RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2025.080

ScienceAsia 51 (5): 2025: ID 2025080: **1–6**

The weak Nekrasov matrices with the chain condition

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Received 18 Oct 2024, Accepted 17 Sep 2025
Available online

ABSTRACT: The nonzero elements chain plays an important role in investigating the non-singularity of matrices. In 1998, Li present a sufficient and necessary condition for a weak Nekrasov matrix to be a nonsingular H-matrix, which is called "the chain condition" [*Linear Algebra Appl* (1998) **281**:87–96]. In this paper, we give it an alternative equivalent version of Li's chain condition which requires no background of graph theory. Adopting this new description, we propose a new criterion for nonsingular H-matrices. Some examples are given to show the effectiveness of the proposed results.

KEYWORDS: weak Nekrasov matrix, nonsingularity, chain condition, H-matrix

MSC2020: 15A45

INTRODUCTION

Let $\mathbb{C}^{n\times n}$ denote the set of all $n\times n$ complex matrices. Denote $N=\{1,2,\dots,n\}$. For a given matrix $A=(a_{ij})\in\mathbb{C}^{n\times n}$, we call A is a (row) diagonally dominant (DD) matrix if

$$|a_{ii}|\geqslant P_i(A),\quad \text{for all } i\in N, \tag{1}$$

where

$$P_i(A)=\sum_{j\neq i}|a_{ij}|.$$

We call A is a strictly diagonally dominant (SDD) matrix if all the inequalities in (1) are strict. We call A is a generalized strictly diagonally dominant matrix if there exists a diagonal matrix X such that AX is an SDD matrix. Let

$$h_1(A)=P_1(A),$$
$$h_i(A)=\sum_{j=1}^{i-1}\frac{h_j(A)}{|a_{jj}|}|a_{ij}|+\sum_{j=i+1}^{n}|a_{ij}|,\quad i\in N\backslash\{1\}.$$

If we replace $P_i(A)$ by $h_i(A)$ in (1), then A is called weak Nekrasov matrix, Nekrasov matrix, and generalized Nekrasov matrix, respectively. The SDD matrices and Nekrasov matrices are important subclasses of nonsingular H-matrices. Nonsingular H-matrices have been widely used in practical problems, such as computational mathematics, matrix theory, control, etc [1, 2]. It is well-known that DD matrices and weak Nekrasov matrices are not necessarily nonsingular. Szulc [3] pointed out that a DD matrix A can be nonsingular if A satisfies: for any $i\in\{i:|a_{ii}|=P_i(A)\}$, there is a nonzero elements chain $a_{ii_1},\cdots,a_{i_kj}$ with $j\in\{i:|a_{ii}|>P_i(A)\}$. A question is posed by Szulc [3]: whether a weak Nekrasov matrix A is nonsingular under the condition that for any $i\in\{i:|a_{ii}|=h_i(A)\}$, there is a nonzero elements chain $a_{ii_1},\cdots,a_{i_kj}$ with $j\in\{i:|a_{ii}|>h_i(A)\}$? Szulc provided a conter-example to show that the answer is negative. In 1998, Li proposed a sufficient and necessary condition for a weak Nekrasov matrix to be a nonsingular H-matrix with the background of graph theory [4].

Recently, some criteria for nonsingular H-matrices were presented, one can see [5–11]. For a nonsingular H-matrix A, we can construct a diagonal matrix D such that AD is an SDD matrix or Nekrasov matrix. Although Li's chain condition is novel and stunning, the notion "chain condition" involves many terms in graph theory, which makes it difficult to understand and hinders its further applications. In this paper, we present an alternative equivalent version of Li's chain condition. By adopting the new expression, we propose a new criterion for nonsingular H-matrices involving the alternative chain condition.

For convenience, let $\frac{1}{0}:=\infty$. We denote

$$N_1(A)=\{i\in N:|a_{ii}|\leqslant h_i(A)\},$$
$$N_2(A)=\{i\in N:|a_{ii}|>h_i(A)\}.$$

It is pointed out in [12] that A is a Nekrasov matrix if $N_2(A)=N$ and A is not an H-matrix if $N_1(A)=N$. Thus, we always assume that $N_1(A)\neq N$ and $N_2(A)\neq N$ in this paper. We also assume $|a_{ii}|\neq 0$ for all $i\in N$, then it holds that $N_1(A)\cup N_2(A)=N$.

AN ALTERNATIVE EQUIVALENT VERSION OF LI'S CHAIN CONDITION

In this section, we first give a brief introduction of the chain condition proposed by Li [4]. Then, we give it a new description.

