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The perturbed Fermat type differential equations

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ABSTRACT: In this paper, we shall extend some of the recent results regarding Fermat type differential equations, which can include several known results for related results obtained earlier as special cases. Finally, we pose some questions for further study.

KEYWORDS: entire solutions, Fermat type differential equations, Nevanlinna theory

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INTRODUCTION AND MAIN RESULTS

Let $f:\mathbb{C}\to\mathbb{C}\cup\{\infty\}$ be a meromorphic function in the complex plane. We use $\rho(f)$ to denote the order of f, which is defined as

$$\rho(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}.$$

Any quantity satisfying $S(r,f)=o(T(r,f))$, as $r\to\infty$ outside a possible exceptional set of finite logarithmic measure, is denoted by $S(r,f)$. Moreover, Nevanlinna theory is an important tool in this paper, its usual notations and basic results come mainly from [1–3].

The classical Fermat-type functional equation is

$$f^n+g^n=1, \tag{1}$$

where n is a positive integer. For $n\geqslant 2$, the entire solutions or meromorphic solutions of (1) were completely analyzed by Baker [4], Gross [5–7] and Montel [8]. For the convenience of the reader, we summarize these proved results as follows.

Theorem 1 *The solutions f and g of the functional Eq. (1) are characterized as follows:*
(1) *if $n=2$, the entire solutions*

$$f=\sin h \quad \textit{and} \quad g=\cos h,$$

where h is entire; the meromorphic solutions

$$f=\frac{1-\beta^2}{1+\beta^2} \quad \textit{and} \quad g=\frac{2\beta}{1+\beta^2},$$

where β is a nonconstant meromorphic function;
(2) *if $n>2$, there are no nonconstant entire solutions;*
(3) *if $n=3$, the meromorphic solutions*

$$f=\frac{1+\frac{\wp'(h(z))}{\sqrt{3}}}{2\wp(h(z))},\qquad g=\frac{1-\frac{\wp'(h(z))}{\sqrt{3}}}{2\wp(h(z))}\eta,$$

where h is a nonconstant entire function, $\eta^3=1$ and $\wp$ remarked as the Weierstrass $\wp$-function that satisfies $(\wp')^2=4\wp^3-1$ under appropriate periods;
(4) *if $n>3$, there are no nonconstant meromorphic solutions.*

In 2004, Yang and Li [9] showed the differential equation

$$(f(z))^2+(f'(z))^2=1 \tag{2}$$

has transcendental entire solutions only with the form $f(z)=\frac{1}{2}(P\,\mathrm{e}^{\alpha z}+\frac{1}{P}\,\mathrm{e}^{-\alpha z})$, where P, α are nonzero constants.

In 2019, Han and Lü [10] proved the next result.

Theorem 2 *The meromorphic solutions f of the following differential equation*

$$(f(z))^n+(f'(z))^n=\mathrm{e}^{\alpha z+\beta} \tag{3}$$

must be entire functions and the following assertions hold.
(i) *For $n=2$, either $\alpha=0$, and*

$$f(z)=\mathrm{e}^{\beta/2}\sin(z+b) \quad \textit{or} \quad f(z)=d\,\mathrm{e}^{(\alpha z+\beta)/2}.$$

(ii) *For $n\geqslant 3$, then $f(z)=d\,\mathrm{e}^{(\alpha z+\beta)/n}$, where α, β, b and d are constants.*

Some related results can be referred to [11, 12] and references therein. It is natural to propose the problem: what about the entire solutions of Eq. (3) when the right hand side of Eq. (3) $\mathrm{e}^{\alpha z+\beta}$ is replaced by $\mathrm{e}^{ng(z)}$, the left-hand side of Eq. (3) f' is replaced by $f^{(k)}$, where g is a polynomial and $k\geqslant 1$ is an integer. In this study, we try to solve the above problem and obtain the following theorem.

