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Identities for sums of floor and ceiling functions of rational numbers

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ABSTRACT: Research into sums involving floor or ceiling functions has been a popular focus in number theory. This paper investigates formulas for sums of the type $\sum_{k=1}^{n} F\left(\frac{pk}{q}\right)$, where F represents either the floor or ceiling function, and p, q are coprime integers. Generalizing from Palatsang et al's work in 2021, we could derive explicit simple formulas, verify their uniqueness, and identify certain conditions of p and q under which these formulas are invalid.

KEYWORDS: floor function, ceiling function, rational numbers, summation identity

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INTRODUCTION

Given $x \in \mathbb{R}$, we define the floor function of x, denoted by $\lfloor x \rfloor$, as the largest integer less than or equal to x. We also define the ceiling function of x, denoted by $\lceil x \rceil$ as the smallest integer greater than or equal to x. It is useful in number theory to study the formula for the sum of floor functions. For instance, a proof of the law of quadratic reciprocity [1] by Eisenstein requires a lemma which states that given an odd prime p and an odd integer a with $p \nmid a$, the Legendre symbol can be computed by the equation

$$\left(\frac{a}{p}\right) = (-1)^{\alpha(a,p)},$$

where

$$\alpha(a,p) = \sum_{k=1}^{\frac{p-1}{2}} \left\lfloor \frac{ka}{p} \right\rfloor.$$

In Apostol's book [2], there is an exercise asking to show that for all $q \in \{1,2,\ldots,7\}$, there exists b depending on q such that

$$\sum_{k=1}^{n} \left\lfloor \frac{k}{q} \right\rfloor = \left\lfloor \frac{(2n+b)^2}{8q} \right\rfloor \tag{1}$$

for every $n \in \mathbb{N}$. Palatsang et al [3] showed that for all $n, q \in \mathbb{N}$, and $r \in \{0,1,\ldots,q-1\}$ with $r \equiv n \pmod q$,

$$\sum_{k=1}^{n} \left\lfloor \frac{k}{q} \right\rfloor = \frac{n(n+2-q) - r(2-q+r)}{2q}.$$

They also proved (1) by choosing $b = 2-q$ and comparing each modulo class. Moreover, they also showed that b must be equal to $2-q$. Additionally, if $q \in \mathbb{N}$ such that $q \geqslant 8$, then the formula does not exist for infinitely many $n \in \mathbb{N}$.

At the end of the paper, they left a question for generalizing the results to the sum of the form

$$\sum_{k=1}^{n} F\left(\frac{pk}{q}\right). \tag{2}$$

In this case, the denominator is a rational number, which includes both positive and negative, instead of only a natural number, and F is the floor or ceiling function. Pongsriiam's book [4] proposes some identities related to these sums. For example, if $p,q,s \in \mathbb{N}$ with $p \leqslant q$, then

$$\sum_{k=1}^{s} \left\lfloor \frac{pk}{q} \right\rfloor + \sum_{1 \leqslant k \leqslant \frac{ps}{q}} \left\lfloor \frac{qk}{p} \right\rfloor = s\left\lfloor \frac{ps}{q} \right\rfloor + \left\lfloor \frac{s\gcd(p,q)}{q} \right\rfloor.$$

In this paper, we study the identities for the sum of the form (2), where $p \in \mathbb{N}, q \in \mathbb{Z}\backslash\{0\}$, and $\gcd(p,q) = 1$. We can find such simple formulas for the sums similar to the previous paper and can find several conditions that make the formulas not exist. For instance, in the case where F is the floor function, $q = 3$, and $p \in \{1,2,4,5,7,8,10,11,13,14,16,17,19\}$, the sum is equal to the formula

$$\left\lfloor \frac{p}{8q}\left(2n + \frac{p-q+1}{p}\right)^2 \right\rfloor$$

for all $n \in \mathbb{N}$. Moreover, if $p \geqslant (\sqrt{2|q|} + \sqrt{|q|+1})^2$ or $p \leqslant (\sqrt{2|q|} - \sqrt{|q|+1})^2$, then the formula does not work for infinitely many $n \in \mathbb{N}$.

PRELIMINARIES

In this section, we introduce some identities regarding floor and ceiling functions that could help us prove the main results.

Lemma 1 ([5]) *Let $p,q\in\mathbb{N}$. Then,*

$$\sum_{k=1}^{q-1}\left\lfloor\frac{pk}{q}\right\rfloor=\frac{(p-1)(q-1)+\gcd(p,q)-1}{2}.$$

Lemma 2 ([5]) *Let $x\in\mathbb{R}$ and $n\in\mathbb{Z}$. Then,*

(i) $\lfloor -x\rfloor=-\lceil x\rceil=\begin{cases}-\lfloor x\rfloor, & x\in\mathbb{Z}\\ -(\lfloor x\rfloor+1), & x\notin\mathbb{Z}.\end{cases}$

(ii) $\lfloor x+n\rfloor=\lfloor x\rfloor+n$.

Next, we wish to extend Lemma 1 for the ceiling function and $q\in\mathbb{Z}\backslash\{0\}$.

Lemma 3 *Let $p\in\mathbb{N}$ and $q\in\mathbb{Z}\backslash\{0\}$ with $\gcd(p,q)=1$. Then,*

$$\sum_{k=1}^{|q|-1}\left\lfloor\frac{pk}{q}\right\rfloor=\frac{(p+1)(q+1)}{2},\quad q\leqslant -1,\quad \textit{and}$$

$$\sum_{k=1}^{|q|-1}\left\lceil\frac{pk}{q}\right\rceil=\begin{cases}\frac{(p+1)(q-1)}{2}, & q\geqslant 1\\ \frac{(p-1)(q+1)}{2}, & q\leqslant -1.\end{cases}$$

Proof: First, we consider the sum of the floor function for $q\leqslant -1$. Note that $|q|=-q$, $\gcd(p,q)=\gcd(p,|q|)$, and $pk/q\notin\mathbb{Z}$ for $k\in\{1,\dots,|q|-1\}$. By Lemma 2, we have

$$\begin{aligned}\sum_{k=1}^{|q|-1}\left\lfloor\frac{pk}{q}\right\rfloor&=-\sum_{k=1}^{|q|-1}\left(\left\lfloor\frac{pk}{|q|}\right\rfloor+1\right)\\&=-\left(\frac{(p-1)(|q|-1)}{2}+|q|-1\right)\\&=\frac{(p-1)(q+1)}{2}+q+1\\&=\frac{(p+1)(q+1)}{2}.\end{aligned}$$

Next, we consider the sum of the ceiling function. By Lemma 2, we obtain

$$\begin{aligned}\sum_{k=1}^{|q|-1}\left\lceil\frac{pk}{q}\right\rceil&=\sum_{k=1}^{|q|-1}\left(\left\lfloor\frac{pk}{q}\right\rfloor+1\right)\\&=\begin{cases}\frac{(p-1)(q-1)}{2}+q-1, & q\geqslant 1\\ \frac{(p+1)(q+1)}{2}-q-1, & q\leqslant -1\end{cases}\\&=\begin{cases}\frac{(p+1)(q-1)}{2}, & q\geqslant 1\\ \frac{(p-1)(q+1)}{2}, & q\leqslant -1\end{cases}\end{aligned}$$

as desired. □

Lemma 4 *Let $n\in\mathbb{N}\cup\{0\}$, $p\in\mathbb{N}$, and $q\in\mathbb{Z}\backslash\{0\}$ with $\gcd(p,q)=1$. Suppose that $n\equiv r\pmod{|q|}$ and $r\in\{0,1,\dots,|q|-1\}$. Then,*

$$\sum_{k=0}^{n}\left\lfloor\frac{pk}{q}\right\rfloor=$$

$$\begin{cases}\frac{p}{2q}\left(n\left(n+\frac{p-q+1}{p}\right)-r\left(r+\frac{p-q+1}{p}\right)\right)+\sum_{k=0}^{r}\left\lfloor\frac{pk}{q}\right\rfloor, & q\geqslant 1\\ \frac{p}{2q}\left(n\left(n+\frac{p-q-1}{p}\right)-r\left(r+\frac{p-q-1}{p}\right)\right)+\sum_{k=0}^{r}\left\lfloor\frac{pk}{q}\right\rfloor, & q\leqslant -1\end{cases}$$

and

$$\sum_{k=0}^{n}\left\lceil\frac{pk}{q}\right\rceil=$$

$$\begin{cases}\frac{p}{2q}\left(n\left(n+\frac{p+q-1}{p}\right)-r\left(r+\frac{p+q-1}{p}\right)\right)+\sum_{k=0}^{r}\left\lceil\frac{pk}{q}\right\rceil, & q\geqslant 1\\ \frac{p}{2q}\left(n\left(n+\frac{p+q+1}{p}\right)-r\left(r+\frac{p+q+1}{p}\right)\right)+\sum_{k=0}^{r}\left\lceil\frac{pk}{q}\right\rceil, & q\leqslant -1.\end{cases}$$