Definition 1 Let $V=N$ be a vertex set. A directed graph Γ is a pair of sets (V,E) where $E\subseteq V\times V$ is the arc set. A path from i_1 to i_k in Γ is a sequence of vertices $\sigma=(i_1,i_2,\dots,i_k)$ such that (i_1,i_2), (i_2,i_3), (i_{k-1},i_k) are arcs in Γ. If $\sigma_1=(i_1,i_2,\dots,i_k)$ and $\sigma_2=(i_k,i_{k+1},\dots,i_t)$ are paths in Γ, then the concatenation path of σ_1 and σ_2 is a path $(i_1, i_2, \ldots, i_k, \ldots, i_t)$, denoted by (σ_1, σ_2).

Definition 2 Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$. The directed graph of A is defined by

$$\Gamma(A) = \{(i, j) : a_{ij} \neq 0\}.$$

Definition 3 For $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, let D, $-L$ and $-U$ be diagonal, lower and upper triangular parts of A, respectively. A path $(i, i_1, \ldots, i_k, j)$ from i to j in $\Gamma(A)$ is called a path with property p if $(i, i_1, \ldots, i_k)$ is a path in $\Gamma(D - L)$ and $(i_k, j) \in \Gamma(U)$.

Definition 4 Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ be a weak Nekrasov matrix. A is said to satisfy the chain condition if for any $i \in N_1(A)$, there exists $j \in N_2(A)$ such that there is a concatenation path $(\bar{i}_0, \ldots, \bar{i}_1, \ldots, \bar{i}_k, \ldots, \bar{i}_{k+1})$ in $\Gamma(A)$ where $\bar{i}_0 = i$ and $\bar{i}_{k+1} = j$ with $\bar{i}_0, \bar{i}_1, \ldots, \bar{i}_{k+1}$ pairwise distinct and $(\bar{i}_t, \ldots \bar{i}_{t+1})$ is a path with property p for $t = 0, 1, \ldots, k$.

Lemma 1 ([4]) *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ be a weak Nekrasov matrix. Then A is a nonsingular H-matrix if and only if A satisfies the chain condition.*

Remark that if $(i, i_1, \ldots, i_k, j)$ is a path with property p from i to j in $\Gamma(A)$, then $i \geqslant i_1 \geqslant \cdots > i_k$, $i_k < j$ and $a_{ii_1} a_{i_1 i_2} \cdots a_{i_{k-1} i_k} a_{i_k j} \neq 0$. Now we present an alternative equivalent version of Li's chain condition.

Definition 5 Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ be a weak Nekrasov matrix. For given $i \in N_1(A)$, if there exists $j \in N_2(A)$ such that $a_{ii_1} a_{i_1 i_2} \cdots a_{i_k j} \neq 0$ with $i_k < j$, A is said to satisfy the chain condition for i. If for any $i \in N_1(A)$, A satisfies the chain condition for i, then we say A satisfies the chain condition.

Theorem 1 *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ be a weak Nekrasov matrix. Then A satisfies the chain condition if and only if for any $i \in N_1(A)$, there exists $j \in N_2(A)$ such that $a_{ii_1} a_{i_1 i_2} \cdots a_{i_k j} \neq 0$ with $i_k < j$.*

Proof: If A satisfies the chain condition, then for any $i \in N_1(A)$, there exists $j \in N_2(A)$ such that there is a concatenation path $(\bar{i}_0, \ldots, \bar{i}_1, \ldots, \bar{i}_s, \ldots, \bar{i}_{s+1})$ in $\Gamma(A)$ where $\bar{i}_0 = i$ and $\bar{i}_{s+1} = j$ with $\bar{i}_0, \bar{i}_1, \ldots, \bar{i}_{s+1}$ pairwise distinct and $(\bar{i}_t, \ldots \bar{i}_{t+1})$ is a path with property p for $t = 0, 1, \ldots, s$. Denote

$$(i, i_1, \ldots, i_k, j) := (\bar{i}_0, \ldots, \bar{i}_1, \ldots, \bar{i}_s, \ldots, \bar{i}_{s+1}).$$

Then by Definition 3, we have $a_{ii_1} a_{i_1 i_2} \cdots a_{i_k j} \neq 0$ and $i_k < j$.