Theorem 3 *Assume that $k\,(\geqslant 1)$, $n\,(\geqslant 2)$ are integers, g is a nonconstant polynomial. The meromorphic solution f of the differential equation*

$$(f(z))^n+(f^{(k)}(z))^n=\mathrm{e}^{ng(z)} \tag{4}$$

must be an entire function and $g(z) = az + b$, where $a\,(\neq 0)$, b are constants. Moreover, f can be characterized as follows:

(1) *if $n = 2$, the entire solutions $f(z) = e^{az+B}$ or $f(z) = e^{az+b}\sin(cz+d)$;*
(2) *if $n \geqslant 3$, then $f(z) = e^{az+B}$, where c, d and B are constants.*

Obviously, Theorem 3 is an extension of Theorem 2. Next, we give some examples to show that the conclusions of Theorem 3 indeed occur.

Example 1 We consider $f(z) = e^{\frac{1}{\sqrt{2}}z}\sin(\frac{1}{\sqrt{2}}z)$, which satisfies the following equation

$$f^2 + (f'')^2 = e^{\sqrt{2}z}.$$

Example 2 The equation

$$f^2 + (f''')^2 = e^{\sqrt{3}z+1}$$

has the entire solution

$$f(z) = e^{\frac{\sqrt{3}}{2}z+\frac{1}{2}}\sin(\frac{1}{2}z+1).$$

Example 3 Let $a = \frac{\sqrt{2+\sqrt{2}}}{2}$, $b = \frac{1}{\sqrt{2}\cdot\sqrt{2+\sqrt{2}}}$. Then the equation

$$f^2 + (f^{(4)})^2 = e^{2az} \tag{5}$$

has the entire solution $f(z) = e^{az}\sin(bz)$.

We will give the basic computation for the readers. In fact, it follows by (5) that

$$f'(z) = [a\sin(bz) + b\cos(bz)]\,e^{az},$$
$$f''(z) = [(a^2 - b^2)\sin(bz) + 2ab\cos(bz)]\,e^{az},$$
$$f'''(z) = [(a^3 - 3ab^2)\sin(bz) + (3a^2b - b^3)\cos(bz)]\,e^{az},$$
$$f^{(4)}(z) = [(a^4 + b^4 - 6a^2b^2)\sin(bz) + (4a^3b - 4ab^3)\cos(bz)]\,e^{az}.$$

According to the values of a and b, after careful calculation, it is immediately obtained

$$a^4 + b^4 - 6a^2b^2 = 0, \quad 4a^3b - 4ab^3 = 1,$$

which gives $f^{(4)}(z) = e^{az}\cos(bz)$. Thus, we find $f(z) = e^{az}\sin(bz)$ is a solution of (5).

In 2016, Liu and Yang [13] generalized (2), and obtained the following result.

Theorem 4 *If $\omega \in \mathbb{C}$ and $\omega^2 \neq 1, 0$, then the equation*

$$(f(z))^2 + 2\omega f(z)f'(z) + (f'(z))^2 = 1 \tag{6}$$

has no transcendental meromorphic solutions.

The motivation of this paper arise from the study of the above result, we will continue to discuss the related questions and prove the following result.

Theorem 5 *Assume that $k\,(\geqslant 1)$ is an integer, g is a nonconstant polynomial, $\omega \in \mathbb{C}$ and $\omega^2 \neq 1, 0$. The meromorphic solution f of the differential equation*

$$(f(z))^2 + 2\omega f(z)f^{(k)}(z) + (f^{(k)}(z))^2 = e^{2g(z)} \tag{7}$$

must be an entire function and $g(z) = az + b$ and f can be characterized as follows:

$$f(z) = e^{az+B} \quad \text{or} \quad f(z) = e^{az+B}\sin(cz+d),$$

where $a\,(\neq 0)$, b, c, d and B are constants.

Now, we provide some examples to show that the conclusions of Theorem 5 indeed occur.