Proof: First, we consider the sum of the floor function. If $|q|=1$, then $r=0$ and

$$\sum_{k=0}^{n}\left\lfloor\frac{pk}{q}\right\rfloor=\frac{p}{q}\sum_{k=0}^{n}k=\frac{pn(n+1)}{2q}$$

$$=\begin{cases}\frac{p}{2q}\left(n\left(n+\frac{p-q+1}{p}\right)-r\left(r+\frac{p-q+1}{p}\right)\right)+\sum_{k=0}^{r}\left\lfloor\frac{pk}{q}\right\rfloor, & q=1\\ \frac{p}{2q}\left(n\left(n+\frac{p-q-1}{p}\right)-r\left(r+\frac{p-q-1}{p}\right)\right)+\sum_{k=0}^{r}\left\lfloor\frac{pk}{q}\right\rfloor, & q=-1.\end{cases}$$

If $n<|q|$, then $r=n$. It is easy to see that the right-hand side is equal to the sum of the floor function for every case of q. So, we assume that $2\leqslant|q|\leqslant n$. By the division algorithm, there exists a unique $s\in\mathbb{N}$ such that $n=s|q|+r$. Then,

$$\sum_{k=0}^{n}\left\lfloor\frac{pk}{q}\right\rfloor=\sum_{k=0}^{s|q|-1}\left\lfloor\frac{pk}{q}\right\rfloor+\sum_{k=s|q|}^{s|q|+r}\left\lfloor\frac{pk}{q}\right\rfloor. \tag{3}$$

By Lemma 2, the first term on the right-hand side of (3) is

$$\begin{aligned}\sum_{k=0}^{s|q|-1}\left\lfloor\frac{pk}{q}\right\rfloor&=\sum_{t=0}^{s-1}\sum_{k=t|q|+1}^{(t+1)|q|-1}\left\lfloor\frac{pk}{q}\right\rfloor+\sum_{t=0}^{s-1}\left\lfloor\frac{p|q|t}{q}\right\rfloor\\&=\sum_{t=0}^{s-1}\sum_{u=1}^{|q|-1}\left\lfloor\frac{p|q|t}{q}+\frac{pu}{q}\right\rfloor+\sum_{t=0}^{s-1}\frac{p|q|t}{q}\\&=\sum_{t=0}^{s-1}\sum_{u=1}^{|q|-1}\left(\frac{p|q|t}{q}+\left\lfloor\frac{pu}{q}\right\rfloor\right)+\frac{p|q|}{q}\sum_{t=0}^{s-1}t\\&=\sum_{t=0}^{s-1}\left(\frac{p|q|(|q|-1)t}{q}+\sum_{u=1}^{|q|-1}\left\lfloor\frac{pu}{q}\right\rfloor\right)+\frac{p|q|}{q}\sum_{t=0}^{s-1}t\\&=\frac{p|q|(|q|-1)}{q}\sum_{t=0}^{s-1}t+s\sum_{u=1}^{|q|-1}\left\lfloor\frac{pu}{q}\right\rfloor+\frac{p|q|}{q}\sum_{t=0}^{s-1}t\\&=pq\sum_{t=0}^{s-1}t+s\sum_{u=1}^{|q|-1}\left\lfloor\frac{pu}{q}\right\rfloor=\frac{pq}{2}s(s-1)+s\sum_{u=1}^{|q|-1}\left\lfloor\frac{pu}{q}\right\rfloor.\end{aligned}$$

For the second term of (3), by Lemma 2, we have

$$\sum_{k=s|q|}^{s|q|+r}\left\lfloor\frac{pk}{q}\right\rfloor=\sum_{l=0}^{r}\left\lfloor\frac{ps|q|}{q}+\frac{pl}{q}\right\rfloor=\frac{p|q|(r+1)}{q}s+\sum_{l=0}^{r}\left\lfloor\frac{pl}{q}\right\rfloor.$$

Summing all the terms and using Lemma 1, we obtain

$$\begin{aligned}\sum_{k=0}^{n}\left\lfloor\frac{pk}{q}\right\rfloor&=\frac{pq}{2}s(s-1)+\frac{p|q|(r+1)}{q}s+s\sum_{u=1}^{|q|-1}\left\lfloor\frac{pu}{q}\right\rfloor+\sum_{l=0}^{r}\left\lfloor\frac{pl}{q}\right\rfloor\\&=\begin{cases}\frac{pq}{2}s^2+\left(\frac{(2r+1)p-q+1}{2}\right)s+\sum_{l=0}^{r}\left\lfloor\frac{pl}{q}\right\rfloor, & q\geqslant 1\\ \frac{pq}{2}s^2-\left(\frac{(2r+1)p-q-1}{2}\right)s+\sum_{l=0}^{r}\left\lfloor\frac{pl}{q}\right\rfloor, & q\leqslant -1.\end{cases}\end{aligned}$$

Substituting $s=(n-r)/|q|$, we have

$$\begin{aligned}&\sum_{k=0}^{n}\left\lfloor\frac{pk}{q}\right\rfloor\\&=\begin{cases}\frac{pq}{2}\left(\frac{n-r}{q}\right)^2+\left(\frac{(2r+1)p-q+1}{2}\right)\left(\frac{n-r}{q}\right)+\sum_{l=0}^{r}\left\lfloor\frac{pl}{q}\right\rfloor, & q\geqslant 1\\ \frac{pq}{2}\left(\frac{n-r}{q}\right)^2+\left(\frac{(2r+1)p-q-1}{2}\right)\left(\frac{n-r}{q}\right)+\sum_{l=0}^{r}\left\lfloor\frac{pl}{q}\right\rfloor, & q\leqslant -1\end{cases}\\&=\begin{cases}\frac{p}{2q}\left(n\left(n+\frac{p-q+1}{p}\right)-r\left(r+\frac{p-q+1}{p}\right)\right)+\sum_{l=0}^{r}\left\lfloor\frac{pl}{q}\right\rfloor, & q\geqslant 1\\ \frac{p}{2q}\left(n\left(n+\frac{p-q-1}{p}\right)-r\left(r+\frac{p-q-1}{p}\right)\right)+\sum_{l=0}^{r}\left\lfloor\frac{pl}{q}\right\rfloor, & q\leqslant -1.\end{cases}\end{aligned}$$

Next, we consider the sum of the ceiling function. By Lemma 2, we obtain

$$\begin{aligned}&\sum_{k=0}^{n}\left\lceil\frac{pk}{q}\right\rceil=-\sum_{k=0}^{n}\left\lfloor-\frac{pk}{q}\right\rfloor\\&=\begin{cases}-\left(\frac{p}{2(-q)}\left(n\left(n+\frac{p+q-1}{p}\right)-r\left(r+\frac{p+q-1}{p}\right)\right)+\sum_{l=0}^{r}\left\lfloor-\frac{pl}{q}\right\rfloor\right), & q\geqslant 1\\ -\left(\frac{p}{2(-q)}\left(n\left(n+\frac{p+q+1}{p}\right)-r\left(r+\frac{p+q+1}{p}\right)\right)+\sum_{l=0}^{r}\left\lfloor-\frac{pl}{q}\right\rfloor\right), & q\leqslant -1\end{cases}\\&=\begin{cases}\frac{p}{2q}\left(n\left(n+\frac{p+q-1}{p}\right)-r\left(r+\frac{p+q-1}{p}\right)\right)+\sum_{l=0}^{r}\left\lceil\frac{pl}{q}\right\rceil, & q\geqslant 1\\ \frac{p}{2q}\left(n\left(n+\frac{p+q+1}{p}\right)-r\left(r+\frac{p+q+1}{p}\right)\right)+\sum_{l=0}^{r}\left\lceil\frac{pl}{q}\right\rceil, & q\leqslant -1.\end{cases}\end{aligned}$$

Thus, the proof is complete. □

Lemma 5 *Let F be the floor or ceiling function, $p\in\mathbb{N}$, $q\in\mathbb{Z}\backslash\{0\}$ with $q\neq 0$, and $\alpha,\beta,\gamma,a,b,c\in\mathbb{R}$ with $\alpha,c\neq 0$. Suppose that $A\subseteq\mathbb{N}$ is an infinite set and*

$$\alpha n^2+\beta n+\gamma=F\left(\frac{p(an+b)^2}{qc}\right)$$

for every $n\in A$. Then,

$$\frac{p(an+b)^2}{qc}=\frac{(2\alpha n+\beta)^2}{4\alpha}$$

for every $n\in A$.