Conversely, if for any $i \in N_1(A)$, there exists $j \in N_2(A)$ such that $a_{ii_1} a_{i_1 i_2} \cdots a_{i_k j} \neq 0$ with $i_k < j$, we will show that A satisfies the chain condition. If $i > i_1 > \cdots > i_k$, then $(i, i_1, \ldots, i_k, j)$ is a path with property p. Then A satisfies the chain condition from i to j. Otherwise, suppose there are m integers satisfying $i_{t+1} > i_t$, denoted by $l_1, l_2, \ldots l_m$. That is,

$$\begin{aligned} i > \cdots > i_{l_1},\ \ i_{l_1} < i_{l_1+1}, \\ i_{l_1+1} > \cdots > i_{l_2},\ \ i_{l_2} < i_{l_2+1}, \\ \cdots \\ i_{l_{m-1}+1} > \cdots > i_{l_m},\ \ i_{l_m} < i_{l_m+1}, \\ i_{l_m+1} > \cdots > i_k. \end{aligned}$$

Then, we denote

$$(i, i_i \ldots, i_k, j) := (i, \ldots, i_{l_1+1}, \ldots, i_{l_2+1}, \ldots, i_{l_{m+1}}, \ldots, j),$$

where $(i, \ldots, i_{l_1}, i_{l_1+1})$, $(i_{l_1+1}, \ldots, i_{l_2}, i_{l_2+1})$, $\ldots$, $(i_{l_m+1}, \ldots, i_k, j)$ are pathes with property p, which implies that A satisfies the chain condition. □

In Definition 5, there are only the entries and the index sets involved. The new description is concise, which may facilitate the application of the chain condition. In what follows, whenever we mention the concept of chain condition, we adopt the expression in Definition 5.

Theorem 2 *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ with $P_i(A) = 0$ for some $i \in N$. Then A is a nonsingular H-matrix if and only if A^i is a nonsingular H-matrix, where A^i is a principal sub-matrix of A lying in the rows and columns indexed by $N \backslash \{i\}$.*

Proof: Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ with $P_i(A) = 0$ for some $i \in N$, without any loss, we suppose $p_n(A) = 0$. Let A be partitioned into 2×2 block as follows:

$$A = \begin{bmatrix} A^n & c^n \\ O & a_{nn} \end{bmatrix},$$

where $c^n = (a_{1n}, a_{2n}, \ldots, a_{n-1n})^T$.

If A is a nonsingular H-matrix, there exists a positive diagonal matrix $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ such that AD is an SDD matrix. Let D be partitioned into 2×2 block as follows:

$$D = \begin{bmatrix} D^n & O \\ O & d_n \end{bmatrix},$$

where $D^n = \mathrm{diag}(d_1, d_2, \ldots, d_{n-1})$. Then

$$AD = \begin{bmatrix} A^n D^n & c^n d_n \\ O & a_{nn} d_n \end{bmatrix},$$

which is an SDD matrix. Then it holds trivially that $A^n D^n$ is an SDD matrix, which implies that A^n is a nonsingular H-matrix.

On the other hand, if A^n is a nonsingular H-matrix, there exists a positive diagonal matrix $D^n = \mathrm{diag}(d_1, d_2, \ldots, d_{n-1})$ such that $A^n D^n$ is an SDD matrix. Then

$$|(A^n D^n)_{ii}| > P_i(A^n D^n), \qquad i \in N \backslash \{1\}$$

i.e.,

$$\min_{i\in N\setminus\{1\}} \frac{|(A^nD^n)_{ii}|-P_i(A^nD^n)}{|a_{in}|} > 0.$$

Construct a matrix

$$D = \begin{bmatrix} D^n & O \\ O & d_n \end{bmatrix}$$

with $d_n \in \Big(0, \min_{i\in N\setminus\{1\}} \frac{|(A^nD^n)_{ii}|-P_i(A^nD^n)}{|a_{in}|}\Big)$. Then D is a positive diagonal matrix. Then

$$AD = \begin{bmatrix} A^nD^n & c^nd_n \\ O & a_{nn}d_n \end{bmatrix}$$

is an SDD matrix. We complete the proof. □

Remark 1 In the following, we always suppose that $P_i(A) \neq 0$ for any matrix A.