Example 4 Let $\omega = \sqrt{2}$. Then $f(z) = \frac{1}{i\sqrt{2}}e^{(i-\sqrt{2})z+1}$ solves the equation

$$f^2 + 2\sqrt{2}ff' + (f')^2 = e^{2[(i-\sqrt{2})z+1]}.$$

Example 5 For any positive integer k, we see that $f(z) = e^{az+b}$ is a solution of the equation

$$f^2 + 2\omega ff^{(k)} + (f^{(k)})^2 = e^{2(az+b)},$$

where a, b are constants with $a^k + 2\omega = 0$.

Example 6 Taking $\omega = 1/2$. Then $f(z) = \frac{2}{\sqrt{3}}e^{-\frac{1}{2}z}\sin(\frac{\sqrt{3}}{2}z)$ is a solution of the equation

$$f^2 + ff' + (f')^2 = e^{-z}.$$

Example 7 Taking $\omega = 3$. Then $f(z) = \frac{1}{2\sqrt{2}i}e^{\sqrt{2}iz}\sin z$ is a solution of the equation

$$f^2 + 6ff'' + (f'')^2 = e^{2\sqrt{2}iz},$$

and $f(z) = \frac{1}{2\sqrt{2}i}e^{iz}\sin(\sqrt{2}z)$ is a solution of the equation

$$f^2 + 6ff'' + (f'')^2 = e^{2iz}.$$

By examining the proof of Theorem 5 in the following section carefully, we have

Theorem 6 *Suppose that $k\,(\geqslant 1)$ is an integer, g is a constant, $\omega \in \mathbb{C}$ and $\omega^2 \neq 1, 0$. Then the differential equation*

$$(f(z))^2 + 2\omega f(z)f^{(k)}(z) + (f^{(k)}(z))^2 = e^{2g} \tag{8}$$

has no transcendental meromorphic solutions.

LEMMAS

We now state some results that will be used to prove Theorem 3 and Theorem 5.

Lemma 1 ([1]) *Suppose that f is meromorphic and has only a finite numbers of poles in the plane, and that f, $f^{(l)}$ have only a finite number of zeros for some $l \geqslant 2$. Then*

$$f(z)=\frac{P_1(z)}{P_2(z)}\,\mathrm{e}^{P_3(z)},$$

where P_1, P_2 and P_3 are polynomials. If, further, f and $f^{(l)}$ have no zeros, then $f(z)=\mathrm{e}^{Az+B}$ or $f(z)=(Az+B)^{-m}$, where A, B are constants such that $A\neq 0$ and m is a positive integer.

Lemma 2 ([1]) *Suppose that g is a transcendental meromorphic function and h is a nonconstant entire function. Then*

$$\lim_{r\to\infty}\frac{T(r,g(h))}{T(r,h)}=\infty.$$

A differential polynomial $P(z,f)$ in f is a finite sum of products of f, derivatives of f, with all the coefficients being small functions of f in the sense of Nevanlinna theory, namely

$$P(z,f)=\sum_{\lambda\in I}a_\lambda f^{\lambda_0}(f')^{\lambda_1}\cdots(f^{(n)})^{\lambda_n}, \tag{9}$$

where I is a finite index set. The *degree* of a single term in $P(z,f)$ will now be defined as

$$\lambda:=\lambda_0+\lambda_1+\cdots+\lambda_n.$$

Of course, the degree of $P(z,f)$ will then be defined as $\Lambda:=\max_{\lambda\in I}\lambda$.

Lemma 3 ([1, 14]) *Let f be a meromorphic solution of*

$$f^nP_1(z,f)=P_2(z,f),$$

where $P_1(z,f)$ and $P_2(z,f)$ are polynomials in f and its derivatives with meromorphic coefficients $\{a_\lambda|\lambda\in I\}$ such that $T(r,a_\lambda)=S(r,f)$ for all $r\in I$. If the total degree of $P_2(z,f)$ as a polynomial in f and its derivatives is less than or equal to n, then $m(r,P_1(r,f))=S(r,f)$.

