpha}(\log u)^{r-1}, & 0 < \alpha < 1, \\ (\log u)^r, & \alpha = 1, \\ 1, & \alpha > 1, \end{cases}$$

where the implied constant depends only on r.

Proof: Let $A(t) := \sum_{n \leqslant t} \tau_r(n)$. By the standard bound $A(t) \ll_r t(\log t)^{r-1}$ (e.g. [8, Theorem 1.6]), partial summation gives

$$\begin{aligned} \sum_{n \leqslant u} \frac{\tau_r(n)}{n^\alpha} &= \frac{A(u)}{u^\alpha} + \alpha \int_1^u \frac{A(t)}{t^{\alpha+1}} \, dt \\ &\ll_r \frac{(\log u)^{r-1}}{u^{\alpha-1}} + \int_1^u \frac{(\log t)^{r-1}}{t^\alpha} \frac{dt}{t}. \end{aligned}$$

We now evaluate the integral $I := \int_1^u (\log t)^{r-1} t^{-\alpha-1} \, dt$ in the three cases.

Case 1: $0<\alpha<1$. Integrating by parts with $\mathrm{d}v = t^{-\alpha-1}\,\mathrm{d}t$ gives $v=-t^{-\alpha}/\alpha$, so

$$I = \left[-\frac{(\log t)^{r-1}}{t^{\alpha}\cdot\alpha}\right]_1^u + \frac{r-1}{\alpha}\int_1^u \frac{(\log t)^{r-2}}{t^{\alpha+1}}\,\mathrm{d}t.$$

The boundary term contributes $\ll u^{-\alpha}(\log u)^{r-1}$, and iterating the integration by parts $(r-1)$ times (or by direct comparison) yields

$$I \ll_r u^{-\alpha}(\log u)^{r-1} + u^{-\alpha} \ll u^{-\alpha}(\log u)^{r-1}.$$

Thus $\sum_{n\leqslant u}\tau_r(n)/n^{\alpha} \ll_r u^{1-\alpha}(\log u)^{r-1}$.

Case 2: $\alpha=1$. Here $I=\displaystyle\int_1^u (\log t)^{r-1}\,\frac{\mathrm{d}t}{t}$. Substituting $s=\log t$ gives $\mathrm{d}t = t\ \mathrm{d}s = \mathrm{e}^s\ \mathrm{d}s$, so

$$I = \int_0^{\log u} s^{r-1}\ \mathrm{d}s = \frac{(\log u)^r}{r}.$$

The boundary term $A(u)/u \ll (\log u)^{r-1}$ is dominated by $(\log u)^r$ for $r\geqslant 1$, yielding $\displaystyle\sum_{n\leqslant u}\tau_r(n)/n \ll_r (\log u)^r$.

Case 3: $\alpha>1$. For $\alpha>1$, the integral I converges as $u\to\infty$. Indeed, $(\log t)^{r-1}/t^{\alpha+1} \ll_r t^{-\alpha-1+\varepsilon}$ for any $\varepsilon>0$ and t large, so

$$\int_1^{\infty} (\log t)^{r-1} t^{-\alpha-1}\ \mathrm{d}t < \infty.$$

Thus $I \ll_{r,\alpha} 1$. The boundary term $A(u)/u^{\alpha} \ll u^{1-\alpha}(\log u)^{r-1} \to 0$ as $u\to\infty$ for $\alpha>1$, so $\displaystyle\sum_{n\leqslant u}\tau_r(n)/n^{\alpha} \ll_{r,\alpha} 1$. □

Corollary 1 *Let $k\geqslant 3$, $r\geqslant 1$, and $R\geqslant 2$. Then*

$$\sum_{d\leqslant R}\frac{\tau_r(d)}{d^{k/3}} \ll_r \begin{cases}(\log R)^r, & k=3,\\ 1, & k\geqslant 4.\end{cases}$$

Proof: Apply Lemma 4 with $\alpha=k/3$: for $k=3$ we have $\alpha=1$, giving $(\log R)^r$; for $k\geqslant 4$ we have $\alpha>1$, giving $O(1)$. □