NEW CRITERION FOR GENERALIZED NEKRASOV MATRICES

Denote

$$l_i(A) = h_i(A) - \sum_{j>i} |a_{ij}|, \quad i \in N;$$

$$r = \max_{i\in N_2(A)} \left\{ \frac{l_i(A) + \sum\limits_{j\in N_1(A),j>i} |a_{ij}|}{|a_{ii}| - \sum\limits_{j\in N_2(A),j>i} |a_{ij}|} \right\};$$

$$\delta_i = \frac{l_i(A) + \sum\limits_{j\in N_1(A),j>i} |a_{ij}| + r \sum\limits_{j\in N_2(A),j>i} |a_{ij}|}{|a_{ii}|}, \quad i \in N_2(A).$$

Lemma 2 ([12]) *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ and $X = \operatorname{diag}(x_1, \dots, x_n)$ with $0 \leqslant x_i \leqslant 1$ for all $i \in N$. Then for all $i \in N$, we have*

$$h_i(AX) \leqslant h_i(A), \qquad l_i(AX) \leqslant l_i(A).$$

Lemma 3 *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$. For any $i \in N_2(A)$, we have $1 > r \geqslant \delta_i(A)$.*

Proof: By the definition of r, it is clear that $r < 1$. For any $i \in N_2(A)$, we have

$$r \geqslant \frac{l_i(A) + \sum\limits_{j\in N_1(A),j>i} |a_{ij}|}{|a_{ii}| - \sum\limits_{j\in N_2(A),j>i} |a_{ij}|}, \quad |a_{ii}| - \sum_{j\in N_2(A),j>i} |a_{ij}| > 0.$$

It follows that

$$r\Big(|a_{ii}| - \sum_{j\in N_2(A),j>i} |a_{ij}|\Big) \geqslant l_i(A) + \sum_{j\in N_1(A),j>i} |a_{ij}|.$$

Then, we obtain

$$r|a_{ii}| \geqslant l_i(A) + \sum_{j\in N_1(A),j>i} |a_{ij}| + r \sum_{j\in N_2(A),j>i} |a_{ij}|,$$

which implies

$$r \geqslant \frac{l_i(A) + \sum\limits_{j\in N_1(A),j>i} |a_{ij}| + r \sum\limits_{j\in N_2(A),j>i} |a_{ij}|}{|a_{ii}|} = \delta_i$$

for all $i \in N_2(A)$. □

Lemma 4 ([13]) *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ with $P_i(A) \neq 0$ for any $i \in N$. Then A is a nonsingular H-matrix if*

$$|a_{ii}| > l_i(A) + \sum_{j\in N_1(A),j>i} |a_{ij}| + \sum_{j\in N_2(A),j>i} |a_{ij}|\delta_j \tag{2}$$

for all $i \in N_1(A)$.

In the following, we will show that some inequalities in (2) can be replaced by

$$\text{“}|a_{ii}| \geqslant l_i(A) + \sum_{j\in N_1(A),j>i} |a_{ij}| + \sum_{j\in N_2(A),j>i} |a_{ij}|\delta_j,\text{”}$$

involving chain condition. We first define an index subset as follows:

$$J(A) = \Big\{ i \in N_1(A) : |a_{ii}| > l_i(A) + \sum_{j\in N_1(A),j>i} |a_{ij}| + \sum_{j\in N_2(A),j>i} |a_{ij}|\delta_j \Big\}.$$

Theorem 3 *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ with $P_i(A) \neq 0$ for any $i \in N$. Then A is a nonsingular H-matrix if the following conditions hold:*

(i) *$|a_{ii}| \geqslant l_i(A) + \sum\limits_{j\in N_1(A),j>i} |a_{ij}| + \sum\limits_{j\in N_2(A),j>i} |a_{ij}|\delta_j$ for all $i \in N_1(A)$;*

(ii) *For any $i \in N_1(A)\backslash J(A)$, there exists $j \in J(A)$ such that $a_{ii_1}a_{i_1i_2}\cdots a_{i_kj} \neq 0$ with $i_k < j$.*

Proof: By the definition of $J(A)$, it holds that

$$|a_{ii}| = l_i(A) + \sum_{j\in N_1(A),j>i} |a_{ij}| + \sum_{j\in N_2(A),j>i} |a_{ij}|\delta_j \tag{3}$$

for $i \in N_1(A)\backslash J(A)$. Let $X = \operatorname{diag}(x_1, \dots, x_n)$ where

$$x_i = \begin{cases} 1, & i \in N_1(A); \\ \delta_i, & i \in N_2(A). \end{cases}$$