Proof: We will only prove the case that F is the floor function on $\mathbb{R}$. Assume that

$$\alpha n^2+\beta n+\gamma=\left\lfloor\frac{p(an+b)^2}{qc}\right\rfloor$$

for all $n\in A$. Hence, for all $n\in A$, we have

$$\alpha n^2+\beta n+\gamma=\frac{pa^2}{qc}n^2+\frac{2pab}{qc}n+\frac{pb^2}{qc}-\theta_n, \qquad (4)$$

with $\theta_n\in[0,1)$. We divide (4) by n^2 and take $n\to\infty$, where $n\in A$. Hence, $\alpha=pa^2/(qc)$. This implies

$$\beta n+\gamma=\frac{2pab}{qc}n+\frac{pb^2}{qc}-\theta_n, \qquad (5)$$

for all $n\in A$. We divide (5) by n and take $n\to\infty$, where $n\in A$ to obtain $\beta=2pab/(qc)$. Hence,

$$\frac{pb^2}{qc}=\left(\frac{4p^2a^2b^2}{q^2c^2}\right)\left(\frac{qc}{4pa^2}\right)=\frac{\beta^2}{4\alpha}.$$

Thus, we have

$$\begin{aligned}\frac{p(an+b)^2}{qc}&=\frac{pa^2}{qc}n^2+\frac{2pab}{qc}n+\frac{pb^2}{qc}\\&=\alpha n^2+\beta n+\frac{\beta^2}{4\alpha}=\frac{(2\alpha n+\beta)^2}{4\alpha}\end{aligned}$$

as desired. The case that F is the ceiling function can be proven analogously. □

MAIN RESULTS

In this section, we state the results regarding the formula for the sum of the form

$$\sum_{k=1}^{n}F\left(\frac{pk}{q}\right),$$

where F is the floor or ceiling function, $p\in\mathbb{N}$, $q\in\mathbb{Z}\backslash\{0\}$, and $\gcd(p,q)=1$. Before stating them, we define

$$\varepsilon=\begin{cases}-1, & F\text{ is the floor function}\\ 1, & F\text{ is the ceiling function.}\end{cases}$$

Additionally, let G be the floor or ceiling function.

Theorem 1 *Let $p\in\mathbb{N}$ and $q\in\mathbb{Z}\backslash\{0\}$ with $\gcd(p,q)=1$, a, b, $c\in\mathbb{R}$ and $c\neq 0$.*

(i) *If $q\geqslant 1$ and there exists an infinite set $A\subseteq\mathbb{N}$ such that*

$$\sum_{k=1}^{n}F\left(\frac{pk}{q}\right)=G\left(\frac{p(an+b)^2}{cq}\right) \qquad (6)$$

for every $n\in A$, then

$$\frac{p(an+b)^2}{cq}=\frac{p}{8q}\left(2n+\frac{p+\varepsilon(q-1)}{p}\right)^2$$

for every $n\in A$.

(ii) *If $q\leqslant -1$ and there exists an infinite set $A\subseteq\mathbb{N}$ such that*

$$\sum_{k=1}^{n}F\left(\frac{pk}{q}\right)=G\left(\frac{p(an+b)^2}{cq}\right) \qquad (7)$$

for every $n\in A$, then

$$\frac{p(an+b)^2}{cq}=\frac{p}{8q}\left(2n+\frac{p+\varepsilon(q+1)}{p}\right)^2$$

for every $n\in A$.

Proof: We will only prove (i) with the case that F is the floor function. Let $n\in A$. By Lemma 4, we can write the left-hand side of (6) as

$$\begin{aligned}\sum_{k=1}^{n}\left\lfloor\frac{pk}{q}\right\rfloor&=\frac{p}{2q}\left(n\left(n+\frac{p-q+1}{p}\right)-r\left(r+\frac{p-q+1}{p}\right)\right)\\&\quad+\sum_{k=0}^{r}\left\lfloor\frac{pk}{q}\right\rfloor\\&=\frac{p}{2q}n^2+\frac{p-q+1}{2q}n+\gamma_n,\end{aligned}$$

where $n \equiv r \pmod q$ and $\gamma_n \in \mathbb{R}$ is bounded. By Lemma 5, set

$$\alpha = \frac{p}{2q} \quad \text{and} \quad \beta = \frac{p-q+1}{2q}.$$

We have

$$\frac{p(an+b)^2}{qc} = \frac{(2(p/(2q))n+((p-q+1)/(2q))^2}{4(p/(2q))} = \frac{p}{8q}\left(2n+\frac{p-q+1}{p}\right)^2.$$

The other cases can be proven analogously. □

Theorem 2 *Let* $p \in \mathbb{N}$ *and* $q \in \mathbb{Z}\backslash\{0\}$ *with* $\gcd(p,q)=1$.
(i) *If* $q \geqslant 1$ *and*

$$\sum_{k=1}^{n} F\left(\frac{pk}{q}\right) = G\left(\frac{p(2n+b)^2}{8q}\right)$$

for infinitely many $n \in \mathbb{N}$*, then* $b = (p+\varepsilon(q-1))/p$.
(ii) *If* $q \leqslant -1$ *and*

$$\sum_{k=1}^{n} F\left(\frac{pk}{q}\right) = G\left(\frac{p(2n+b)^2}{8q}\right)$$

for infinitely many $n \in \mathbb{N}$*, then* $b = (p+\varepsilon(q+1))/p$.

Proof: The idea of this proof is based on [3]. We will only show (i) with the case that F is the floor function. By Lemma 4,

$$\begin{aligned}\sum_{k=1}^{n}\left\lfloor\frac{pk}{q}\right\rfloor &= \frac{p}{2q}\left(n\left(n+\frac{p-q+1}{p}\right)-r\left(r+\frac{p-q+1}{p}\right)\right) + \sum_{k=0}^{r}\left\lfloor\frac{pk}{q}\right\rfloor \\ &= \frac{p}{2q}n^2+\frac{p}{2q}\left(\frac{p-q+1}{p}\right)n+\gamma_n,\end{aligned}$$

where $n \equiv r \pmod q$ and $\gamma_n \in \mathbb{R}$ is bounded. By Lemma 5, set

$$\alpha = \frac{p}{2q} \quad \text{and} \quad \beta = \frac{p}{2q}\left(\frac{p-q+1}{p}\right).$$

We have

$$\frac{p(2n+b)^2}{8q} = \frac{p}{8q}\left(2n+\frac{p-q+1}{p}\right)^2$$

for infinitely many $n \in \mathbb{N}$. Thus, there exist $n_1, n_2 \in \mathbb{N}$ with $n_1 \neq n_2$ such that

$$4n_1b+b^2 = 4\left(\frac{p-q+1}{p}\right)n_1+\left(\frac{p-q+1}{p}\right)^2,$$
$$4n_2b+b^2 = 4\left(\frac{p-q+1}{p}\right)n_2+\left(\frac{p-q+1}{p}\right)^2.$$

This implies $b = (p-q+1)/p$. The other cases can be proven analogously. □

Theorem 1 and Theorem 2 verify the uniqueness for the formulas of the sums.

Theorem 3 *Let* $p \in \mathbb{N}$ *and* $q \in \mathbb{Z}\backslash\{0\}$ *with* $\gcd(p,q)=1$*. Then, for every* $a,b,c \in \mathbb{R}$ *with* $c \neq 0$*, there are infinitely many* $n \in \mathbb{N}$ *such that each of the following holds.*
(i) *If* $q \geqslant 1$*, then*

$$\sum_{k=1}^{n} F\left(\frac{pk}{q}\right) \neq \left\lceil\frac{p(an+b)^2}{qc}\right\rceil.$$

(ii) *If* $q \leqslant -1$*, then*

$$\sum_{k=1}^{n} F\left(\frac{pk}{q}\right) \neq \left\lfloor\frac{p(an+b)^2}{qc}\right\rfloor.$$

Proof: We will first prove (i) with the case that F is the floor function. Assume on the contrary that there are $a,b,c \in \mathbb{R}$ with $c \neq 0$ such that the inequality holds for finitely many $n \in \mathbb{N}$. By Theorem 1, there exists a sufficiently large $N \in \mathbb{N}$ such that

$$\sum_{k=1}^{n}\left\lfloor\frac{pk}{q}\right\rfloor = \left\lceil\frac{p}{8q}\left(2n+\frac{p-q+1}{p}\right)^2\right\rceil$$

for $n \geqslant N$. There are two cases to consider.