PROOF OF THEOREM 3

From (4), after a routine operation, it is immediately concluded that f must be an entire function. Further, we rewrite (4) as

$$[f\,\mathrm{e}^{-g}]^n+[f^{(k)}\,\mathrm{e}^{-g}]^n=1. \tag{10}$$

Suppose that $f\,\mathrm{e}^{-g}$ is a constant, say c_1. Then $c_1\neq 0$ since f is a transcendental entire function and (10) shows that $f^{(k)}\mathrm{e}^{-g}=c_2$, where c_2 is a nonzero constant. Moreover, we see that f and $f^{(k)}$ have no zeros. If $k\geqslant 2$, it follows by Lemma 1 that $f(z)=\mathrm{e}^{Az+B}$, where A, B are constants such that $A^k=c_2/c_1$. So, we can infer that $g(z)=Az+b$, where b is a constant. In the case of $k=1$, using $f'/f=c_2/c_1:=a$, we can immediately deduce $f(z)=\mathrm{e}^{az+B}$ and $g(z)=\frac{c_2}{c_1}z+b=az+b$.

Next, let's focus on considering that $f\,\mathrm{e}^{-g}$ is not a constant. As long as we pay attention to Theorem 1, n must be equal to 2. Otherwise $f\,\mathrm{e}^{-g}$ and $f^{(k)}\mathrm{e}^{-g}$ must be constants for $n\geqslant 3$. In this case, (10) and Theorem 1 imply

$$f\,\mathrm{e}^{-g}=\sin h,\quad f^{(k)}\,\mathrm{e}^{-g}=\cos h, \tag{11}$$

where h is an entire function.

Set $\alpha_1=g'$, $\beta_1=h'$. Then from (11), we have

$$f'=(\alpha_1\sin h+\beta_1\cos h)\,\mathrm{e}^g, \tag{12}$$

which tells us

$$f''=(\alpha_2\sin h+\beta_2\cos h)\,\mathrm{e}^g, \tag{13}$$

where $\alpha_2=g''+(g')^2-(h')^2$, $\beta_2=2g'h'+h''$.

Again, it follows by (13) that

$$\begin{aligned}
f'''&=(\alpha_3\sin h+\beta_3\cos h)\,\mathrm{e}^g,\\
\alpha_3&=\alpha_2'+\alpha_1\alpha_2-\beta_1\beta_2\\
&=g'''+3g'g''-3h'h''-3g'(h')^2+(g')^3,\\
\beta_3&=\beta_2'+\alpha_1\beta_2+\alpha_2\beta_1\\
&=h'''+3g'h''+3g''h'+3(g')^2h'-(h')^3.
\end{aligned} \tag{14}$$

Thus, using the same way, for $k\geqslant 3$, we deduce

$$\begin{aligned}
f^{(k)}&=(\alpha_k\sin h+\beta_k\cos h)\,\mathrm{e}^g,\\
\alpha_k&=\alpha_{k-1}'+\alpha_1\alpha_{k-1}-\beta_1\beta_{k-1},\\
\beta_k&=\beta_{k-1}'+\alpha_1\beta_{k-1}+\alpha_{k-1}\beta_1.
\end{aligned} \tag{15}$$

Moreover, it follows by (15) that the expressions for α_k and β_k can be given using g', g'', h', h'', ... For example

$$\begin{aligned}
\alpha_4={}&g^{(4)}+4g'g'''+3(g'')^2+6(g')^2g''+(g')^4-6(g')^2(h')^2\\
&-6g''(h')^2-12g'h'h''-4h'h'''+(h')^4-3(h'')^2,
\end{aligned}$$

$$\begin{aligned}
\beta_4={}&h^{(4)}+4g'h'''+6g''h''+4g'''h'+12g'g''h'+4(g')^3h'\\
&+6(g')^2h''-6(h')^2h''-4g'(h')^3,
\end{aligned}$$

$$\begin{aligned}
\alpha_5={}&g^{(5)}+5g'g^{(4)}+10g''g'''+10(g')^2g'''+15g'(g'')^2\\
&+10(g')^3g''+(g')^5-10(g')^3(h')^2-30g'g''(h')^2\\
&-10g'''(h')^2-30(g')^2h'h''-30g''h'h''-20g'h'h'''\\
&+10(h')^3h''-5h'h^{(4)}-15g'(h'')^2-10h''h'''+5g'(h')^4,
\end{aligned}$$