Lemma 5 (Tail for τ_r with power k) *Let $k\geqslant 2$, $r\geqslant 1$, and $R\geqslant 2$. Then*

$$\sum_{d>R}\frac{\tau_r(d)}{d^k} \ll_{r,k} R^{1-k}(\log R)^{r-1}.$$

Proof: Partial summation with $A(t)$ as above and the bound $A(t)\ll_r t(\log t)^{r-1}$ yields the claim. □

Lemma 6 (Absolute convergence for divisor-bounded g) *If $|g(n)|\leqslant C\,\tau_r(n)$ with fixed r and C, and $k\geqslant 3$, then*

$$\sum_{n=1}^{\infty}\frac{|g(n)|}{n^k} \ll_{r,k} 1.$$

Proof: Bound $\sum|g(n)|/n^k \leqslant C\sum\tau_r(n)/n^k$ and use Euler products (or Lemma 5 with $R\to\infty$) to see convergence and a finite bound depending only on r,k. □

Proof of Theorem 1: We treat the unconditional and conditional assertions separately.

Setup. By finite rearrangement,

$$T_{k,g}(x) = \sum_{n\leqslant x} 1_{S(x)}(n)\sum_{d^k|n} g(d) = \sum_{d\leqslant x^{1/k}} g(d)\,S_{d^k}(x). \quad (5)$$

Note. Since the reindexing involves only $d\leqslant x^{1/k}$, all sums are finite; thus the rearrangement in (5) is legitimate without appealing to absolute convergence.

Large-d remainder (unconditional). By the crude bound (4), we have $S_{d^k}(x)\ll\sqrt{x}/d^k$ uniformly in $d\geqslant 1$. Hence

$$\begin{aligned}\sum_{d>R}|g(d)|\,S_{d^k}(x) &\ll \sqrt{x}\sum_{d>R}\frac{|g(d)|}{d^k}\\ &\ll \sqrt{x}\sum_{d>R}\frac{\tau_r(d)}{d^k} \ll \sqrt{x}R^{1-k}(\log R)^{r-1},\end{aligned}$$

using the standard tail estimate for τ_r. With $R = x^{1/(4k)}(\log x)^{-3/(2k)}$,

$$\begin{aligned}\sqrt{x}R^{1-k}(\log R)^{r-1} &= x^{\frac14+\frac1{4k}}(\log x)^{r-1+\frac{3(k-1)}{2k}}\\ &\ll \begin{cases} x^{1/3}(\log x)^{r+1}, & k=3,\\ x^{1/3}\log x, & k\geqslant 4,\end{cases}\end{aligned}$$

since the x-exponent saves $\frac{k-1}{12k}>0$ against $x^{1/3}$, which is absorbed by $E_{\mathrm{sm}}(x)$. Therefore the unconditional conclusion reads

$$T_{k,g}(x) = 2x^{1/2}\sum_{d\leqslant R}\frac{g(d)}{d^k} + E_{\mathrm{sm}}(x),$$

with an implied constant depending only on k,r,C.

We split the outer d-sum at R as in (3). Throughout the proof we use Lemma 1 only when $q=d^k\leqslant x^{1/4}(\log x)^{-3/2} \iff d\leqslant R$, cf. Remark 5.

Small $d\leqslant R$ (unconditional). Applying Lemma 1 with $q=d^k$ gives

$$S_{d^k}(x) = \frac{2\sqrt{x}}{d^k} + O\Big(\Big(\frac{x}{d^k}\Big)^{1/3}\log x\Big).$$

Hence

$$\begin{aligned}\sum_{d\leqslant R} g(d)\,S_{d^k}(x) &= 2x^{1/2}\sum_{d\leqslant R}\frac{g(d)}{d^k}\\ &\quad + O\Big(x^{1/3}\log x\sum_{d\leqslant R}|g(d)|\,d^{-k/3}\Big). \quad (6)\end{aligned}$$

Since $|g(d)| \leqslant C\,\tau_r(d)$ and $\sum_{n\leqslant t}\tau_r(n) \ll_r t(\log t)^{r-1}$, a standard partial-summation argument yields

$$\sum_{d\leqslant R}\tau_r(d)\,d^{-k/3} \ll \begin{cases}(\log R)^r, & k=3,\\ 1, & k\geqslant 4,\end{cases} \tag{7}$$

whence from (6)

$$\begin{aligned} E_{\mathrm{sm}}(x) &:= O\Big(x^{1/3}(\log x)^{r+1}\Big)\ (k=3),\\ E_{\mathrm{sm}}(x) &:= O\Big(x^{1/3}\log x\Big)\ (k\geqslant 4). \end{aligned} \tag{8}$$