Case $q \geqslant 2$. We choose $m \in \mathbb{N}$ such that $m \geqslant N$ and $m \equiv q-1 \pmod q$. By Lemma 4, we have

$$\sum_{k=1}^{m}\left\lfloor\frac{pk}{q}\right\rfloor = \frac{m(pm+p-q+1)-(q-1)}{2q}.$$

Note that

$$\left\lceil\frac{p}{8q}\left(2m+\frac{p-q+1}{p}\right)^2\right\rceil = \sum_{k=1}^{m}\left\lfloor\frac{pk}{q}\right\rfloor+\left\lceil\frac{q-1}{2q}+\frac{(p-q+1)^2}{8pq}\right\rceil.$$

Since $q \geqslant 2$,

$$\frac{q-1}{2q}+\frac{(p-q+1)^2}{8pq} \geqslant \frac{q-1}{2q} > 0.$$

Thus,

$$\left\lceil\frac{p}{8q}\left(2m+\frac{p-q+1}{p}\right)^2\right\rceil \geqslant \sum_{k=1}^{m}\left\lfloor\frac{pk}{q}\right\rfloor+1 > \sum_{k=1}^{m}\left\lfloor\frac{pk}{q}\right\rfloor.$$

Hence, we arrive at a contradiction.

Case $q=1$. It is easy to see that

$$\frac{pn(n+1)}{2} = \sum_{k=1}^{n}\lfloor pk\rfloor = \left\lceil\frac{p}{8}(2n+1)^2\right\rceil = \frac{pn(n+1)}{2}+\left\lceil\frac{p}{8}\right\rceil.$$

Note that $0 = \lceil p/8\rceil \geqslant 1$, since $p \in \mathbb{N}$. Here, we also arrive at a contradiction.

Next, we will prove (i) for the case that F is the ceiling function. We may follow a similar logic by using a proof by contradiction. There are two cases to consider.

Case $p \neq q-1$. We choose $m \in \mathbb{N}$ such that $m \geqslant N$ and $m \equiv q-1 \pmod{q}$. By Lemma 4, we have

$$\sum_{k=1}^{m} \left\lceil \frac{pk}{q} \right\rceil = \frac{m(pm+p+q-1)+(q-1)}{2q}.$$

Note that

$$\begin{aligned}\left\lceil \frac{p}{8q}\left(2m+\frac{p+q-1}{p}\right)^2 \right\rceil &= \sum_{k=1}^{m} \left\lceil \frac{pk}{q} \right\rceil + \left\lceil \frac{1-q}{2q} + \frac{(p+q-1)^2}{8pq} \right\rceil \\ &= \sum_{k=1}^{m} \left\lceil \frac{pk}{q} \right\rceil + \left\lceil \frac{(p-q+1)^2}{8pq} \right\rceil.\end{aligned}$$

Thus,

$$\left\lceil \frac{p}{8q}\left(2m+\frac{p+q-1}{p}\right)^2 \right\rceil \geqslant \sum_{k=1}^{m} \left\lceil \frac{pk}{q} \right\rceil + 1 > \sum_{k=1}^{m} \left\lceil \frac{pk}{q} \right\rceil.$$

Hence, we arrive at a contradiction.

Case $p = q-1$. So, $q \geqslant 2$. We choose $m \in \mathbb{N}$ such that $m \geqslant N$ and $m \equiv 0 \pmod{q}$. By Lemma 4, we have

$$\sum_{k=1}^{m} \left\lceil \frac{pk}{q} \right\rceil = \frac{m(pm+p+q-1)}{2q}.$$

Thus,

$$\begin{aligned}\left\lceil \frac{p}{8q}\left(2m+\frac{p+q-1}{p}\right)^2 \right\rceil &= \sum_{k=1}^{m} \left\lceil \frac{pk}{q} \right\rceil + \left\lceil \frac{(p+q-1)^2}{8pq} \right\rceil \\ &= \sum_{k=1}^{m} \left\lceil \frac{pk}{q} \right\rceil + 1 > \sum_{k=1}^{m} \left\lceil \frac{pk}{q} \right\rceil\end{aligned}$$

since $0 < (p+q-1)^2/(8pq) = p/(2(p+1)) < 1$. Here, we also arrive at a contradiction.

To prove (ii), we may proceed with a similar logic as in the proof of (i). For instance, when F is the floor function, we separate the problem into two cases: $p \neq |q|-1$ and $p = |q|-1$. We first suppose for contradiction that the inequality holds for only a finite number of n. For $p \neq |q|-1$, we choose a sufficiently large m such that $m \equiv |q|-1 \pmod{|q|}$ to create a contradiction. In the other case, we choose m such that $m \equiv 0 \pmod{|q|}$. □

Theorem 3 restricts whether the formula should involve the ceiling or floor function depending on F and q. For example, when F is the floor function and $q > 0$, the formula must involve the floor function by Theorem 3(i).

Theorem 4 *Let $p \in \mathbb{N}$ and $q \in \mathbb{Z}\backslash\{0\}$ such that $p \geqslant (\sqrt{2|q|}+\sqrt{|q|+1})^2$ or $p \leqslant (\sqrt{2|q|}-\sqrt{|q|+1})^2$ and* $\gcd(p,q)=1$. *Then, for every $a,b,c \in \mathbb{N}$, there exist infinitely many $n \in \mathbb{N}$ such that each of the following holds.*

(i) *If $q \geqslant 1$, then*

$$\sum_{k=1}^{n} F\left(\frac{pk}{q}\right) \neq \left\lfloor \frac{p(an+b)^2}{qc} \right\rfloor.$$

(ii) *If $q \leqslant -1$, then*

$$\sum_{k=1}^{n} F\left(\frac{pk}{q}\right) \neq \left\lceil \frac{p(an+b)^2}{qc} \right\rceil.$$

Proof: We first prove (i) for the case that F is the floor function. Assume on the contrary that there are $a,b,c \in \mathbb{R}$ with $c \neq 0$ such that the inequality holds for finitely many $n \in \mathbb{N}$. By Theorem 1, there exists $N \in \mathbb{N}$ such that

$$\frac{p(an+b)^2}{qc} = \frac{p}{8q}\left(2n+\frac{p-q+1}{p}\right)^2$$

for $n \geqslant N$. We choose $m \in \mathbb{N}$ such that $m \geqslant N$ and $m \equiv q-1 \pmod{q}$. Then, by Lemma 3 and Lemma 4, we have

$$\sum_{k=1}^{m} \left\lfloor \frac{pk}{q} \right\rfloor = \frac{m(pm+p-q+1)-(q-1)}{2q}.$$

However,

$$\left\lfloor \frac{p}{8q}\left(2m+\frac{p-q+1}{p}\right)^2 \right\rfloor = \sum_{k=1}^{m} \left\lfloor \frac{pk}{q} \right\rfloor + \left\lfloor \frac{(p+q-1)^2}{8pq} \right\rfloor.$$

Since $p \geqslant (\sqrt{2q}+\sqrt{q+1})^2$ or $p \leqslant (\sqrt{2q}-\sqrt{q+1})^2$, we have $(p+q-1)^2/(8pq) \geqslant 1$. Hence, we arrive at a contradiction.

Next, we will prove (i) for the case that F is the ceiling function. We may follow a similar logic by using a proof by contradiction. Then, we choose $m \in \mathbb{N}$ such that $m \geqslant N$ and $m \equiv 0 \pmod{q}$. By Lemma 3 and Lemma 4,

$$\sum_{k=1}^{m} \left\lceil \frac{pk}{q} \right\rceil = \frac{m(pm+p+q-1)}{2q}.$$

However,

$$\left\lfloor \frac{p}{8q}\left(2m+\frac{p+q-1}{p}\right)^2 \right\rfloor = \sum_{k=1}^{m} \left\lceil \frac{pk}{q} \right\rceil + \left\lfloor \frac{(p+q-1)^2}{8pq} \right\rfloor.$$

As $(p+q-1)^2/(8pq) \geqslant 1$, we arrive at a contradiction.