Denote $B = AX = (b_{ij})$. Now we show that B is a weak Nekrasov matrix. For $i \in N_1(A)\backslash J(A)$, by (3), we have

$$\begin{aligned} |b_{ii}| &= |a_{ii}| \\ &= l_i(A) + \sum_{j\in N_1(A),j>i} |a_{ij}| + \sum_{j\in N_2(A),j>i} |a_{ij}|\delta_j \\ &= l_i(A) + \sum_{j\in N_1(A),j>i} |b_{ij}| + \sum_{j\in N_2(A),j>i} |b_{ij}| \\ &\geqslant l_i(B) + \sum_{j\in N_1(A),j>i} |b_{ij}| + \sum_{j\in N_2(A),j>i} |b_{ij}| = h_i(B), \end{aligned} \tag{4}$$

by Lemma 2. For $i \in J(A)$, we have

$$
\begin{aligned}
|b_{ii}| &= |a_{ii}| \\
&> l_i(A) + \sum_{j \in N_1(A), j>i} |a_{ij}| + \sum_{j \in N_2(A), j>i} |a_{ij}|\delta_j \\
&= l_i(A) + \sum_{j \in N_1(A), j>i} |b_{ij}| + \sum_{j \in N_2(A), j>i} |b_{ij}| \\
&\geqslant l_i(B) + \sum_{j \in N_1(A), j>i} |b_{ij}| + \sum_{j \in N_2(A), j>i} |b_{ij}| = h_i(B),
\end{aligned} \tag{5}
$$

by Lemma 2. For $i \in N_2(A)$, by the definition of r and δ_i, we have

$$
\begin{aligned}
|b_{ii}| &= |a_{ii}|\delta_i \\
&= l_i(A) + \sum_{j \in N_1(A), j>i} |a_{ij}| + r \sum_{j \in N_2(A), j>i} |a_{ij}| \\
&\geqslant l_i(A) + \sum_{j \in N_1(A), j>i} |a_{ij}| + \sum_{j \in N_2(A), j>i} |a_{ij}|\delta_j \\
&= l_i(A) + \sum_{j \in N_1(A), j>i} |b_{ij}| + \sum_{j \in N_2(A), j>i} |b_{ij}| \\
&\geqslant l_i(B) + \sum_{j \in N_1(A), j>i} |b_{ij}| + \sum_{j \in N_2(A), j>i} |b_{ij}| = h_i(B),
\end{aligned} \tag{6}
$$

by Lemma 2. By (4), (5), and (6), B is a weak Nekrasov matrix.

In what follows, we show that B satisfies the chain condition, that is, for any $i \in N_1(B)$, there exists $j \in N_2(B)$ such that $b_{ii_1} b_{i_1 i_2} \cdots b_{i_k j} \neq 0$ with $i_k < j$. It follow from by (5) that

$$J(A) \subseteq N_2(B),$$

then

$$N_1(B) \subseteq N \backslash J(A) = N_2(A) \cup \big(N_1(A) \backslash J(A)\big).$$

To show that B satisfies the chain condition for any $i \in N_1(B)$, it is sufficient to show that, "for all $i \in N_2(A) \cup (N_1(A) \backslash J(A)$, either there exists $j \in N_2(B)$ such that $b_{ii_1} b_{i_1 i_2} \cdots b_{i_k j} \neq 0$ with $i_k < j$ or $|b_{ii}| > h_i(B)$". Notice that

$$
\begin{aligned}
N_2(A) \cup \big(N_1(A) \backslash J(A)\big) \\
&= \big(N_1(A) \backslash J(A)\big) \cup \{i \in N_2(A) : P_i(A) \neq 0\} \\
&= \big(N_1(A) \backslash J(A)\big) \cup \Big\{i \in N_2(A) : \sum_{j \in N_1(A)} |a_{ij}| \neq 0\Big\} \\
&\quad \cup \Big\{i \in N_2(A) : \sum_{j \in N_1(A)} |a_{ij}| = 0, \sum_{j \neq i, j \in N_2(A)} |a_{ij}| \neq 0\Big\}.
\end{aligned}
$$

Then we consider the following three cases.

Case 1: $i \in N_1(A) \backslash J(A)$. By (ii), there exists $j \in J(A) \subseteq N_2(B)$ such that $a_{ii_1} a_{i_1 i_2} \cdots a_{i_k j} \neq 0$ with $i_k < j$. Thus, we have $b_{ii_1} b_{i_1 i_2} \cdots b_{i_k j} \neq 0$ with $i_k < j$, which implies that B satisfies the chain condition for any $i \in N_1(A) \backslash J(A)$.

Case 2: $i \in N_2(A)$ with $\sum_{j \in N_1(A)} |a_{ij}| \neq 0$. Under this situation, we have

$$\sum_{j>i, j \in N_1(A)} |a_{ij}| + \sum_{j<i, j \in N_1(A)} |a_{ij}| \neq 0,$$

which implies that at least one of

$$\sum_{j>i, j \in N_1(A)} |a_{ij}| \neq 0 \tag{7}$$

and

$$\sum_{j<i, j \in N_1(A)} |a_{ij}| \neq 0 \tag{8}$$

holds.