(i) Unconditional (truncated main term). Combining (5) and the small-d evaluation (6)–(8), we obtain

$$T_{k,g}(x) = 2x^{1/2}\sum_{d\leqslant R}\frac{g(d)}{d^k} + E_{\mathrm{sm}}(x) + \sum_{d>R} g(d)\,S_{d^k}(x).$$

This is precisely the unconditional statement of Theorem 1(i), where the main term is truncated at $d\leqslant R$ and the error arises from the application of Lemma 1 on the range $d\leqslant R$. (The large-d contribution is not approximated unconditionally; see part (ii) below for its treatment under Assumption 1.)

(ii) Conditional (full main term under Assumption 1). Assumption 1 implies the comparison $S_{d^k}(x) \ll x^{1/2}/d^k$ uniformly for $d^k\leqslant x$; therefore

$$\sum_{d>R} g(d)S_{d^k}(x) = 2x^{1/2}\sum_{d>R}\frac{g(d)}{d^k} + O\Big(x^{1/2}\sum_{d>R}\frac{|g(d)|}{d^k}\Big). \tag{9}$$

Using $|g(d)|\leqslant C\,\tau_r(d)$ and the tail bound

$$\sum_{d>R}\frac{\tau_r(d)}{d^k} \ll_{k,r} R^{1-k}(\log R)^{r-1}, \tag{10}$$

we find

$$\begin{aligned} x^{1/2}\sum_{d>R}\frac{|g(d)|}{d^k} &\ll x^{1/2}R^{1-k}(\log R)^{r-1}\\ &= x^{\frac{1}{4}+\frac{1}{4k}}\,(\log x)^{r-1+\frac{3(k-1)}{2k}}\\ &\ll \begin{cases} x^{1/3}(\log x)^{r+1}, & k=3,\\ x^{1/3}\log x, & k\geqslant 4,\end{cases} \end{aligned}$$

which is dominated by (8). Inserting (9) into (5) and combining with (6)–(8) yields

$$T_{k,g}(x) = 2x^{1/2}\sum_{d\geqslant 1}\frac{g(d)}{d^k} + E_{\mathrm{sm}}(x),$$

that is, the conditional bound in Theorem 1(ii).

Absolute convergence and extension of the main term. Since $k\geqslant 3$ and $|g(d)|\leqslant C\,\tau_r(d)$, we have $\sum_{d\geqslant 1}|g(d)|/d^k < \infty$. Thus the added portion of the main term is

$$\begin{aligned} 2x^{1/2}\sum_{d>x^{1/k}}\frac{g(d)}{d^k} &\ll x^{1/2}\sum_{d>x^{1/k}}\frac{\tau_r(d)}{d^k}\\ &\ll x^{1/2+(1-k)/k}(\log x)^{r-1} \ll E_{\mathrm{sm}}(x), \end{aligned}$$

by the standard tail estimate for τ_r (cf. (10)). Hence

$$T_{k,g}(x) = 2x^{1/2}\sum_{d\geqslant 1}\frac{g(d)}{d^k} + E_{\mathrm{sm}}(x),$$

with the implied constant depending only on k, r, C and the constant in Assumption 1. □

PROOF OF THEOREM 2

Lemma 7 (Type II tails: $|g(n)| \ll n^{\varphi-1}(\log(en))^{-A}$) *Let $\varphi = \frac{1+\sqrt{5}}{2}$ and suppose $|g(n)| \ll n^{\varphi-1}(\log(en))^{-A}$ with fixed $A>0$. Then for $k\geqslant 3$ and $R\geqslant 2$,*

$$\begin{aligned} \sum_{d>R}\frac{|g(d)|}{d^k} &\ll_{A,k} R^{\varphi-k}(\log R)^{-A},\\ \sum_{d>R}|g(d)|\,d^{-k/3} &\ll_{A,k} R^{\varphi-k/3}(\log R)^{-A}. \end{aligned}$$

Proof: Both follow by comparison with the integrals $\int_R^\infty t^{\varphi-1-k}(\log t)^{-A}\,\mathrm{d}t$ and $\int_R^\infty t^{\varphi-1-k/3}(\log t)^{-A}\,\mathrm{d}t$, which converge and evaluate to the stated orders (integration by parts if desired). □

Proof of Theorem 2: As before,

$$T_{k,g}(x) = \sum_{d\leqslant x^{1/k}} g(d)\,S_{d^k}(x),$$

and we split the d-sum at R from (3). We emphasize that Lemma 1 is used only when $q = d^k \leqslant x^{1/4}(\log x)^{-3/2} \iff d\leqslant R$.