To prove (ii), we may proceed with a similar logic as in the proof of (i). When F is the floor or ceiling function, we choose a large enough m such that $m \equiv 0 \pmod{|q|}$ or $m \equiv |q|-1 \pmod{|q|}$, respectively, to create a contradiction. □

Theorem 4 finds a bound for p depending on q which makes the formulas not hold. This is very helpful when finding all $p \in \mathbb{N}$ for each $q \in \mathbb{Z}\backslash\{0\}$ that make the formulas hold as we only need to consider a finite number of candidates of p. The theorem below provides all pairs (p,q) with $1 \leqslant |q| \leqslant 8$ where the formulas hold for every $n \in \mathbb{N}$ in the four cases.

Theorem 5 *Let $p \in \mathbb{N}$, $q \in \mathbb{Z}\backslash\{0\}$, and $\gcd(p,q)=1$. Then, the formulas*

$$\sum_{k=1}^{n}\left\lfloor \frac{pk}{q}\right\rfloor = \begin{cases} \left\lfloor \frac{p}{8q}\left(2n+\frac{p-q+1}{p}\right)^2\right\rfloor, & q \geqslant 1 \\ \left\lceil \frac{p}{8q}\left(2n+\frac{p-q-1}{p}\right)^2\right\rceil, & q \leqslant -1 \end{cases}$$

$$\sum_{k=1}^{n}\left\lceil \frac{pk}{q}\right\rceil = \begin{cases} \left\lfloor \frac{p}{8q}\left(2n+\frac{p+q-1}{p}\right)^2\right\rfloor, & q \geqslant 1 \\ \left\lceil \frac{p}{8q}\left(2n+\frac{p+q+1}{p}\right)^2\right\rceil, & q \leqslant -1 \end{cases}$$

hold for every $n \in \mathbb{N}$ when $(p, |q|)$ are given in the table below.

$\lvert q\rvert$	*p*												
1	1	2	3	4	5	6	7						
2	1	3	5	7	9	11	13						
3	1	2	4	5	7	8	10	11	13	14	16	17	19
4	1	3	7	9	11	13	15	17	19	21	25		
5	1	2	3	4	7	8	9	12	13	14	16	17	18
	19	21	22	23	26	27	28	31					
6	1	5	11	17	25	31	37						
7	1	2	3	5	6	10	12	13	17	18	19	23	24
	25	26	30	31	32	33	36	37	39	43			
8	3	5	13	19	21	27	29	35	37	43			

Proof: The idea of this proof is straightforward. We will only prove the case for the sum of the floor function where $p=3$ and $q=4$ as an example. That is, we wish to show that

$$\sum_{k=1}^{n}\left\lfloor \frac{3k}{4}\right\rfloor = \left\lfloor \frac{3n^2}{8}\right\rfloor \tag{8}$$

for every $n \in \mathbb{N}$. By Lemma 4, we can re-write the left-hand side of (8) as

$$\begin{aligned}\sum_{k=1}^{n}\left\lfloor \frac{3k}{4}\right\rfloor &= \frac{3}{8}(n^2-r^2)+\sum_{k=0}^{r}\left\lfloor \frac{3k}{4}\right\rfloor \\ &= \begin{cases} \frac{3n^2}{8}, & n \equiv 0 \pmod 4 \\ \frac{3n^2-3}{8}, & n \equiv 1,3 \pmod 4 \\ \frac{3n^2-4}{8}, & n \equiv 2 \pmod 4. \end{cases}\end{aligned}$$

Now, consider the right-hand side of (8). By the division algorithm, there are four cases to consider.

If $n \equiv 0 \pmod 4$, that means $n = 4\ell$ for some $\ell \in \mathbb{N}$. So

$$\left\lfloor \frac{3n^2}{8}\right\rfloor = \left\lfloor \frac{3(4\ell)^2}{8}\right\rfloor = 6\ell^2 = \frac{3n^2}{8}.$$

If $n \equiv 1 \pmod 4$, that means $n = 4\ell+1$ for some $\ell \in \mathbb{N}$. So

$$\left\lfloor \frac{3n^2}{8}\right\rfloor = \left\lfloor \frac{3(4\ell+1)^2}{8}\right\rfloor = 6\ell^2+3\ell = \frac{3n^2-3}{8}.$$

If $n \equiv 2 \pmod 4$, that means $n = 4\ell+2$ for some $\ell \in \mathbb{N}$. So

$$\left\lfloor \frac{3n^2}{8}\right\rfloor = \left\lfloor \frac{3(4\ell+2)^2}{8}\right\rfloor = 6\ell^2+6\ell+1 = \frac{3n^2-4}{8}.$$

If $n \equiv 3 \pmod 4$, that means $n = 4\ell+3$ for some $\ell \in \mathbb{N}$. So

$$\left\lfloor \frac{3n^2}{8}\right\rfloor = \left\lfloor \frac{3(4\ell+3)^2}{8}\right\rfloor = 6\ell^2+9\ell+3 = \frac{3n^2-3}{8}.$$

By comparing the four cases, we can see that (8) holds for every $n \in \mathbb{N}$. □

As a remark, it is not guaranteed that if q is in the given bound, then the formulas hold. For instance, let $p, q \in \mathbb{N}$ with $\gcd(p,q)=1$, it is not always true that there exist $a, b, c \in \mathbb{R}$ with $c \neq 0$ such that

$$\sum_{k=1}^{n}\left\lfloor \frac{pk}{q}\right\rfloor \neq \left\lfloor \frac{p(an+b)^2}{qc}\right\rfloor$$

for finitely many $n \in \mathbb{N}$ when $p \in ((\sqrt{2q}-\sqrt{q+1})^2, (\sqrt{2q}+\sqrt{q+1})^2)$. For example, if $(p,q)=(5,4)$, it is easy to see that $5 \in ((2\sqrt{2}-\sqrt{5})^2, (2\sqrt{2}+\sqrt{5})^2)$, but the formula does not hold for infinitely many $n \in \mathbb{N}$.

The interval $((\sqrt{2|q|}-\sqrt{|q|+1})^2, (\sqrt{2|q|}+\sqrt{|q|+1})^2)$ gets larger as $|q|$ increases. Hence, it is not practical to find all p that make the formulas hold for a large $|q|$. Instead, we will give some criteria for the formulas not to hold. In fact, if $p \equiv \pm 1 \pmod{|q|}$, $|q| \geqslant 8$, and $\gcd(p,q)=1$, then the formulas are not valid. Before we show our result, we need the following lemma.

Lemma 6 *Let $p \in \mathbb{N}$, $q \in \mathbb{Z}\backslash\{-1,0,1\}$ such that $\gcd(p,q)=1$. Suppose that $r \in \{1,\dots,|q|-1\}$.*

(i) *If $q \geqslant 2$ and $p \equiv 1 \pmod q$, then*

$$\sum_{k=1}^{r}\left\lfloor \frac{pk}{q}\right\rfloor = \frac{(p-1)r(r+1)}{2q} \quad \textit{and}$$
$$\sum_{k=1}^{r}\left\lceil \frac{pk}{q}\right\rceil = \frac{(p-1)r(r+1)}{2q}+r.$$

(ii) *If $q \geqslant 2$ and $p \equiv -1 \pmod q$, then*

$$\sum_{k=1}^{r}\left\lfloor \frac{pk}{q}\right\rfloor = \frac{(p+1)r(r+1)}{2q}-r \quad \textit{and}$$
$$\sum_{k=1}^{r}\left\lceil \frac{pk}{q}\right\rceil = \frac{(p+1)r(r+1)}{2q}.$$

(iii) *If $q \leqslant -2$ and $p \equiv 1 \pmod{|q|}$, then*

$$\sum_{k=1}^{r}\left\lfloor \frac{pk}{q}\right\rfloor = \frac{(p-1)r(r+1)}{2q}-r \quad \textit{and}$$
$$\sum_{k=1}^{r}\left\lceil \frac{pk}{q}\right\rceil = \frac{(p-1)r(r+1)}{2q}.$$

(iv) *If $q \leqslant -2$ and $p \equiv -1 \pmod{|q|}$, then*

$$\sum_{k=1}^{r} \left\lfloor \frac{pk}{q} \right\rfloor = \frac{(p+1)r(r+1)}{2q} \quad \text{and}$$
$$\sum_{k=1}^{r} \left\lceil \frac{pk}{q} \right\rceil = \frac{(p+1)r(r+1)}{2q} + r.$$