$$\begin{aligned}
\beta_5={}&h^{(5)}+5g'h^{(4)}+10g''h'''+10(g')^2h'''+10g'''h''\\
&+30g'g''h''+5g^{(4)}h'+10(g')^3h''+15(g'')^2h'\\
&+30(g')^2g''h'+20g'g'''h'-30g'(h')^2h''-10(g')^2(h')^3\\
&-10g''(h')^3-10(h')^2h'''+5(g')^4h'-15h'(h'')^2+(h')^5.
\end{aligned}$$

Since g is a nonconstant polynomial and h is a nonconstant entire function, according to Lemma 2, when $\alpha_j \not\equiv 0$ and $\beta_j \not\equiv 0$, we claim that α_k, β_k are small functions of $\sin h$, $\cos h$ and $\cot h$.

Case 1 If $k = 1$, then by making use of (11) and (12), we obtain that

$$g' \sin h = (1 - h') \cos h. \tag{16}$$

It is easy to see that $g' \not\equiv 0$ because we have assumed that g is a nonconstant polynomial. Applying Lemma Lemma 2 to (16) gives

$$\begin{aligned} T(r, \cot h) &= T(r, \frac{g'}{1-h'}) \\ &\leqslant T(r, h') + O(\log r) = S(r, \cot h), \end{aligned} \tag{17}$$

which is a contradiction.

Case 2 Suppose that $k = 2$. Now, by using (11) and (13), we have

$$\alpha_2 \sin h + \beta_2 \cos h = \cos h. \tag{18}$$

Firstly, we assume that $\alpha_2 \not\equiv 0$. Now, applying Lemma 2 to (18) implies

$$\begin{aligned} T(r, \cot h) &= T(r, \frac{\alpha_2}{1-\beta_2}) \\ &\leqslant 3T(r, h') + O(\log r) = S(r, \cot h), \end{aligned} \tag{19}$$

which is a contradiction. Therefore, we must have $\alpha_2 = g'' + (g')^2 - (h')^2 \equiv 0$ and $2g'h' + h'' - 1 \equiv 0$. Note that g is a nonconstant polynomial, so that h can only be a nonconstant polynomial after calculation. Moreover, $\deg h = \deg g = 1$. Thus, $g(z) = az + b$, $h(z) = cz + d$ and $f(z) = \mathrm{e}^{az+b} \sin(cz + d)$, where a, b, c, d are constants with $a = \pm c$, $2ac = 1$.

Case 3 Assume that $k \geqslant 3$. Let us start with $k = 3$. If $\alpha_3 \not\equiv 0$, then $\beta_3 - 1 \not\equiv 0$. It follows by (14), (11) and Lemma 2 that

$$\begin{aligned} T(r, \cot h) &= T(r, \frac{\alpha_3}{1-\beta_3}) \\ &= O\{T(r, h')\} + O(\log r) = S(r, \cot h), \end{aligned}$$

this is impossible. Consequently, we get $\alpha_3 \equiv 0$, and $\beta_3 - 1 \equiv 0$, that is

$$g''' + 3g'g'' - 3h'h'' - 3g'(h')^2 + (g')^3 \equiv 0 \tag{20}$$

and

$$h''' + 3g'h'' + 3g''h' + 3(g')^2h' - (h')^3 - 1 \equiv 0. \tag{21}$$

Suppose that h' is a transcendental entire function. From (21), we deduce

$$(h')^2h' = h''' + 3g'h'' + 3g''h' + 3(g')^2h' - 1,$$

which and Lemma 3 show that $m(r, h') = S(r, h')$, this is absurd. Thus, h' is a polynomial. Next, we will prove g' and h' are nonzero constants. If not, we may set

$$\begin{aligned} g'(z) &= a_s z^s + a_{s-1} z^{s-1} + \cdots + a_0, \\ h'(z) &= b_t z^t + b_{t-1} z^{t-1} + \cdots + b_0, \end{aligned}$$