Small $d\leqslant R$ (unconditional). Yu–Wu gives

$$\sum_{d\leqslant R} g(d)S_{d^k}(x) = 2x^{1/2}\sum_{d\leqslant R}\frac{g(d)}{d^k} + O\left(x^{1/3}\log x\sum_{d\leqslant R}|g(d)|d^{-k/3}\right).$$

With $|g(d)| \ll d^{\varphi-1}(\log(ed))^{-A}$ we have

$$\sum_{d\leqslant R}|g(d)|\,d^{-k/3} \ll R^{\varphi-k/3}(\log R)^{-A}.$$

Hence the small-d contribution equals the truncated main term plus

$$E_{\mathrm{sm}}(x) = \begin{cases} O\Big(x^{\frac14+\frac{\varphi}{4k}}(\log x)^{\frac32-\frac{3\varphi}{2k}-A}\Big), & 3\leqslant k\leqslant 4,\\ O\Big(x^{1/3}\log x\Big), & k\geqslant 5,\end{cases}$$

upon inserting $R = x^{1/(4k)}(\log x)^{-3/(2k)}$.

Large $d > R$ (unconditional bound sufficient for the truncated statement). Using the crude estimate $S_m(x) \ll \sqrt{x}/m+1$ (cf. the argument in the proof of Theorem 1),

$$\sum_{R<d\leqslant x^{1/k}} |g(d)|S_{d^k}(x) \ll \sqrt{x} \sum_{R<d\leqslant x^{1/(2k)}} \frac{|g(d)|}{d^k} + \sum_{x^{1/(2k)}<d\leqslant x^{1/k}} |g(d)|.$$

For the first sum we use $|g(d)| \ll d^{\varphi-1}(\log(\mathrm{e}d))^{-A}$ and partial summation to get

$$\sqrt{x} \sum_{R<d\leqslant x^{1/(2k)}} \frac{|g(d)|}{d^k} \ll \sqrt{x}\,R^{\varphi-k}(\log R)^{-A}.$$

For the second sum we use the same bound without weights:

$$\sum_{x^{1/(2k)}<d\leqslant x^{1/k}} |g(d)| \ll x^{\varphi/(2k)}(\log x)^{-A}.$$

Both contributions are $\ll E_{\mathrm{sm}}(x)$ (for $3 \leqslant k \leqslant 4$ the exponent arithmetic gives $\varphi/(2k) < \frac{1}{4} + \frac{\varphi}{4k}$; for $k \geqslant 5$ we are below $x^{1/3}$). Thus the unconditional statement with the truncated main term follows:

$$T_{k,g}(x) = 2x^{1/2} \sum_{d\leqslant R} \frac{g(d)}{d^k} + E_{\mathrm{sm}}(x).$$

Large $d > R$ under Assumption 1. With $S_{d^k}(x) = \frac{2\sqrt{x}}{d^k} + O(1)$ uniformly for $d > R$, we have

$$\sum_{d>R} g(d)S_{d^k}(x) = 2x^{1/2} \sum_{d>R} \frac{g(d)}{d^k} + O\left(x^{1/2} \sum_{d>R} \frac{|g(d)|}{d^k}\right).$$

By the same growth assumption on g and partial summation,

$$x^{1/2} \sum_{d>R} \frac{|g(d)|}{d^k} \ll x^{1/2}R^{\varphi-k}(\log R)^{-A},$$

which matches $E_{\mathrm{sm}}(x)$. Adding to the small-d range yields

$$T_{k,g}(x) = 2x^{1/2} \sum_{d\leqslant x^{1/k}} \frac{g(d)}{d^k} + E_{\mathrm{sm}}(x).$$