Proof: For (i), by the division algorithm, there exists $s \in \mathbb{N}$ such that $p = sq + 1$. By using Lemma 2 and plugging in $s = (p-1)/q$, we have

$$\sum_{k=1}^{r} \left\lfloor \frac{pk}{q} \right\rfloor = \sum_{k=1}^{r} \left\lfloor sk + \frac{k}{q} \right\rfloor = s\sum_{k=1}^{r} k + \sum_{k=1}^{r} \left\lfloor \frac{k}{q} \right\rfloor = \frac{(p-1)r(r+1)}{2q} + \sum_{k=1}^{r} \left\lfloor \frac{k}{q} \right\rfloor,$$

and

$$\sum_{k=1}^{r} \left\lceil \frac{pk}{q} \right\rceil = \sum_{k=1}^{r} \left\lceil sk + \frac{k}{q} \right\rceil = s\sum_{k=1}^{r} k + \sum_{k=1}^{r} \left\lceil \frac{k}{q} \right\rceil = \frac{(p-1)r(r+1)}{2q} + \sum_{k=1}^{r} \left\lceil \frac{k}{q} \right\rceil.$$

Now, $\lfloor k/q \rfloor = 0$ and $\lceil k/q \rceil = 1$ for all $1 \leqslant k \leqslant r$ since $1 \leqslant k \leqslant r \leqslant q-1$. So, we obtain (i). For (ii), by the division algorithm, there exists $t \in \mathbb{N}$ such that $p = tq - 1$. By using Lemma 2 and plugging in $t = (p+1)/q$, we have

$$\sum_{k=1}^{r} \left\lfloor \frac{pk}{q} \right\rfloor = t\sum_{k=1}^{r} k + \sum_{k=1}^{r} \left\lfloor -\frac{k}{q} \right\rfloor = \frac{(p+1)r(r+1)}{2q} + \sum_{k=1}^{r} \left\lfloor -\frac{k}{q} \right\rfloor,$$

and

$$\sum_{k=1}^{r} \left\lceil \frac{pk}{q} \right\rceil = t\sum_{k=1}^{r} k + \sum_{k=1}^{r} \left\lceil -\frac{k}{q} \right\rceil = \frac{(p+1)r(r+1)}{2q} + \sum_{k=1}^{r} \left\lceil -\frac{k}{q} \right\rceil.$$

Now, $\lfloor -k/q \rfloor = -1$ and $\lceil -k/q \rceil = 0$ for all $1 \leqslant k \leqslant r$ since $1 \leqslant k \leqslant r \leqslant q-1$. So, we obtain (ii). To prove (iii) and (iv), we may use (i) and (ii) together with Lemma 2. □

Now we can tackle our problem.

Theorem 6 *Let $p \in \mathbb{N}$ and $q \in \mathbb{Z}\setminus\{0\}$ with $\gcd(p,q)=1$. Suppose that $|q| \geqslant 8$ and $p \equiv 1 \pmod{|q|}$. Then, for every $a, b, c \in \mathbb{N}$, there exist infinitely many $n \in \mathbb{N}$ such that each of the following holds.*

(i) *If $q \geqslant 1$, then*

$$\sum_{k=1}^{n} F\left(\frac{pk}{q}\right) \neq \left\lfloor \frac{p(an+b)^2}{qc} \right\rfloor.$$

(ii) *If $q \leqslant -1$, then*

$$\sum_{k=1}^{n} F\left(\frac{pk}{q}\right) \neq \left\lceil \frac{p(an+b)^2}{qc} \right\rceil.$$

Proof: If $p \notin ((\sqrt{2|q|} - \sqrt{|q|+1})^2, (\sqrt{2|q|} + \sqrt{|q|+1})^2)$, then the statement holds by Theorem 4. So, we will only consider p that lie in this interval. We first prove (i) for the case that F is the floor function. Assume on the contrary that there are $a, b, c \in \mathbb{R}$ with $c \neq 0$ such that the inequality holds for a finite number of $n \in \mathbb{N}$. By Theorem 1, there exists $N \in \mathbb{N}$ such that

$$\frac{p(an+b)^2}{qc} = \frac{p}{8q}\left(2n + \frac{p-q+1}{p}\right)^2 = \frac{4n(pn+p-q+1)}{8q} + \frac{(p-q+1)^2}{8pq}$$

for $n \geqslant N$. There are two cases to consider.

Case q is even. Let $m \geqslant N$ such that $m \equiv (q-2)/2 \pmod{q}$. By Lemma 4 and Lemma 6, we have

$$\begin{aligned}
&\sum_{k=1}^{m} \left\lfloor \frac{pk}{q} \right\rfloor \\
&= \frac{p}{2q}\left(m\left(m + \frac{p-q+1}{p}\right) - \frac{q-2}{2}\left(\frac{q-2}{2} + \frac{p-q+1}{p}\right)\right) + \sum_{k=0}^{(q-2)/2} \left\lfloor \frac{pk}{q} \right\rfloor \\
&= \frac{4m(pm+p-q+1) - (q-2)(pq-2q+2)}{8q} + \frac{(p-1)(q-2)}{8} \\
&= \frac{4m(pm+p-q+1) + (q-2)^2}{8q}.
\end{aligned}$$

Thus,

$$\left\lfloor \frac{p}{8q}\left(2m + \frac{p-q+1}{p}\right)^2 \right\rfloor = \sum_{k=1}^{m} \left\lfloor \frac{pk}{q} \right\rfloor + \left\lfloor \frac{(p-q+1)^2 - p(q-2)^2}{8pq} \right\rfloor.$$

Since $q \geqslant 8$, we have

$$p \in ((\sqrt{2q} - \sqrt{q-1})^2, (\sqrt{2q} + \sqrt{q+1})^2) \subseteq (1, (q-1)^2).$$

So,

$$\frac{-p(q-2)^2 + (p-q+1)^2}{8pq} < 0.$$

This implies

$$\left\lfloor \frac{p}{8q}\left(2m + \frac{p-q+1}{p}\right)^2 \right\rfloor \leqslant \sum_{k=1}^{m} \left\lfloor \frac{pk}{q} \right\rfloor - 1 < \sum_{k=1}^{m} \left\lfloor \frac{pk}{q} \right\rfloor.$$

Hence, we arrive at a contradiction.

Case q is odd. Let $m \geqslant N$ such that $m \equiv (q-1)/2 \pmod{q}$. By Lemma 4 and Lemma 6, we have

$$\begin{aligned}\sum_{k=1}^{m}\left\lfloor\frac{pk}{q}\right\rfloor &= \frac{p}{2q}\left(m\left(m+\frac{p-q+1}{p}\right)\right.\\ &\quad\left.-\frac{q-1}{2}\left(\frac{q-1}{2}+\frac{p-q+1}{p}\right)\right)+\sum_{k=1}^{(q-1)/2}\left\lfloor\frac{pk}{q}\right\rfloor\\ &= \frac{4m(pm+p-q+1)-(q-1)(pq+p-2q+2)}{8q}\\ &\quad+\frac{(p-1)(q-1)(q+1)}{8q}\\ &= \frac{4m(pm+p-q+1)+(q-1)(q-3)}{8q}.\end{aligned}$$

Thus,

$$\begin{aligned}&\left\lfloor\frac{p}{8q}\left(2m+\frac{p-q+1}{p}\right)^2\right\rfloor\\ &\quad=\sum_{k=1}^{m}\left\lfloor\frac{pk}{q}\right\rfloor+\left\lfloor\frac{(p-q+1)^2-p(q-1)(q-3)}{8pq}\right\rfloor.\end{aligned}$$

Since $q \geqslant 9$, we have

$$\begin{aligned}&p\in((\sqrt{2q}-\sqrt{q+1})^2,(\sqrt{2q}+\sqrt{q+1})^2)\\ &\subseteq\left(\frac{(q-1)(\sqrt{q+1}-\sqrt{q-3})^2}{4},\frac{(q-1)(\sqrt{q+1}+\sqrt{q-3})^2}{4}\right).\end{aligned}$$

So,

$$\frac{(p-q+1)^2-p(q-1)(q-3)}{8pq}<0.$$

This implies

$$\left\lfloor\frac{p}{8q}\left(2m+\frac{p-q+1}{p}\right)^2\right\rfloor\leqslant\sum_{k=1}^{m}\left\lfloor\frac{pk}{q}\right\rfloor-1<\sum_{k=1}^{m}\left\lfloor\frac{pk}{q}\right\rfloor.$$

Here, we also arrive at a contradiction.

Next, we will prove (i) for the case that F is the ceiling function. We may follow a similar logic by using a proof by contradiction. There are two cases to consider.