Subcase 2.1: (7) holds. Then there exists $j_0 \in N_1(A)$ with $j_0 > i$ such that $|a_{ij_0}| \neq 0$. Then

$$|b_{ij_0}| \neq 0. \tag{9}$$

If $j_0 \in J(A)$, then $j_0 \in N_2(B)$, which implies that B satisfies the chain condition for i by (9). If $j_0 \in N_1(A) \backslash J(A)$, by Case 1, B satisfies the chain condition for j_0, which implies that B satisfies the chain condition for i. Overall, B satisfies the chain condition for $i \in N_2(A)$ with $\sum_{j>i, j \in N_1(A)} |a_{ij}| \neq 0$.

Subcase 2.2: (8) holds. Then it must hold that $\sum_{j<i, j \in N_1(A)} |a_{ij}| \neq 0$, which implies that there exists $j_0 \in N_1(A)$ satisfying $|a_{ij_0}| \neq 0$ with $j_0 < i$. Then

$$|b_{ij_0}| \neq 0, \qquad h_{j_0}(A) \geqslant |a_{j_0 j_0}| = |b_{j_0 j_0}|. \tag{10}$$

If $j_0 \in J(A)$, by $J(A) \subseteq N_2(B)$ and (10), we know that

$$h_{j_0}(A) > h_{j_0}(B).$$

By (6) and (10), we have

$$
\begin{aligned}
|b_{ii}| &\geqslant l_i(A) + \sum_{j>i} |b_{ij}| \\
&= \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j>i} |b_{ij}| \\
&= \sum_{j=1, j \neq j_0}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + |a_{ij_0}| \frac{h_{j_0}(A)}{|a_{j_0 j_0}|} + \sum_{j>i} |b_{ij}| \\
&\geqslant \sum_{j=1, j \neq j_0}^{i-1} |a_{ij}| \frac{h_j(B)}{|a_{jj}|} + |a_{ij_0}| \frac{h_{j_0}(A)}{|a_{j_0 j_0}|} + \sum_{j>i} |b_{ij}| \\
&= \sum_{j=1, j \neq j_0}^{i-1} |b_{ij}| \frac{h_j(B)}{|b_{jj}|} + |b_{ij_0}| \frac{h_{j_0}(A)}{|b_{j_0 j_0}|} + \sum_{j>i} |b_{ij}| \\
&> \sum_{j=1, j \neq j_0}^{i-1} |b_{ij}| \frac{h_j(B)}{|b_{jj}|} + |b_{ij_0}| \frac{h_{j_0}(B)}{|b_{j_0 j_0}|} + \sum_{j>i} |b_{ij}| \\
&= l_i(B) + \sum_{j>i} |b_{ij}| = h_i(B),
\end{aligned}
$$

which implies that $|b_{ii}| > h_i(B)$.

If $j_0 \in N_1(A)\backslash J(A)$, by Case 1, B satisfies the chain condition for j_0. By (10) again, B satisfies the chain condition for i. Overall, for any $i \in N_2(A)$ with $\sum_{j\in N_1(A)} |a_{ij}| \neq 0$, B satisfies the chain condition for i or $|b_{ii}| > h_i(B)$.

Case 3: $i \in N_2(A)$ with $\sum_{j\in N_1(A)} |a_{ij}| = 0$. Then $\sum_{j\in N_2(A), j\neq i} |a_{ij}| \neq 0$. Denote

$$\left\{i \in N_2(A) : \sum_{j\in N_1(A)} |a_{ij}| = 0, \sum_{j\neq i, j\in N_2(A)} |a_{ij}| \neq 0\right\} := \{i_1, \ldots, i_t\},$$

where $i_1 < \cdots < i_t$.

We first consider the case $i = i_t$. Then $\sum_{j\in N_1(A)} |a_{i_t j}| = 0$ and $\sum_{j\neq i_t, j\in N_2(A)} |a_{i_t j}| \neq 0$. Then there exists $j_0 \in N_2(A)$ with $j_0 \neq i_t$ such that $|a_{i_t j_0}| \neq 0$. Considering that i_t is the biggest number in $N_2(A)$, we have $j_0 < i_t$. Now instigate the j_0-th row of A by considering the following four cases.