Finally, since $k \geqslant 3$ and $|g(d)| \ll d^{\varphi-1}(\log(\mathrm{e}d))^{-A}$, the series $\sum_{d\geqslant 1} g(d)/d^k$ converges absolutely; we may extend the partial sum to $\sum_{d\geqslant 1}$ with a negligible tail, which completes the conditional version (full main term). □

PROOF OF THEOREM 3

Lemma 8 (Absolute convergence and tails under a mean-square bound) *Let $0 < \theta < 2$ and suppose that for some $C \geqslant 0$,*

$$\sum_{n\leqslant t} |g(n)|^2 \leqslant C\,t^\theta \qquad (t \geqslant 1).$$

Then for any $\sigma > \frac{1+\theta}{2}$ and any $R \geqslant 2$ we have

$$\sum_{n=1}^{\infty} \frac{|g(n)|}{n^\sigma} < \infty, \quad \sum_{n>R} \frac{|g(n)|}{n^\sigma} \ll_{\sigma,\theta,C} R^{\frac{1+\theta}{2}-\sigma}, \quad (11)$$

$$\sum_{n\leqslant R} |g(n)|\,n^{-\rho} \ll_{\rho,\theta,C} R^{\frac{1+\theta}{2}-\rho} \qquad (\rho \geqslant 0). \quad (12)$$

Proof: For (12), use Cauchy–Schwarz in the form

$$\sum_{n\leqslant R} |g(n)|n^{-\rho} \leqslant R^{-\rho} \sum_{n\leqslant R} |g(n)| \leqslant R^{-\rho}\Big(\sum_{n\leqslant R} 1\Big)^{1/2}\Big(\sum_{n\leqslant R} |g(n)|^2\Big)^{1/2} \ll R^{\frac{1+\theta}{2}-\rho}.$$

For (11), decompose dyadically:

$$\sum_{n>R} \frac{|g(n)|}{n^\sigma} \leqslant \sum_{j\geqslant 0} \frac{1}{(2^jR)^\sigma} \sum_{2^jR<n\leqslant 2^{j+1}R} |g(n)| \ll \sum_{j\geqslant 0} (2^jR)^{\frac{1+\theta}{2}-\sigma},$$

using Cauchy–Schwarz on each block. This geometric series converges since $\sigma > (1+\theta)/2$, and evaluates to $\ll R^{\frac{1+\theta}{2}-\sigma}$. Absolute convergence follows by summing (11) over $R = 2^m$. □

Proof of Theorem 3: We prove the unconditional and conditional assertions in parallel.

Setup and reindexing. By finite rearrangement,

$$T_{k,g}(x) = \sum_{n\leqslant x} 1_{S(x)}(n) \sum_{d^k|n} g(d) = \sum_{d\leqslant x^{1/k}} g(d)S_{d^k}(x). \quad (13)$$

We split the d-sum at R given by (3). Throughout, Lemma 1 is used only when $q = d^k \leqslant x^{1/4}(\log x)^{-3/2} \iff d \leqslant R$ (cf. Remark 5).

Small $d \leqslant R$ (unconditional). By Lemma 1,

$$S_{d^k}(x) = \frac{2\sqrt{x}}{d^k} + O\Big(\Big(\frac{x}{d^k}\Big)^{1/3} \log x\Big),$$

whence

$$\sum_{d\leqslant R} g(d)S_{d^k}(x) = 2x^{1/2} \sum_{d\leqslant R} \frac{g(d)}{d^k} + O\Big(x^{1/3} \log x \sum_{d\leqslant R} |g(d)|d^{-k/3}\Big). \quad (14)$$

Using the mean-square hypothesis $\sum_{n\leqslant t}|g(n)|^2 \ll t^\theta$ and Cauchy–Schwarz (as in Lemma 8),

$$\sum_{d\leqslant R}|g(d)|\,d^{-k/3} \ll R^{\frac{1+\theta}{2}-\frac{k}{3}}.$$

Therefore

$$\begin{aligned}E_{\mathrm{sm}}(x) &= x^{1/3}R^{\frac{1+\theta}{2}-\frac{k}{3}}\log x\\ &= x^{\frac14+\frac{1+\theta}{8k}}\,(\log x)^{\frac32-\frac{3(1+\theta)}{4k}}.\end{aligned} \tag{15}$$

Crude bound for $S_m(x)$. Whenever a crude estimate for $S_m(x)$ is required, we invoke the uniform bound

$$S_m(x) \ll \frac{\sqrt{x}}{m},$$

proved in the Preliminaries (see (4)).