Case q is even. We consider $m \geqslant N$ such that $m \equiv q/2 \pmod{q}$. By Lemma 4 and Lemma 6, we have

$$\begin{aligned}&\sum_{k=1}^{m}\left\lceil\frac{pk}{q}\right\rceil\\ &=\frac{p}{2q}\left(m\left(m+\frac{p+q-1}{p}\right)-\frac{q}{2}\left(\frac{q}{2}+\frac{p+q-1}{p}\right)\right)+\sum_{k=0}^{q/2}\left\lceil\frac{pk}{q}\right\rceil\\ &=\frac{4m(pm+p+q-1)-q(pq+2p+2q-2)}{8q}+\frac{(p-1)q(q+2)+4q^2}{8q}\\ &=\frac{4m(pm+p+q-1)+q^2}{8q}.\end{aligned}$$

Thus,

$$\left\lfloor\frac{p}{8q}\left(2m+\frac{p+q-1}{p}\right)^2\right\rfloor=\sum_{k=1}^{m}\left\lceil\frac{pk}{q}\right\rceil+\left\lfloor\frac{(p+q-1)^2-pq^2}{8pq}\right\rfloor.$$

Since $q \geqslant 8$, we have

$$p\in((\sqrt{2q}-\sqrt{q+1})^2,(\sqrt{2q}+\sqrt{q+1})^2)\subseteq(1,(q-1)^2).$$

So,

$$\frac{(p+q-1)^2-pq^2}{8pq}<0.$$

This implies

$$\left\lfloor\frac{p}{8q}\left(2m+\frac{p+q-1}{p}\right)^2\right\rfloor\leqslant\sum_{k=1}^{m}\left\lceil\frac{pk}{q}\right\rceil-1<\sum_{k=1}^{m}\left\lceil\frac{pk}{q}\right\rceil.$$

Hence, we arrive at a contradiction.

Case q is odd. We consider $m \geqslant N$ such that $m \equiv (q-1)/2 \pmod{q}$. By Lemma 4 and Lemma 6, we have

$$\begin{aligned}\sum_{k=1}^{m}\left\lceil\frac{pk}{q}\right\rceil &= \frac{p}{2q}\left(m\left(m+\frac{p+q-1}{p}\right)\right.\\ &\quad\left.-\left(\frac{q-1}{2}\right)\left(\frac{q-1}{2}+\frac{p+q-1}{p}\right)\right)+\sum_{k=0}^{(q-1)/2}\left\lceil\frac{pk}{q}\right\rceil\\ &=\frac{4m(pm+p+q-1)-(q-1)(pq+p+2q-2)}{8q}\\ &\quad+\frac{(p-1)(q-1)(q+1)}{8q}+\frac{q-1}{2}\\ &=\frac{4m(pm+p+q-1)+q^2-1}{8q}.\end{aligned}$$

Thus,

$$\begin{aligned}&\left\lfloor\frac{p}{8q}\left(2m+\frac{p+q-1}{p}\right)^2\right\rfloor\\ &\quad=\sum_{k=1}^{m}\left\lceil\frac{pk}{q}\right\rceil+\left\lfloor\frac{(p+q-1)^2-p(q^2-1)}{8pq}\right\rfloor.\end{aligned}$$

Since $q \geqslant 9$, we have

$$\begin{aligned}&p\in((\sqrt{2q}-\sqrt{q+1})^2,(\sqrt{2q}+\sqrt{q+1})^2)\\ &\subseteq\left(\frac{(q-1)\left(\sqrt{q+1}-\sqrt{q-3}\right)^2}{4},\frac{(q-1)\left(\sqrt{q+1}+\sqrt{q-3}\right)^2}{4}\right).\end{aligned}$$

So,

$$\frac{(p+q-1)^2-p(q^2-1)}{8pq}<0.$$

This implies

$$\left\lfloor\frac{p}{8q}\left(2m+\frac{p+q-1}{p}\right)^2\right\rfloor\leqslant\sum_{k=1}^{m}\left\lceil\frac{pk}{q}\right\rceil-1<\sum_{k=1}^{m}\left\lceil\frac{pk}{q}\right\rceil.$$

Here, we also arrive at a contradiction.

To prove (ii), we may proceed with a similar logic as in the proof of (i). When F is the floor function, we choose a sufficiently large m such that $m \equiv |q|/2 \pmod{|q|}$ or $m \equiv (|q|-1)/2 \pmod{|q|}$ when q is even or odd, respectively, to create a contradiction. When F is the ceiling function, we choose a large enough m such that $m \equiv (|q|-2)/2 \pmod{|q|}$ or $m \equiv (|q|-1)/2 \pmod{|q|}$ when q is even or odd, respectively, to create a contradiction. □

Theorem 7 *Let $p \in \mathbb{N}$ and $q \in \mathbb{Z}\backslash\{0\}$ with $\gcd(p,q)=1$. Suppose that $|q| \geqslant 8$ and $p \equiv -1 \pmod{|q|}$. Then, for every $a,b,c \in \mathbb{N}$, there exist infinitely many $n \in \mathbb{N}$ such that each of the following holds.*
(i) *If $q \geqslant 1$, then*

$$\sum_{k=1}^{n} F\left(\frac{pk}{q}\right) \neq \left\lfloor \frac{p(an+b)^2}{qc} \right\rfloor.$$

(ii) *If $q \leqslant -1$, then*

$$\sum_{k=1}^{n} F\left(\frac{pk}{q}\right) \neq \left\lceil \frac{p(an+b)^2}{qc} \right\rceil.$$

Proof: If $p \notin ((\sqrt{2|q|}-\sqrt{|q|+1})^2, (\sqrt{2|q|}+\sqrt{|q|+1})^2)$, then the statement holds by Theorem 4. So, we will only consider p that lie in this interval. We will first prove (i) for the case that F is the floor function. Assume on the contrary that there are $a,b,c \in \mathbb{R}$ with $c \neq 0$ such that the inequality holds for a finite number of $n \in \mathbb{N}$. By Theorem 1, there exists $N \in \mathbb{N}$ such that

$$\begin{aligned}\frac{p(an+b)^2}{qc} &= \frac{p}{8q}\left(2n+\frac{p-q+1}{p}\right)^2\\ &= \frac{4n(pn+p-q+1)}{8q}+\frac{(p-q+1)^2}{8pq}\end{aligned}$$

for $n \geqslant N$. There are two cases to consider.

Case q is even. Let $m \geqslant N$ such that $m \equiv q/2 \pmod{q}$. Then, by Lemma 4 and Lemma 6, we have

$$\begin{aligned}\sum_{k=1}^{m}\left\lfloor\frac{pk}{q}\right\rfloor &= \frac{p}{2q}\Bigg(m\left(m+\frac{p-q+1}{p}\right)\\ &\quad -\frac{q}{2}\left(\frac{q}{2}+\frac{p-q+1}{p}\right)\Bigg)+\sum_{k=1}^{q/2}\left\lfloor\frac{pk}{q}\right\rfloor\\ &= \frac{4m(pm+p-q+1)-q(pq+2p-2q+2)}{8q}\\ &\quad +\frac{(p+1)(q+2)-4q}{8}\\ &= \frac{4m(pm+p-q+1)-q^2}{8q}.\end{aligned}$$

Therefore,

$$\left\lfloor\frac{p}{8q}\left(2m+\frac{p-q+1}{p}\right)^2\right\rfloor = \sum_{k=1}^{m}\left\lfloor\frac{pk}{q}\right\rfloor+\left\lfloor\frac{(p-q+1)^2}{8pq}+\frac{q}{8}\right\rfloor.$$

Since $q \geqslant 8$, we have

$$\frac{(p-q+1)^2}{8pq}+\frac{q}{8} \geqslant \frac{q}{8} \geqslant 1.$$

This implies

$$\left\lfloor\frac{p}{8q}\left(2m+\frac{p-q+1}{p}\right)^2\right\rfloor \geqslant \sum_{k=1}^{m}\left\lfloor\frac{pk}{q}\right\rfloor+1 > \sum_{k=1}^{m}\left\lfloor\frac{pk}{q}\right\rfloor.$$

Hence, we arrive at a contradiction.