Subcase 3.1: $\sum_{j\in N_1(A)} |a_{j_0 j}| \neq 0$. By the similar deduction of Case 2, B satisfies the chain condition for j_0 or $|b_{j_0 j_0}| > h_{j_0}(B)$. If B satisfies the chain condition for j_0, then B satisfies the chain condition for i_t. If $|b_{j_0 j_0}| > h_{j_0}(B)$, it holds that

$$\begin{aligned}|a_{i_t j_0}|\frac{h_{j_0}(A)}{|a_{j_0 j_0}|} &\geqslant |a_{i_t j_0}|\frac{l_{j_0}(A) + \sum_{j\in N_1(A), j>j_0} |a_{j_0 j}| + r\sum_{j\in N_2(A), j>j_0} |a_{j_0 j}|}{|a_{j_0 j_0}|}\\ &= |a_{i_t j_0}|\delta_{j_0} = |b_{i_t j_0}|\\ &> |b_{i_t j_0}|\frac{h_{j_0}(B)}{|b_{j_0 j_0}|},\end{aligned} \tag{11}$$

which, together with (6), we have by (11)

$$\begin{aligned}|b_{i_t i_t}| &\geqslant l_{i_t}(A) + \sum_{j>i_t} |b_{i_t j}|\\ &= \sum_{j=1}^{i_t-1} |a_{i_t j}|\frac{h_j(A)}{|a_{jj}|} + \sum_{j>i_t} |b_{i_t j}|\\ &= \sum_{j=1, j\neq j_0}^{i_t-1} |a_{i_t j}|\frac{h_j(A)}{|a_{jj}|} + |a_{i_t j_0}|\frac{h_{j_0}(A)}{|a_{j_0 j_0}|} + \sum_{j>i_t} |b_{i_t j}|\\ &\geqslant \sum_{j=1, j\neq j_0}^{i_t-1} |a_{i_t j}|\frac{h_j(B)}{|a_{jj}|} + |a_{i_t j_0}|\frac{h_{j_0}(A)}{|a_{j_0 j_0}|} + \sum_{j>i_t} |b_{i_t j}|\\ &> \sum_{j=1, j\neq j_0}^{i_t-1} |b_{i_t j}|\frac{h_j(B)}{|b_{jj}|} + |b_{i_t j_0}|\frac{h_{j_0}(B)}{|b_{j_0 j_0}|} + \sum_{j>i_t} |b_{i_t j}|\\ &= l_{i_t}(B) + \sum_{j>i_t} |b_{i_t j}| = h_{i_t}(B).\end{aligned} \tag{12}$$

Overall, if $\sum_{j\in N_1(A)} |a_{j_0 j}| \neq 0$, then B satisfies the chain condition for i_t or $|b_{i_t i_t}| > h_{i_t}(B)$.

Subcase 3.2: $\sum_{j\in N_1(A)} |a_{j_0 j}| = 0$ and $\sum_{j\in N_2(A), j>j_0} |a_{j_0 j}| \neq 0$. Since $r < 1$, we have

$$\frac{h_{j_0}(A)}{|a_{j_0 j_0}|} > \frac{l_{j_0}(A) + \sum_{j\in N_1(A), j>j_0} |a_{i_t j}| + r\sum_{j\in N_2(A), j>j_0} |a_{i_t j}|}{|a_{j_0 j_0}|} = \delta_{j_0}.$$

Since B is a weak Nekrasov matrix, we obtain

$$\begin{aligned}|a_{i_t j_0}|\frac{h_{j_0}(A)}{|a_{j_0 j_0}|} &> |a_{i_t j_0}|\delta_{j_0} = |b_{i_t j_0}|\\ &\geqslant |b_{i_t j_0}|\frac{h_{j_0}(B)}{|b_{j_0 j_0}|} = |a_{i_t j_0}|\frac{h_{j_0}(B)}{|a_{j_0 j_0}|},\end{aligned}$$

then

$$h_{j_0}(A) > h_{j_0}(B).$$

By the similar deduction of Subcase 2.2, we have $|b_{i_t i_t}| > h_{i_t}(B)$.