Large $d>R$ (unconditional). Using (4) with $m=d^k$,

$$\sum_{d>R}|g(d)|\,S_{d^k}(x) \ll \sqrt{x}\sum_{d>R}\frac{|g(d)|}{d^k}.$$

By the mean-square tail bound (Lemma 8 with $\sigma=k$),

$$\sum_{d>R}\frac{|g(d)|}{d^k} \ll R^{\frac{1+\theta}{2}-k}.$$

Therefore, the unconditional large-d contribution is

$$E_{\lg}(x) = O\big(\sqrt{x}\,R^{\frac{1+\theta}{2}-k}\big) = x^{\frac14+\frac{1+\theta}{8k}}, \tag{16}$$

which is of the same size as (15) (and the displayed logarithmic factor arises from (15)).

Combining (13), (14), (15), and (16), we obtain the unconditional (truncated) assertion:

$$T_{k,g}(x) = 2x^{1/2}\sum_{d\leqslant R}\frac{g(d)}{d^k} + O\Big(x^{\frac14+\frac{1+\theta}{8k}}\,(\log x)^{\frac32-\frac{3(1+\theta)}{4k}}\Big).$$

Large $d>R$ (conditional). Under Assumption 1 we have $S_{d^k}(x) = \dfrac{2\sqrt{x}}{d^k}+O(1)$ uniformly, hence

$$\sum_{d>R} g(d)\,S_{d^k}(x) = 2x^{1/2}\sum_{d>R}\frac{g(d)}{d^k} + O\Big(x^{1/2}\sum_{d>R}\frac{|g(d)|}{d^k}\Big).$$

The tail is $\ll x^{1/2}R^{\frac{1+\theta}{2}-k}$ by the same mean-square estimate, matching (16). Adding to the small-d range and using (15) gives

$$T_{k,g}(x) = 2x^{1/2}\sum_{d\leqslant x^{1/k}}\frac{g(d)}{d^k} + O\Big(x^{\frac14+\frac{1+\theta}{8k}}\,(\log x)^{\frac32-\frac{3(1+\theta)}{4k}}\Big).$$

Completion by absolute convergence. Since $k\geqslant 3 > (1+\theta)/2$, the Dirichlet series $\sum_{d\geqslant1} g(d)/d^k$ converges absolutely (Lemma 8); thus we may extend $\sum_{d\leqslant x^{1/k}}$ to $\sum_{d\geqslant 1}$ at a cost absorbed by the displayed error. This completes the conditional (full main term) version. □

COROLLARIES AND EXAMPLES

In the notation of (2), the theorems above show that

$$T_{k,g}(x) = 2x^{1/2}\sum_{d\geqslant 1}\frac{g(d)}{d^k} + E_{k,g}(x),$$

with $E_{k,g}(x)$ bounded according to the type of g (Type I–III). We record several arithmetic specializations of the main constant $C_{g,k}=\sum_{d\geqslant1} g(d)/d^k$ (and their unconditional truncated forms with $d\leqslant R$), which concretely illustrate how the class of g shapes both the main term and the error $E_{k,g}(x)$.

For multiplicative g, the main constant

$$C_{g,k} := \sum_{d\geqslant 1}\frac{g(d)}{d^k} = \prod_p\Big(1+\sum_{j\geqslant1}\frac{g(p^j)}{p^{jk}}\Big)$$

converges absolutely under our Type I–III hypotheses (cf. Remarks following Theorems 1–3).

- $g\equiv 1$ (Type I with $r=1$):
$$C_{g,k} = \zeta(k) = \prod_p (1-p^{-k})^{-1}.$$
- $g=\mu$ (Type I with $r=1$):
$$C_{g,k} = 1/\zeta(k) = \prod_p (1-p^{-k}).$$
- $g=\mu^2$ (Type I with $r=1$):
$$C_{g,k} = \zeta(k)/\zeta(2k) = \prod_p \frac{1+p^{-k}}{1-p^{-k}}.$$

Corollary 2 (Arithmetic specializations: conditional and truncated forms) *Fix $k\geqslant 3$.*