Case q is odd. Let $m \geqslant N$ such that $m \equiv (q-1)/2 \pmod{q}$. Then, by Lemma 4 and Lemma 6, we have

$$\begin{aligned}\sum_{k=1}^{m}\left\lfloor\frac{pk}{q}\right\rfloor &= \frac{p}{2q}\Bigg(m\left(m+\frac{p-q+1}{p}\right)\\ &\quad -\frac{q-1}{2}\left(\frac{q-1}{2}+\frac{p-q+1}{p}\right)\Bigg)+\sum_{k=1}^{(q-1)/2}\left\lfloor\frac{pk}{q}\right\rfloor\\ &= \frac{4m(pm+p-q+1)-(q-1)(pq+p-2q+2)}{8q}\\ &\quad +\frac{(p+1)(q-1)(q+1)}{8q}-\frac{q-1}{2}\\ &= \frac{4m(pm+p-q+1)-(q^2-1)}{8q}.\end{aligned}$$

Hence,

$$\begin{aligned}&\left\lfloor\frac{p}{8q}\left(2m+\frac{p-q+1}{p}\right)^2\right\rfloor\\ &\qquad = \sum_{k=1}^{m}\left\lfloor\frac{pk}{q}\right\rfloor+\left\lfloor\frac{(p-q+1)^2}{8pq}+\frac{q^2-1}{8q}\right\rfloor.\end{aligned}$$

Since $q \geqslant 9$, we have

$$\frac{(p-q+1)^2}{8pq}+\frac{q^2-1}{8q} \geqslant \frac{q^2-1}{8q} > 1.$$

This implies

$$\left\lfloor\frac{p}{8q}\left(2m+\frac{p-q+1}{p}\right)^2\right\rfloor \geqslant \sum_{k=1}^{m}\left\lfloor\frac{pk}{q}\right\rfloor+1 > \sum_{k=1}^{m}\left\lfloor\frac{pk}{q}\right\rfloor.$$

Here, we also arrive at a contradiction.

Next, we will prove (i) for the case that F is the ceiling function. We may follow a similar logic by using a proof by contradiction. There are two cases to consider.

Case q is even. We consider $m \geqslant N$ such that $m \equiv$

$(q-2)/2 \pmod q$. By Lemma 4 and Lemma 6, we have

$$\sum_{k=1}^{m}\left\lceil \frac{pk}{q}\right\rceil = \frac{p}{2q}\left(m\left(m+\frac{p+q-1}{p}\right)\right.$$
$$\left.-\frac{q-2}{2}\left(\frac{q-2}{2}+\frac{p+q-1}{p}\right)\right)+\sum_{k=0}^{(q-2)/2}\left\lceil \frac{pk}{q}\right\rceil$$
$$=\frac{4m(pm+p+q-1)-(q-2)(pq+2q-2)}{8q}$$
$$+\frac{(p+1)(q-2)}{8}$$
$$=\frac{4m(pm+p+q-1)-(q-2)^2}{8q}.$$

Thus,

$$\left\lfloor \frac{p}{8q}\left(2m+\frac{p+q-1}{p}\right)^2\right\rfloor = \sum_{k=1}^{m}\left\lceil \frac{pk}{q}\right\rceil + \left\lfloor \frac{(p+q-1)^2+p(q-2)^2}{8pq}\right\rfloor.$$

By expansion, we can see that

$$\frac{(p+q-1)^2+p(q-2)^2}{8pq}=\frac{p}{8q}+\frac{q}{8p}+\frac{1}{8pq}+\frac{1}{4q}-\frac{1}{4p}+\frac{q}{8}-\frac{1}{4}.$$

Note that $p \geqslant q-1$ since $p \equiv -1 \pmod q$ and $p \in \mathbb{N}$. If $q=8$, then $p \geqslant 7$ and so

$$\frac{(p+q-1)^2+p(q-2)^2}{8pq}=\frac{p}{64}+\frac{49}{64p}+\frac{25}{32}\geqslant 1.$$

If $q \geqslant 10$, then

$$\frac{(p+q-1)^2+p(q-2)^2}{8pq}\geqslant \frac{p}{8q}+\frac{1}{p}+\frac{1}{8pq}+\frac{1}{4q}+1\geqslant 1.$$

This implies

$$\left\lfloor \frac{p}{8q}\left(2m+\frac{p+q-1}{p}\right)^2\right\rfloor \geqslant \sum_{k=1}^{m}\left\lceil \frac{pk}{q}\right\rceil+1 > \sum_{k=1}^{m}\left\lceil \frac{pk}{q}\right\rceil.$$

Hence, we arrive at a contradiction.

Case q is odd. We consider $m \geqslant N$ such that $m \equiv (q-1)/2 \pmod q$. By Lemma 4 and Lemma 6, we have

$$\sum_{k=1}^{m}\left\lceil \frac{pk}{q}\right\rceil = \frac{p}{2q}\left(m\left(m+\frac{p+q-1}{p}\right)\right.$$
$$\left.-\frac{q-1}{2}\left(\frac{q-1}{2}+\frac{p+q-1}{p}\right)\right)+\sum_{k=0}^{(q-1)/2}\left\lceil \frac{pk}{q}\right\rceil$$
$$=\frac{4m(pm+p+q-1)-(q-1)(pq+p+2q-2)}{8q}$$
$$+\frac{(p+1)(q-1)(q+1)}{8q}$$
$$=\frac{4m(pm+p+q-1)-(q-1)(q-3)}{8q}.$$

Thus,

$$\left\lfloor \frac{p}{8q}\left(2m+\frac{p+q-1}{p}\right)^2\right\rfloor = \sum_{k=1}^{m}\left\lceil \frac{pk}{q}\right\rceil + \left\lfloor \frac{(p+q-1)^2+p(q-1)(q-3)}{8pq}\right\rfloor.$$

By expansion, we can see that

$$\frac{(p+q-1)^2+p(q-1)(q-3)}{8pq}=\frac{p}{8q}+\frac{q}{8p}+\frac{1}{8pq}+\frac{1}{8q}-\frac{1}{4p}+\frac{q}{8}-\frac{1}{4}.$$

Note that $p \geqslant q-1$ since $p \equiv -1 \pmod q$ and $q \in \mathbb{N}$. If $q=9$, then $p \geqslant 8$ and so

$$\frac{(p+q-1)^2+p(q-1)(q-3)}{8pq}=\frac{p}{72}+\frac{8}{9p}+\frac{8}{9}\geqslant 1.$$

If $q \geqslant 11$, then

$$\frac{(p+q-1)^2+p(q-1)(q-3)}{8pq}\geqslant \frac{p}{8q}+\frac{9}{8p}+\frac{1}{8pq}+\frac{1}{8q}+\frac{9}{8}\geqslant 1.$$

This implies

$$\left\lfloor \frac{p}{8q}\left(2m+\frac{p+q-1}{p}\right)^2\right\rfloor \geqslant \sum_{k=1}^{m}\left\lceil \frac{pk}{q}\right\rceil+1 > \sum_{k=1}^{m}\left\lceil \frac{pk}{q}\right\rceil.$$

Here, we also arrive at a contradiction.

To prove (ii), we may proceed with a similar logic as in the proof of (i). When F is the floor function, we choose a sufficiently large m such that $m \equiv (|q|-2)/2 \pmod{|q|}$ or $m \equiv (|q|-1)/2 \pmod{|q|}$ when q is even or odd, respectively, to create a contradiction. When F is the ceiling function, we choose a large enough m such that $m \equiv |q|/2 \pmod{|q|}$ or $m \equiv (|q|-1)/2 \pmod{|q|}$ when q is even or odd, respectively, to create a contradiction. □

DISCUSSION AND CONCLUSION

In this work, we proposed how the simple formulas for the sum of the floor or the ceiling function must look like in Theorem 1, Theorem 2, and Theorem 3. Then we discovered a bound for p depending on q that makes the formula not hold for infinitely many $n \in \mathbb{N}$ as presented in Theorem 4. We listed the pairs (p,q) with $1 \leqslant |q| \leqslant 8$ that make the formulas hold for all $n \in \mathbb{N}$ as shown in Theorem 5. Lastly, we showed that the formula does not hold when $p \equiv \pm 1 \pmod{|q|}$ and $|q| \geqslant 8$ in Theorem 6 and Theorem 7.

For suggestions in future research, based on the last two theorems, we predict that for every $p \in \mathbb{N}$ and $q \in \mathbb{Z}\backslash\{0\}$, there exists $\rho_r \in \mathbb{Z}$, where $r \in \{0,1,\dots,|q|-1\}$, such that if $p \equiv r \pmod{|q|}$ and $|q| \geqslant \rho_r$, then the formulas do not exist for infinitely many $n \in \mathbb{N}$. It might also be possible to generalize the results for a sum with irrational denominators (for example, by considering continued fractions).

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