Subcase 3.3: $\sum_{j\in N_1(A)} |a_{j_0 j}| = 0$, $\sum_{j\in N_2(A), j>j_0} |a_{j_0 j}| = 0$ and $\sum_{j\in N_2(A), j<j_0} |a_{j_0 j}| \neq 0$. Then there exists $j_1 \in N_2(A)$ with $j_1 < j_0$ such that $|a_{j_0 j_1}| \neq 0$. We need to consider the j_1-th row of A with the similar argument to Subcase 4.1–Subcase 4.4. If the j_1-th row of A satisfies the conditions of Subcase 3.1 or Subcase 3.2, it holds that B satisfies the chain condition for i_t or $|b_{i_t i_t}| > h_{i_t}(B)$. If the j_1-th row of A satisfies the conditions of Subcase 3.3, then there exists $j_2 \in N_2(A)$ with $j_2 < j_1$ such that $|a_{j_1 j_2}| \neq 0$, and we consider the j_2-th row of A with the similar discussion to Subcase 3.1–Subcase 3.3. Without loss of generality, we assume that there are finite integers $j_0 > j_1 > \cdots > j_l$ where $j_l > i_1$ and $\sum_{j\in N_2(A), j<j_l} |a_{j_l j}| = 0$ (i.e., the j_l-th row of A at least satisfies one of the conditions of Subcase 3.1 and Subcase 3.2). Then by similar deduction of (11) and (12), B satisfies the chain condition for i_t or $|b_{i_t i_t}| > h_{i_t}(B)$.

By the similar argument for $i = i_t$, we discuss the case $i = i_s$ for $s = t-1, \ldots, 1$. We can obtain that for any $i \in N_2(A)$ with $\sum_{j\in N_1(A)} |a_{ij}| = 0$, B satisfies the upper chain condition for i or $|b_{ii}| > h_i(B)$.

From the discussion of Case 1–Case 3, we conclude that for any $i \in N_2(A) \cup (N_1(A)\backslash J(A))$, B satisfies the chain condition for i or $|b_{ii}| > h_i(B)$. Hence, B satisfies the chain condition. Since B is a weak Nekrasov matrix, then B is a nonsingular H-matrix by Lemma 1. Then A is a nonsingular H-matrix. □

Remark that if $J(A) = N_1(A)$, then Theorem 3 coincides with Lemma 4. Generally, the chain condition makes it possible to replace "$>$" with "$\geqslant$" in the criteria for nonsingular H-matrices.

Example 1 Consider the matrix

$$A = \begin{bmatrix} 6 & 1 & 0 & 1 & 4 \\ 1 & 10 & 5.4 & 3.6 & 1 \\ 1 & 2 & 12 & 4 & 6 \\ 3 & 10 & 12 & 35 & 3 \\ 1 & 2 & 2 & 0 & 10 \end{bmatrix}.$$

It is easy to compute that $h_1(A) = 6$, $h_2(A) = 11$, $h_3(A) = 13.2$, $h_4(A) = 32.2$, $h_5(A) = 5.4$; $l_1(A) = 0$, $l_2(A) = 1$, $l_3(A) = 3.2$, $l_4(A) = 27.2$, $l_5(A) = 5.4$. By computation, we have

$$\gamma = 0.85,\ \delta_4 = 0.85,\ \delta_5 = 0.54;$$
$$N_1(A) = \{1,2,3\},\ N_2(A) = \{4,5\}.$$

It can be testified that

$$|a_{11}| = 6 > 4.01 = l_1(A) + \sum_{j\in N_1(A), j>1} |a_{1j}| + \sum_{j\in N_2(A), j>1} |a_{1j}|\delta_j,$$
$$|a_{22}| = 10 = 10 = l_2(A) + \sum_{j\in N_1(A), j>2} |a_{2j}| + \sum_{j\in N_2(A), j>2} |a_{2j}|\delta_j,$$
$$|a_{33}| = 12 > 9.84 = l_3(A) + \sum_{j\in N_1(A), j>3} |a_{3j}| + \sum_{j\in N_2(A), j>3} |a_{3j}|\delta_j,$$

which implies that $J(A) = \{1,3\}$ and then $N_1(A)\backslash J(A) = \{2\}$. Moreover, for $i = 2$, there exists $j = 3 \in J(A)$ such that $a_{21}a_{12}a_{23} \neq 0$. Hence, the situations in Theorem 3 hold. Hence, by Theorem 3, A is a nonsingular H-matrix. In fact, we can find a diagonal matrix $X = \mathrm{diag}\{1,1,1,0.84,0.53\}$ such that

$$AX = \begin{bmatrix} 6 & 1 & 0 & 0.84 & 2.12 \\ 1 & 10 & 0 & 3.024 & 0.53 \\ 1 & 2 & 12 & 3.36 & 3.18 \\ 3 & 10 & 12 & 29.4 & 1.59 \\ 1 & 2 & 2 & 0 & 5.3 \end{bmatrix}.$$

It is obvious that AX is an SDD matrix, i.e., A is a nonsingular H-matrix.

Acknowledgements: This work was supported by the National Natural Science Foundation of China (No. 12171323), Liaoning Provincial Department of Education Program (JYTMS20230281), and the Fundamental Research Funds for the Universities of Liaoning Province.

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