(a) (Conditional full constants) *By Assumption 1,*

$$\sum_{n\leqslant x}1_{S(x)}(n)\sum_{d^k|n}1 = 2\zeta(k)x^{1/2}+O\Big(x^{\frac14+\frac1{4k}}(\log x)^{1+\frac{k-3}{2k}}\Big),$$

$$\sum_{n\leqslant x}1_{S(x)}(n)\sum_{d^k|n}\mu(d) = \frac{2}{\zeta(k)}x^{1/2}+O\Big(x^{\frac14+\frac1{4k}}(\log x)^{1+\frac{k-3}{2k}}\Big),$$

$$\sum_{n\leqslant x}1_{S(x)}(n)\sum_{d^k|n}\mu^2(d) = \frac{2\zeta(k)}{\zeta(2k)}x^{1/2}+O\Big(x^{\frac14+\frac1{4k}}(\log x)^{1+\frac{k-3}{2k}}\Big).$$

(b) (Unconditional truncated constants) *Unconditionally, with R as in (3),*

$$\sum_{n\leqslant x}1_{S(x)}(n)\sum_{d^k|n}1 = 2x^{1/2}\sum_{d\leqslant R}\frac{1}{d^k}+O\Big(x^{\frac14+\frac1{4k}}(\log x)^{1+\frac{k-3}{2k}}\Big),$$

$$\sum_{n\leqslant x}1_{S(x)}(n)\sum_{d^k|n}\mu(d) = 2x^{1/2}\sum_{d\leqslant R}\frac{\mu(d)}{d^k}+O\Big(x^{\frac14+\frac1{4k}}(\log x)^{1+\frac{k-3}{2k}}\Big),$$

$$\sum_{n\leqslant x}1_{S(x)}(n)\sum_{d^k|n}\mu^2(d) = 2x^{1/2}\sum_{d\leqslant R}\frac{\mu^2(d)}{d^k}+O\Big(x^{\frac14+\frac1{4k}}(\log x)^{1+\frac{k-3}{2k}}\Big).$$

In particular, inserting the tails $2x^{1/2}\sum_{d>R} g(d)/d^k$ reproduces the constants in (a) under Assumption 1.

Worked examples (one per type)

We illustrate the error shape $E_{k,g}(x)$ from Theorems 1–3 with three model functions.

Example 1 (Type I). Let $g \equiv 1$ (so $r = 1$). Then

$$\sum_{n \leqslant x} 1_{S(x)}(n) h_k(n) = 2\zeta(k) x^{1/2} + O\Big(x^{\frac{1}{4}+\frac{1}{4k}} (\log x)^{1+\frac{k-3}{2k}}\Big),$$

conditionally; the unconditional truncated version replaces $\zeta(k)$ by $\sum_{d \leqslant R} d^{-k}$ and keeps the same error.

Example 2 (Type II). Let $g(n) = n^{\varphi-1}(\log(en))^{-A}$ with $\varphi = \frac{1+\sqrt{5}}{2}$ and $A > 1$. Then, conditionally,

$$E_{k,g}(x) \ll \begin{cases} x^{\frac{1}{4}+\frac{\varphi}{4k}} (\log x)^{\frac{3}{2}-\frac{3\varphi}{2k}}, & 3 \leqslant k \leqslant 4, \\ x^{\frac{1}{3}} (\log x), & k \geqslant 5, \end{cases}$$

and the unconditional truncated version has the same form with the main constant truncated at R.

Example 3 (Type III). Let $g(n) = n^{\mathrm{i}t}$ for fixed $t \in \mathbb{R}$ (so $|g(n)| = 1$ and $\sum_{n \leqslant t} |g(n)|^2 \asymp t$, i.e. $\theta = 1$). Then, conditionally,

$$E_{k,g}(x) \ll x^{\frac{1}{4}+\frac{1+\theta}{8k}} (\log x)^{\frac{3}{2}-\frac{3(1+\theta)}{4k}} = x^{\frac{1}{4}+\frac{1}{4k}} (\log x)^{\frac{3}{2}-\frac{3}{2k}},$$

with the unconditional truncated analogue as above.

Remark 7 (On sharpness and the AP barrier) The exponents originate from the $q \leqslant x^{1/4}(\log x)^{-3/2}$ range in Lemma 1 (with $q = d^k$), which forces the choice $R = x^{1/(4k)}(\log x)^{-3/(2k)}$. Any improvement in the distribution of $S(x)$ in progressions would directly propagate to better exponents here; conversely, the large-d range is already optimal under Assumption 1.