where $a_j \, (j = 0, 1, \ldots, s)$, $b_l \, (l = 0, 1, \ldots, t)$ are constants with $a_s b_t \neq 0$ and $s \geqslant 1$, $t \geqslant 1$. By carefully comparing the coefficients of Eqs. (20) and (21) to the same power, it is not difficult to work out

$$3a_s b_t^2 = a_s^3 \quad \text{and} \quad 3a_s^2 b_t = b_t^3,$$

which, of course, is impossible. By the above discussion, we see that g' and h' are nonzero constants. Let $g(z) = az + b$, $h(z) = cz + d$. Further, by (20) and (21), we obtain $a^2 = 3c^2$, $8c^3 = 1$, and $f(z) = e^{az+b} \sin(cz + d)$.

For $k \geqslant 4$, using exactly the same ideas as above, it can be shown that $\alpha_k \equiv 0$ and $\beta_k \equiv 1$, which yield g and h are linear (first-order) polynomials. Thus, $g(z) = az+b$, $h(z) = cz+d$ and $f(z) = e^{az+b} \sin(cz+d)$.

This completes the proof of Theorem 3.

PROOF OF THEOREM 5

Assume that f is a solution of (8), it is easy to see that f must be an entire function. To prove this Theorem, first of all, we rewrite (8) as follows

$$[\mathrm{i}\sqrt{\omega^2 - 1} f]^2 + [\omega f + f^{(k)}]^2 = \mathrm{e}^{2g},$$

which gives

$$[\mathrm{i}\sqrt{\omega^2 - 1} f \ \mathrm{e}^{-g}]^2 + [(\omega f + f^{(k)}) \mathrm{e}^{-g}]^2 = 1. \tag{22}$$

Obviously, by (22) and Theorem 1, we have

$$\mathrm{i}\sqrt{\omega^2 - 1} f \ \mathrm{e}^{-g} = \sin h, \ (\omega f + f^{(k)}) \mathrm{e}^{-g} = \cos h, \tag{23}$$

where h is an entire function.

Moreover, it follows by (23) that

$$\begin{aligned} f &= \frac{1}{\mathrm{i}\sqrt{\omega^2 - 1}} \mathrm{e}^g \sin h, \\ f^{(k)} &= (\cos h + \frac{\mathrm{i}\omega}{\sqrt{\omega^2 - 1}} \sin h) \mathrm{e}^g. \end{aligned} \tag{24}$$

Thus, we apply the same idea as proving Theorem 3 and obtain

$$f' = \frac{1}{i\sqrt{\omega^2 - 1}} (\alpha_1 \sin h + \beta_1 \cos h) e^g, \tag{25}$$

where $\alpha_1 = g'$, $\beta_1 = h'$.

$$f'' = \frac{1}{\mathrm{i}\sqrt{\omega^2 - 1}} (\alpha_2 \sin h + \beta_2 \cos h) \mathrm{e}^g, \tag{26}$$

where $\alpha_2 = g'' + (g')^2 - (h')^2$, $\beta_2 = 2g'h' + h''$.

Consequently, when $k \geqslant 2$, we have

$$\begin{aligned} f^{(k)} &= \frac{1}{\mathrm{i}\sqrt{\omega^2-1}}(\alpha_k \sin h + \beta_k \cos h)\,\mathrm{e}^g, \\ \alpha_k &= \alpha'_{k-1} + \alpha_1\alpha_{k-1} - \beta_1\beta_{k-1}, \\ \beta_k &= \beta'_{k-1} + \alpha_1\beta_{k-1} + \alpha_{k-1}\beta_1. \end{aligned} \tag{27}$$

Suppose that h is a constant. Then, by (24), we see f and $f^{(k)}$ have no zeros. If $k \geqslant 2$, it follows by Lemma 1 that $f(z) = \mathrm{e}^{Az+B}$, where A, B are constants. So, we can infer that $g(z) = Az + b$, where b is a constant. If $k = 1$, then (24) and (25) give $g' = \mathrm{i}\sqrt{\omega^2-1}\cot h - \omega$. Therefore, $g(z) = (\mathrm{i}\sqrt{\omega^2-1}\cot h - \omega)z + b := az + b$ and $f(z) = \mathrm{e}^{az+B}$.