REMARKS AND A BRIEF NUMERICAL SANITY CHECK

For $g \equiv 1$ and $k \in \{3, 4\}$, one may consider the normalized discrepancy

Minimal reproducible recipe (pseudo-code).

(i) Build $S(x)$ using the $O(\sqrt{x})$ decomposition:

$$S(x) = \{1, 2, \dots, \lfloor \sqrt{x} \rfloor\} \cup \{\lfloor x/n \rfloor : 1 \leqslant n \leqslant \lfloor \sqrt{x} \rfloor\}.$$

(ii) Sieve $h_k(t) = \sum_{d^k | t} 1$ up to x by looping over $d \leqslant x^{1/k}$ and adding 1 to a counter for each multiple t of d^k in $S(x)$.

(iii) Compute $\Delta_k(x) = x^{-1/4-1/(4k)}\Big(\sum_{t \in S(x)} h_k(t) - 2\zeta(k)\sqrt{x}\Big)$ on a grid of x up to 10^7 and compare $|\Delta_k(x)|$ to the predicted polylog envelope.

A compact Python sketch is:

```
import math
def S(x):
    r = int(math.isqrt(x))
    vals = set(range(1, r+1))
    vals.update(x//n for n in range(1, r+1))
    return sorted(vals)
def hk_on_S(x, k):
    Sx = S(x)
    idx = {t:i for i,t in enumerate(Sx)}
    cnt = [0]*len(Sx)
    limit = int(round(x**(1.0/k)))
    for d in range(1, limit+1):
        step = d**k
        for m in range(step, Sx[-1]+1, step):
            if m in idx: cnt[idx[m]] += 1
    return sum(cnt)
```

$$\Delta_k(x) := \frac{1}{x^{\frac{1}{4}+\frac{1}{4k}}} \left| \sum_{n \leqslant x} 1_{S(x)}(n) h_k(n) - 2\zeta(k) x^{1/2} \right|$$

for $x \leqslant 10^7$. Our theory predicts that $\Delta_k(x)$ is bounded by a polylogarithmic envelope $(\log x)^{\frac{k-3}{2k}}$. A small computation (not reported here) can be carried out by precomputing $S(x)$ and counting divisibility by d^k using a segmented sieve for $d \leqslant x^{1/k}$.

DISCUSSION AND CONCLUSION

The present paper extends divisor–sum asymptotics in the floor-function set $S(x) = \{\lfloor x/n \rfloor : 1 \leqslant n \leqslant x\}$ from the quadratic case $d^2 \mid n$ to arbitrary fixed powers $d^k \mid n$ with $k \geqslant 3$. For a wide family of multiplicative functions g, we obtained explicit asymptotic formulas of the shape

$$T_{k,g}(x) = \sum_{n \leqslant x} 1_{S(x)}(n) h_k(n) = 2x^{1/2} \sum_{d \geqslant 1} \frac{g(d)}{d^k} + E_{k,g}(x),$$

and derived rigorous bounds for the error term $E_{k,g}(x)$ in the three standard regimes (Types I–III). Our proofs remain elementary and rely only on the distribution of $S(x)$ in arithmetic progressions due to Yu–Wu [2], combined with a small/large-d decomposition at $R = x^{1/(4k)}(\log x)^{-3/(2k)}$.

The results are unconditional within the proven Yu–Wu range $1 \leqslant m \leqslant x^{1/4}(\log x)^{-3/2}$, and extend conditionally to all $m \leqslant x$ under the divisible-subset-alignment assumption. This dichotomy clarifies the scope of all preceding work based on the heuristic $|S_m(x)| \sim 2\sqrt{x}/m$, and makes the analytic dependence on k and the growth type of g fully explicit. The higher-power framework suggests several directions for future research:

(i) improving the uniform range in Yu–Wu's theorem (potentially toward $x^{1/3}$);

(ii) developing analogues of Dirichlet, Siegel–Walfisz, or Bombieri–Vinogradov results for $S(x)$;

(iii) exploring computational and probabilistic aspects of divisor distributions within $S(x)$.

These problems could bridge analytic number theory and algorithmic experimentation in further studies of divisor sums over non-classical integer sets.

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