In the following, we may suppose that h is a nonconstant entire function, according to Lemma 2, we claim that α_j, β_j $(j = 1, 2, \dots, k)$ are small functions of $\sin h$, $\cos h$ and $\cot h$.

Case 1 If $k = 1$, then (24) and (25) imply

$$(\omega + g')\sin h = \mathrm{i}(\sqrt{\omega^2-1} + \mathrm{i}h')\cos h. \tag{28}$$

Now, applying Lemma 2 to (28), one has $g' = -\omega$, $h' = \mathrm{i}\sqrt{\omega^2-1}$ and thus $g(z) = -\omega z + b$, $h(z) = \mathrm{i}\sqrt{\omega^2-1}z + d$ and $f(z) = \frac{1}{\mathrm{i}\sqrt{\omega^2-1}}\,\mathrm{e}^{-\omega z+b}\sin(\mathrm{i}\sqrt{\omega^2-1}\,z + d)$, where b and d are constants.

Case 2 If $k \geqslant 2$, then (24) and (27) imply

$$(\omega + \alpha_k)\sin h = i(\sqrt{\omega^2-1} + i\beta_k)\cos h. \tag{29}$$

Now, applying Lemma 2 to (29), one has $\alpha_k = -\omega$, $\beta_k = \mathrm{i}\sqrt{\omega^2-1}$. Now, by the similar methods employed as in the proof of Theorem 3, we can obtain $g(z) = az + b$, $h(z) = cz + d$ and $f(z) = \frac{1}{\mathrm{i}\sqrt{\omega^2-1}}\,\mathrm{e}^{az+b}\sin(cz + d)$, where a, b, c and d are constants.

The proof of Theorem 5 is completed.

CONCLUSION

By examining the proof of Theorem 3 carefully, we will find that if $n \geqslant 3$ or $k = 1$, the condition that g is a nonconstant polynomial is not necessary. In other words, the conclusion of Theorem 3 still holds if we replace the nonconstant polynomial g with a general nonconstant entire function when $n \geqslant 3$.

Now, an important and interesting question is raised as follows:

Question 1 What can be said if the nonconstant polynomial g is replaced by a general nonconstant entire function when $n = 2$ in Theorem 3?

Clearly, Theorem 5 is an extension and supplement of Theorem 4. In addition, Zhang, Yang and Ng [15] proved the following conclusion:

Theorem 7 *Let $k \geqslant 2$ be an integer and a be a nonconstant entire function. Then the differential equation $f^2 + a(f^{(k)})^2 = 1$ has no admissible entire solution.*

Some related questions were studied by Zhang and Liao [16] and some related results can be referred references therein. In [17], Gross, Osgood and Yang gave necessary and sufficient conditions for the existence of entire solutions to the functional equation $\phi^2 + g\varphi^2 = h$, where g, h are given nonzero polynomials in z. We think such arguments can then used to produce transcendental entire function solutions of $pf^2 + qg^2 = \mathrm{e}^{2g}$. The special case is $pf^2 + q(f^{(k)})^2 = \mathrm{e}^{2g}$, where p, q and g are polynomials with $pq \not\equiv 0$. For possible future discussion, we are very interested in the following question:

Question 2 How to find out all admissible solutions to the following equation

$$[f(z)]^n + r(z)[f(z)]^s[L(z,f)]^t + [L(z,f)]^n = \mathrm{e}^{ng(z)},$$

where n, k, s, t are positive integers with $s + t \leqslant n$, $L(z,f) = \sum_{j=1}^{k} b_j f^{(j)}$, $b_1, \dots, b_k$ are polynomials with $b_k \not\equiv 0$, and r, g are entire functions?

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